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GEOMETRICAL-ACOUSTICS CONSIDERATION OF THE FLEXURAL MODES IN IMMERSED ANISOTROPIC WEDGES

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Geometrical-acoustics approach interprets vibration modes localized at the edge of wedges as the quasi-plane flexural waves propagating in a plate of variable thickness. This approach is combined with the dispersion relation for flexural wave in a thin anisotropic fluid-loaded plate to analytically determine the subsonic velocities c of the localized modes in anisotropic immersed wedges. The transcendent equation in c is established for an arbitrarily anisotropic wedge material and a general case of the wedge–fluid coupling. An approximate explicit solution for c is obtained in the cases when the parameter of the wedge-fluid coupling $\theta n/r$ is either small or large (here θ is the apex angle, n is the modal order, and r is the ratio of the fluid density and the wedge density). In both cases, the ratio of the wedge-mode velocities c/c_0 in the immersed and free wedge is a corresponding function of the coupling parameter $\theta n/r$. Provided that the wedge–fluid coupling is sufficiently pronounced, the ratio c/c_0 in the presence of anisotropy acquires the scaling factor, which depends appropriately on elastic coefficients of the wedge and turns to unity in the isotropic limit.

1. INTRODUCTION

Flexural wedge waves propagating along the edges of free elastic wedges were predicted in 1972 by Lagasse [1] and Maradudin *et al.* [2], and have been studied since then both theoretically and experimentally (see references [3–12]). The absence of dispersion, low velocity, and localization near the wedge tip make these waves attractive for possible applications in signal processing, non-destructive testing of special engineering constructions, structural dynamics, etc. Because of the complexity of the boundary problem, no exact analytical theory of wedge waves is available even for the case of elastically isotropic material of a wedge. The approximate analytical model, based on the geometrical acoustics approach, has been put forward for free slender wedges consisting of isotropic [5, 6] and anisotropic [7, 12] materials.

More recently, flexural localized waves in fluid-loaded wedges have been also investigated [11, 13–17]. It was shown in reference [16] that the geometrical-acoustics method for isotropic immersed wedges provides good agreement with experiment. In the present paper, this method is further developed and applied to obtain explicit analytical approximation for the case of an anisotropic immersed wedge.

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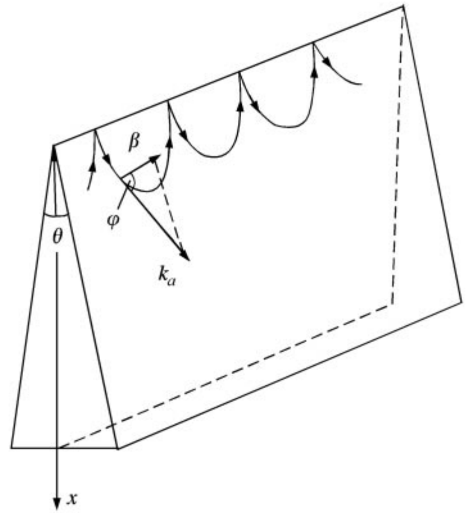

Figure 1. Geometry of the problem.

2. GENERAL CASE OF WEDGE-FLUID COUPLING

Consider a fluid-loaded elastic wedge with an acute apex angle θ. According to reference [16], the velocity c of the nth order antisymmetric localized mode, propagating along the edge, may be defined by the equation in its wavenumber $\beta = \omega/c$,

$$\int_C [k_a^2(x, \varphi) - \beta^2]^{1/2}\, \mathrm{d}x = 2\pi n \quad (n = 1, 2, \dots), \tag{1}$$

where

$$\beta = k_a(x, \varphi) \cos \varphi, \tag{2}$$

x is the co-ordinate orthogonal to the wedge edge; φ is the polar angle varying in the mid-plane of the wedge, and k_a is the wavenumber of the flexural mode in a fluid-loaded plate of the small variable thickness $h = \theta x$. The integration path C follows the ray trajectory, which starts from the edge $x = 0$ at $\varphi = \pi/2$, passes the turning point x_t corresponding to $\varphi = 0$, and returns to the edge $x = 0$ with $\varphi = -\pi/2$ (Figure 1).

Let c_f be the speed of sound in fluid, ρ_f the density of fluid, and ρ the density of wedge material. Denote their ratio by

$$r = \frac{\rho_f}{\rho}. \tag{3}$$

Regarding the flexural wavenumber k_a for a thin ($k_a h \ll 1$) fluid-loaded plate of unrestricted anisotropy, it can be shown that the corresponding approximate dispersion equation in the subsonic region $\omega/k_a < c_f$ may be written similar to the case of isotropy as

$$k_a^5 h^3 - \frac{\omega^2}{\gamma^2} k_a h - 2r \frac{\omega^2}{\gamma^2} = 0, \tag{4}$$

where

$$\gamma = \frac{c_{0a}}{k_{0a}h} \tag{5}$$

is the coefficient of linear dependence of the velocity c_{0a} upon the wavenumber k_{0a} for the free-plate flexural mode at $k_{0a}h \ll 1$. (Throughout the paper, subscript 0 indicates reference to the mechanically free state as opposed to the fluid-loaded state for both a plate and a wedge.) The angular dependence $\gamma = \gamma(\varphi)$ accounts for the effects of anisotropy. For an arbitrary anisotropic elastic material it follows that [18]

$$\gamma(\varphi) = \frac{\sqrt{\mathbf{m}\cdot[(mm)-(mn)(nn)^{-1}(nm)]\,\mathbf{m}}}{2\sqrt{3\rho}}, \tag{6}$$

where $\mathbf{m} = \mathbf{m}(\varphi)$ is the unit vector turning about the angle φ in the mid-plane, $\mathbf{n}$ is the unit vector normal to the mid-plane, and $(\cdots)$ are the matrices written by means of notation: $(ab) \equiv a_j c_{ijkl} b_k$ for any vectors $\mathbf{a}$, $\mathbf{b}$. In the generic case, the quadratic form under the radical in equation (6) is a homogeneous polynomial of the fourth degree in $\sin\varphi$, $\cos\varphi$ with the coefficients depending on elastic moduli. It is strictly positive definite for an arbitrary elasticity tensor c_{ijkl} and any vectors $\mathbf{m}$, $\mathbf{n}$ [19]. In the case of isotropy in the mid-plane, the coefficient γ has the constant value $\gamma = c_{0p}/2\sqrt{3}$, where

$$c_{0p} = \sqrt{\frac{1}{\rho}\left(c_{11} - \frac{c_{13}^2}{c_{33}}\right)} \tag{7}$$

is the velocity of longitudinal dilatational mode in a free transversely isotropic plate with the principal symmetry axis orthogonal to the faces (for brevity, this setting is hereafter referred to as isotropic).

By equation (2), equation (1) may be cast into the form

$$\beta \int_C \tan\varphi \,\mathrm{d}x = 2\pi n. \tag{8}$$

Correspondingly, substituting $h = \theta x$ and equation (2) into equation (4) leads to the equation, which involves variables x, φ and the unknown constant parameter $\beta = \omega/c$. Taking φ as a free variable specifies this equation as a cubic one in $x(\varphi)$. The correctly chosen root which tends to the appropriate solution $x(\varphi) = c^2\cos^2\varphi/\omega\theta\gamma(\varphi)$ of equation (4) in the limit of vanishing fluid density $r \to 0$, is

$$x(\varphi) = \frac{(rc^5)^{1/3}}{\omega\theta\gamma^{2/3}(\varphi)}\left[\left(1+\sqrt{1-\frac{c^2\cos^2\varphi}{27r^2\gamma^2(\varphi)}}\right)^{1/3} + \left(1-\sqrt{1-\frac{c^2\cos^2\varphi}{27r^2\gamma^2(\varphi)}}\right)^{1/3}\right]\cos^{5/3}\varphi. \tag{9}$$

Rewriting equation (8) in the form

$$\frac{\omega}{c}\int_{\pi/2}^{-\pi/2} x'(\varphi)\tan\varphi \,\mathrm{d}\varphi = 2\pi n \tag{10}$$

and inserting the derivative of equation (9) yields the transcendent equation in c, which is applicable for an arbitrary anisotropic material of the slender wedge and any orientation of its mid-plane. (Once the latter is fixed, integral (10) does not depend upon the orientation of the edge in the given plane due to the period π of $\gamma(\varphi)$.) At the same time, the unknown

c appears under the integral, which cannot be evaluated explicitly even in the case of isotropy, so this equation may be solved only numerically.

Seeking explicit analytical solution for the wedge-mode velocity, one has to implement further approximation of the thin-plate dispersion relation (4). The approximation may be stipulated by two alternative strong inequalities $k_a h \ll r$ and $k_a h \gg r$. When applied for the immersed-wedge problem, those two limiting cases may be interpreted as, respectively, strong and weak wedge-fluid coupling.

3. STRONG WEDGE-FLUID COUPLING

Suppose that $k_a h \ll r$. Then (4) yields the approximate solution

$$k_a \approx \frac{(2r)^{1/5}}{(\theta x)^{3/5}} \left(\frac{\omega}{\gamma}\right)^{2/5}, \tag{11}$$

where $h = \theta x$ has been taken into account. If the wedge is isotropic, then inserting equation (11) into equation (2) allows resolving it for $\tan\varphi$ as a function of x and taking the integral in equation (8), as has been done in reference [16]. This procedure can no longer be carried out in the presence of anisotropy, when $\gamma = \gamma(\varphi)$. At the same time, combining equations (11) and (2) readily supplies x as a function of φ.

$$x(\varphi) = \frac{(2rc)^{1/3}}{\theta\omega} \frac{\cos^{5/3}\varphi}{\gamma^{2/3}(\varphi)}, \tag{12}$$

which approximates equation (9). On inserting the derivative $x'(\varphi)$ into equation (10), one obtains the wedge-mode velocity c in the following explicit form:

$$c = \frac{2\pi^{3/2}}{I} \frac{(\theta n)^{3/2}}{r^{1/2}}, \tag{13}$$

in which

$$I = \left\{ \int_{\pi/2}^{-\pi/2} \left[\frac{\cos^{5/3}\varphi}{\gamma^{2/3}(\varphi)} \right]' \tan\varphi \, \mathrm{d}\varphi \right\}^{3/2}. \tag{14}$$

Recall that, according to reference [12], the velocity c_0 in the free (dry) anisotropic wedge is

$$c_0 = \frac{\pi}{\sqrt{3}J} \theta n, \tag{15}$$

where

$$J = \frac{1}{2\sqrt{3}} \int_{\pi/2}^{-\pi/2} \left[\frac{\cos^2\varphi}{\gamma(\varphi)} \right]' \tan\varphi \, \mathrm{d}\varphi. \tag{16}$$

Hence, equation (13) may be presented in the form

$$\frac{c}{c_0} = \frac{2\sqrt{3}\pi J}{I} \sqrt{\frac{\theta n}{r}}, \tag{17}$$

where the first fraction on the right-hand side depends (only) on elastic coefficients of an anisotropic wedge.

In the case of isotropic wedge,

$$I^{(iso)} = \frac{2\sqrt{3}\xi^{3/2}}{c_{0p}}, \tag{18}$$

where c_{0p} is the plate velocity (7), and the notation ξ is used for the table integral

$$\xi \equiv 5\int_0^1 \sqrt{1-y^3}\,\mathrm{d}y \approx 4. \tag{19}$$

Then equations (13) and (17) reduce to the result obtained in reference [16], namely,

$$c^{(iso)} = \left(\frac{\pi}{\xi}\right)^{3/2}\frac{c_{0p}}{\sqrt{3}}\frac{(\theta n)^{3/2}}{r^{1/2}}, \qquad \frac{c^{(iso)}}{c_0^{(iso)}} = \left(\frac{\pi}{\xi}\right)^{3/2}\sqrt{\frac{\theta n}{r}}, \tag{20a, b}$$

in which

$$c_0^{(iso)} = \frac{c_{0p}}{\sqrt{3}}\theta n \tag{21}$$

is the velocity of the nth mode in the free isotropic wedge [5]. Conjunction of equations (13) and (20a) leads to the relation

$$c = \chi c^{(iso)}, \tag{22}$$

where $c^{(iso)}$ is the velocity for the reference isotropic material, which is characterized by the value c_{0p} in it, and

$$\chi = 2\sqrt{3}\xi^{3/2}\frac{1}{c_{0p}I} \tag{23}$$

is the scaling factor describing the impact of the wedge material anisotropy on the velocity in the immersed wedge. For comparison, a similar factor χ_0 for the free wedge is, by equations (15) an (21),

$$c_0 = \chi_0 c_0^{(iso)}, \quad \chi_0 = \pi\frac{1}{c_{0p}J}. \tag{24}$$

Let us evaluate the range of validity of the obtained solution which is stipulated by the initial assumption $k_a h \ll r$. Combining equations (11)–(13) gives $k_a h = (2\pi\cos^{2/3}\varphi/(\gamma I)^{2/3})\,\theta n$, so that $(k_a h)_{max} \sim \theta n$, where the numerical factor is of the order of unity (it is $2\pi/\xi \approx 1{\cdot}5$ in the case of isotropy). Hence, a good approximation for the velocity of the nth wedge mode is guaranteed in the range

$$\frac{\theta n}{r} \ll 1. \tag{25}$$

Regarding the range of values of the apex angle θ, recall that the geometrical acoustics approach is well justified provided that θ is small [5, 6]. At the same time, it has been noted in reference [16] that the theoretical expression (20b) for the ratio of velocities in the isotropic immersed and free wedge manifests a remarkably good agreement with the experimental results also for large θ, which is probably due to the fact that the application of the thin-plate approximation to the case of large θ brings in the same relative distortions into the free- and immersed-wedge velocities, so that they cancel when the ratio of these velocities is taken. Anyhow, this conjecture seems to be less likely in the case of generic anisotropy, which is prone to entail dissimilar dispersion dependencies of the velocities in the free and immersed thick plates (as distinct from the thin plates, for which anisotropy retains the dependence on kh, affecting only the coefficients).

As an example, consider a wedge made of the tetragonal material, with the mid-plane orthogonal to the four-fold axis. Then

$$\gamma(\varphi) = \frac{c_{0p}}{2\sqrt{3}}\sqrt{1 - A\sin^2 2\varphi}, \tag{26}$$

where $A = (c_{11} - c_{12} - 2c_{66})/2\rho c_{0p}^2$ is the anisotropy parameter, which shows the departure of a given tetragonal material from the reference (transversely) isotropic material with the same value c_{0p}. Assuming weak anisotropy ($|A| \ll 1$) readily allows explicit evaluation of the integral (14), so that, by equation (23), the scaling factor for the immersed wedge is $\chi \approx 1 + 0{\cdot}23A$. For comparison, the scaling factor for the same wedge in vacuum is, by equation (24b), $\chi_0 \approx 1 - 0{\cdot}25A$.

4. WEAK WEDGE-FLUID COUPLING

Now we suppose that $r \ll k_a h$ ($\ll 1$). Then the dispersion equation (4) for a flexural-mode wavenumber k_a in a thin immersed plate may be approximately replaced by the equation

$$k_a^3 h^2 - \frac{\omega}{\gamma} k_a h - \frac{\omega}{\gamma} r = 0. \tag{27}$$

Seeking the solution k_a as a disturbance of the corresponding wavenumber $k_{0a} = \sqrt{\omega/\gamma h}$ in a free plate (see equation (5)), we obtain

$$k_a = k_{0a} + \frac{r}{2h}. \tag{28}$$

Inserting $h = \theta x$ and combining equation (28) with equation (2) leads to the relation

$$x(\varphi) = \frac{c^2\cos^2\varphi}{\omega\theta\gamma(\varphi)} + r\frac{c\cos\varphi}{\omega\theta}, \tag{29}$$

which approximates equation (9) for the weak-coupling limiting case. In this relation, we observe the fluid-uncoupled term and its perturbation by fluid loading, and we note that the presence of anisotropy affects the first term, but not the second one. Substituting the derivative of equation (29) into equation (10) yields the equation

$$2\pi n\frac{c}{c_0} + 2\frac{r}{\theta}\int_0^{\pi/2-\delta}\frac{\sin^2\varphi}{\cos\varphi}\,d\varphi = 2\pi n. \tag{30}$$

On the left-hand side of equation (30), c_0 is the velocity in the free wedge, given by equation (15), and c is the unknown velocity in the immersed wedge, whose value is affected by the fluid-loading perturbation described by the second term. Cutting off the integration path close to the edge (taking the upper limit in the integral in equation (30) to be $\pi/2 - \delta$, where $0 \neq \delta \ll 1$) in the considered approximation is stipulated by the initial assumption $r \ll k_a h$, which prevents h from reaching zero. The value δ in the leading approximation may be evaluated using the limiting estimate $r \sim k_{0a} h$ and neglecting anisotropy in it. Then, recalling that $\omega/c_0 = k_{0a} \cos\varphi$ (see equation (2)) and invoking equations (5) and (21), we may put

$$\delta = \frac{r}{2\theta n} \ll 1. \tag{31}$$

Applying equation (31) to the integral in equation (30) and confining to the principal order of approximation at $r/\theta n \ll 1$ gives

$$\frac{c}{c_0} = 1 - \frac{r}{\pi\theta n} \ln\left(\frac{4}{\mathrm{e}}\frac{\theta n}{r}\right). \tag{32}$$

It is clear that equation (32) applies to the range of values $\theta n/r$, certainly exceeding $e^2/4$, for which c/c_0 monotonically increases with growing $\theta n/r$ and tends to 1 at $\theta n/r \gg 1$. Comparison of the dependence (32) with the numerical solution of the transcendent equation [16] and experimental results [14], obtained for the mode $n = 1$ in the isotropic case (brass wedge immersed in water, $r = \rho_f/\rho \approx 0{\cdot}11$), shows a good agreement for apex angles $\theta > 20^\circ$ ($\theta n/r > 3$). Note that the dependence of c/c_0 solely on the ratio $\theta n/r$, where $r \ll 1$, may explain why in this case equation (32), based on the thin-plate approximation, applies for θ being not small.

It should be noted that the numerical factor ($4/e$) under the logarithm in equation (32) is a result of the rather rough approximation. However, the possible theoretical inaccuracy in predicting this factor is not critical in the range $\theta n/r \gg 1$, where this logarithm is a very slowly changing function anyway. With this reservation borne in mind, it follows from equation (32) that the weak wedge–fluid coupling in the leading approximation does not depend on elastic coefficients and hence on the elastic anisotropy. In other words, in this case the effect of anisotropy on the velocity c in the immersed wedge is described by the same scaling factor as on its value c_0 in the free wedge (see equations (24)).

5. CONCLUSIONS

Localized modes in immersed anisotropic wedges have been considered. The transcendent equation for wedge-mode velocity c is obtained for an arbitrary material of the wedge and a general case of the wedge–fluid coupling. The approximate explicit solution for the ratio of wedge wave velocities c/c_0 in the immersed and free wedge has been established in the cases of strong and weak wedge–fluid coupling as functions of the coupling parameter $\theta n/r$. In the case of strong coupling ($\theta n/r \ll 1$), the effect of wedge material anisotropy on the value c/c_0 is represented by the additional scaling factor, which depends on the elastic coefficients of the wedge and turns to unity in the isotropic limit. In the case of weak coupling ($\theta n/r \gg 1$), elastic anisotropy in the leading approximation considered has the same scaling effect on the velocities c and c_0, and therefore does not affect the ratio c/c_0.

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