Journal of Sound and Vibration (2000) **237**(3), 542–549
doi:10.1006/jsvi.2000.3004, available online at http://www.idealibrary.com on IDEAL®

FREE VIBRATION OF A RESTRAINED SHEAR-DEFORMABLE TAPERED BEAM WITH A TIP MASS AT ITS FREE END*

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(*Received* 5 *January* 2000, *and in final form* 22 *March* 2000)

1. INTRODUCTION

Movable arms, tall buildings, towers and antennae are the most typical examples that can be reduced to a Timoshenko beam variable cross-section. By using this theory, free vibration frequencies have been obtained by many authors, employing finite element techniques. For example, To [1, 2] examined a beam with varying cross-section, for various boundary conditions, by using a cubic–linear interpolating function. A similar approach was used by Cleghorn and Tabarrok [3], who were able to obtain the exact stiffness matrix of the element and therefore more accurate results were obtained, with less effort. Rossi *et al.* [4] have presented a refined finite element formulation for tapered beams elements. Laura and Gutierrez [5] employ a refined Rayleigh–Ritz method and a sophisticated finite element model, but their results are limited to the fundamental frequency. For a cantilever uniform beam with a tip mass at the free end, Bruch and Mitchell [6] have obtained the exact solution. Shortly after, Abramovich and Hamburger [7] extended the analysis to eccentric masses. If the cross-section is supposed to vary according to a continuous law, Laura *et al.* [8] proposed an FEM-like algorithm, which was illustrated earlier by Prezemieniecki [9]. Both in reference [5] and references [8–10], upper bounds to the true results for the fundamental frequencies are obtained.

In this article, as already emphasized in reference [11], a Lagrangian approach is used. The structure is reduced to a set of rigid bars linked together by means of elastic constraints, and consequently a stiffer structure than the real one is obtained, whereas a displacement-based FE method leads to a more flexible system.

2. ANALYSIS OF THE MODEL

Consider the beam in Figure 1, in which the width remains constant and the height of the cross-section varies linearly, according to the following law:

$$h(z) = h_0\left(1 + \frac{h^* - 1}{L}\,z\right), \qquad I(z) = I_0\left(1 + \frac{h^* - 1}{L}\,z\right)^3, \tag{1, 2}$$

where h_0 and I_0 are the height of the cross-section and the cross-sectional inertia at the left end, and $h^* = h(L)/h_0$.

* Work partially supported by Italian M.U.R.S.T. 60%

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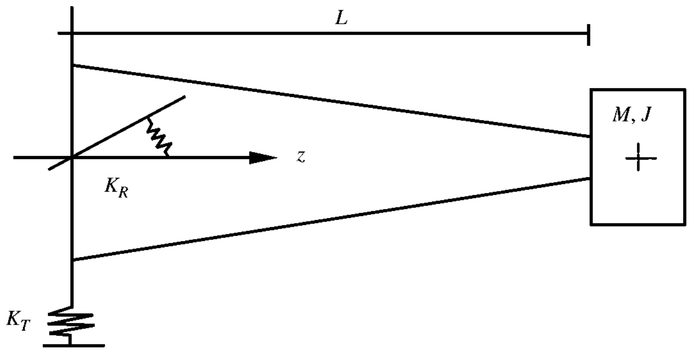

Figure 1. The structural system under study for the vibration problem.

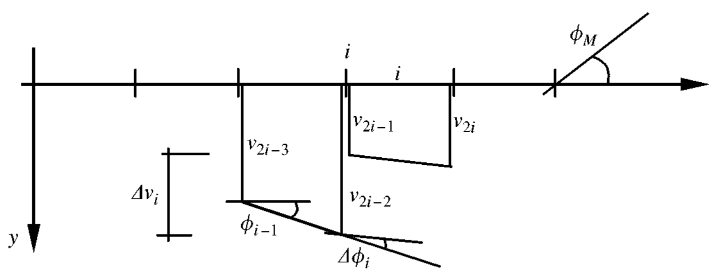

Figure 2. The discretization model.

The beam is supposed to be divided into t rigid bars, linked together by means of elastic elements which allow relative rotations and relative vertical displacements. Therefore, the structure is reduced to a finite-degree-of-freedom system. The displacements of the ith rigid bar can be easily deduced if the vertical displacements of the both its ends are known (Figure 2).

The elastic constraints take into account both the bending deformation and the shear deformation. The strain energy of the system is given by the sum of the bending strain and of the shear strain energy. At the ith abscissa, the following linear relationship holds:

$$U_i = \tfrac{1}{2}\,(M_i \Delta\varphi_i + T_i \Delta \mathrm{v}_i), \tag{3}$$

where M_i and T_i are the bending moment and the shear stress. The strain energy of the structure is

$$U = \frac{1}{2}\sum_{i=1}^{t+1}\left(2E\,\frac{I_i I_{i+1}}{I_{i+1}l_i + I_i l_i}\right)\Delta\varphi_i^2 + \frac{1}{2}\sum_{i=1}^{t+1}\left(2\,Gk\,\frac{A_i A_{i+1}}{A_{i+1}l_i + A_i l_{i+1}}\right)\Delta v_i^2$$
$$= \frac{1}{2}\sum_{i=1}^{t+1}(k_{fi}\,\Delta\varphi_i^2 + k_{si}\Delta v_i^2), \tag{4}$$

where E and G are the Young's and the shear moduli respectively, A is the cross-sectional area, I is the moment of inertia and k the shear factor.

Substituting

$$\mathbf{c}^{\mathrm{T}} = [v_1, v_2, \ldots, v_{2t}, \varphi_M] \tag{5}$$

the relative rotations are written as

$$\Delta\varphi_1 = \frac{(v_1 - v_2)}{l_1} - \varphi_1, \quad \Delta\varphi_i = \frac{(v_{2i-1} - v_{2i})}{l_i} + \frac{(v_{2i-2} - v_{2i-3})}{l_{i-1}}, \quad \Delta\varphi_{t+1} = v_{2t} - \frac{v_{2t-1}}{l_t} + \varphi_M \tag{6}$$

and the relative displacements as

$$\Delta v_1 = v_1, \qquad \Delta v_1 = v_{2i-1} - v_{2i-2}, \qquad \Delta v_{t+1} = 0 \tag{7}$$

or, in matrix from as

$$\Delta\boldsymbol{\varphi} = \mathbf{Ac}, \qquad \Delta\mathbf{V} = \mathbf{Bc}. \tag{8, 9}$$

The kinetic energy can be written as

$$T = \frac{1}{2}\sum_{i=1}^{2t} m_i\dot{v}_i^2 + \frac{1}{2}\sum_{i=1}^{t} \rho l_i I_i \dot{r}_i^2 + \frac{1}{2} I_M \dot{\varphi}_M^2. \tag{10}$$

The strain energy of the whole structure, in matrix form, can therefore be calculated as

$$U = \tfrac{1}{2}\,\mathbf{c}^{\mathrm{T}}\left[\mathbf{A}^{\mathrm{T}}\mathbf{D}_f\mathbf{A} + \mathbf{B}^{\mathrm{T}}\mathbf{D}_s\mathbf{B}\right]\mathbf{c}, \tag{11}$$

where $\mathbf{D}_f$ and $\mathbf{D}_s$ are the diagonal matrices of the coefficients k_{fi} and k_{si} respectively. The absolute rotations can be expressed as a function of the vector $\mathbf{c}$ as

$$\varphi_i = v_{2i} - v_{2i-1}, \qquad i = 1, \ldots, t, \quad \varphi_{t+1} = \varphi_M$$

TABLE 1

First three frequency coefficients p_i for various values of r and Y^* Laura et al. [8]

		p_1		p_2		p_3		p_4	
r	Y^*	Present	[8]	Present	[8]	Present	[8]	Present	[8]
0·02	0	3·584	3·59	19·984	20·17	52·445	53·48	97·210	100·32
	0·2	2·479	2·61	16·060	16·44	45·236	46·23	87·525	89·97
	0·4	2·002	2·14	15·228	15·52	44·219	45·03	86·508	88·63
	0·6	1·727	1·86	14·868	15·11	43·820	44·54	86·125	88·22
	0·8	1·537	1·67	14·667	14·87	43·606	44·28	85·924	87·85
	1	1·402	1·52	14·542	14·72	43·474	44·12	85·801	87·75
0·04	0	3·552	3·56	18·855	19·01	46·693	47·43	81·483	83·48
	0·2	2·458	2·59	15·328	15·67	40·845	41·55	74·449	75·84
	0·4	1·990	2·13	14·559	14·82	39·474	40·52	73·650	74·82
	0·6	1·714	1·85	14·225	14·43	39·628	40·10	73·346	74·37
	0·8	1·528	1·66	14·039	14·21	39·444	39·86	73·187	74·27
	1	1·393	1·51	13·921	14·07	39·328	39·72	73·088	74·09
0·08	0	3·415	3·42	15·744	15·84	34·960	35·35	55·881	56·91
	0·2	2·390	2·51	13·188	13·42	31·326	31·66	52·231	52·81
	0·4	1·940	2·07	12·585	12·76	30·708	30·92	51·737	52·13
	0·6	1·674	1·80	12·320	12·45	30·457	30·60	51·544	51·84
	0·8	1·494	1·62	12·170	12·28	30·322	30·42	51·442	51·74
	1	1·362	1·48	12·074	12·16	30·237	30·32	51·370	51·64

or, in matrix form as

$$\boldsymbol{\varphi} = \mathbf{Rc}.$$

Henceforth, the kinetic energy becomes

$$\tfrac{1}{2}\dot{\mathbf{c}}^{\mathrm{T}}[\mathbf{M}_M + \mathbf{R}^{\mathrm{T}}\bar{\mathbf{M}}\mathbf{R}]\dot{\mathbf{c}} = \tfrac{1}{2}\dot{\mathbf{c}}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{c}}, \qquad (12)$$

where $\mathbf{M}_M$ is a diagonal matrix of the masses, and

$$\bar{M}_i = \rho I_i l_i, \quad i = 1, \dots, t,$$

$$\bar{M}_{i+1} = \rho I_M, \quad i = t + 1.$$

The equation of motion can be written as

$$\mathbf{M}\ddot{\mathbf{c}} + \mathbf{Kc} = \mathbf{0} \qquad (13)$$

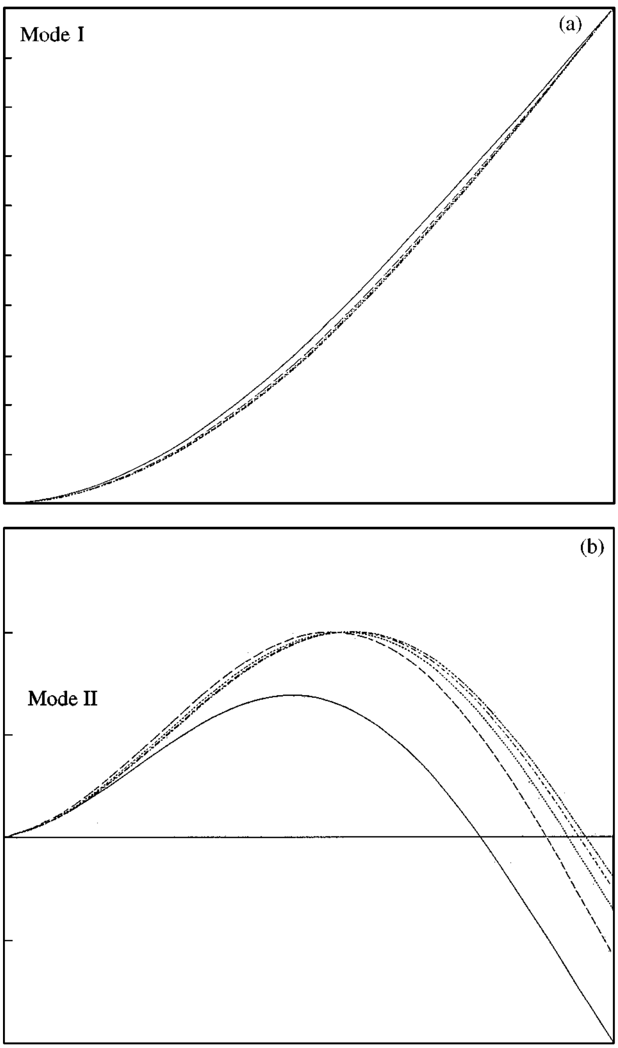

Figure 3. (a) First mode: influence of tip mass for $r = 0{\cdot}08$, $h^* = 0{\cdot}8$ and $Z = 0$. (b) Second mode: as a Figure 3(a). (c) Third mode; as Figure 3(a): ——, $Y^* = 0$; - - - - - -, $Y^* = 0{\cdot}2$; ··········, $Y^* = 0{\cdot}4$; – · – · – · –, $Y^* = 0{\cdot}6$; – ·· – ·· – ·· –, $Y^* = 0{\cdot}8$.

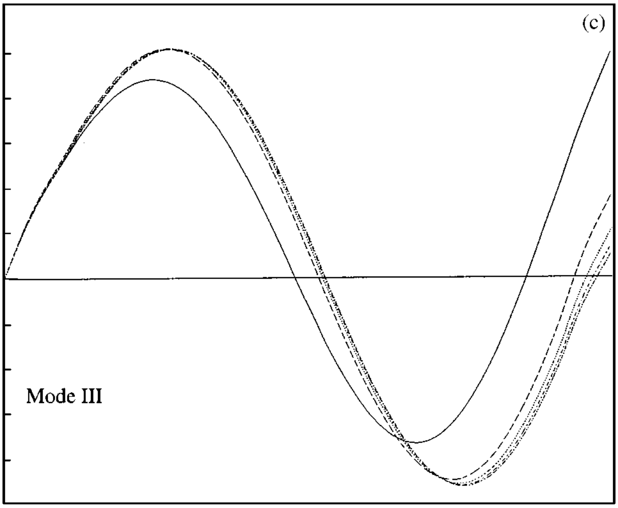

Figure 3. Continued.

and the free vibration frequencies are calculated as the eigenvalue problem imposing

$$\det(\mathbf{K} - \omega^2 \mathbf{M}) = 0. \tag{14}$$

4. RESULTS AND CONCLUSION

The natural frequencies of the structure can be calculated from equation (14). More particularly, the non-dimensional coefficients

$$p_i = \omega_i L^2 \sqrt{\frac{\rho A_0}{E I_0}}$$

are given as function of the parameters

$$Y^* = \frac{M}{m_t}, \qquad Z = \frac{J^*}{L}, \qquad r = \sqrt{\frac{I_0}{A_0 L^2}}.$$

As a numerical examples let us consider the beam with ratio $E/G = 2{\cdot}6$, shear factor $k = 5/6$ and $J = M\ J^{*2}$. The beam is discretized into 20 rigid bars. In Table 1 are shown the frequency coefficients p_i for various factors r and for increasing Y^*. For the sake of comparison, the same results are also given, as obtained by means of the finite element method (FEM): see Laura *et al.* [8]. Due to the nature of the two methods, it is evident that the free vibration frequencies, as obtained by means of the Lagrangian procedure, will be slightly lower than the corresponding frequencies obtained by adopting the finite element approach. The discrepancies can become significant for the higher mode, and increase for increasing Y^* value. On the other hand, they can be reduced by increasing both the discretization levels, so that a narrow lower–upper bound to the true solution can be obtained.

The three mode shapes of the beam for $r = 0{\cdot}08$, $h^* = 0{\cdot}8$, $Z = 0$ and various values of Y^* are shown in Figure 3(a)–(c). It can be noticed that the presence of the tip mass becomes noticeable for the higher modes, and obviously it reduces the free end displacement. For

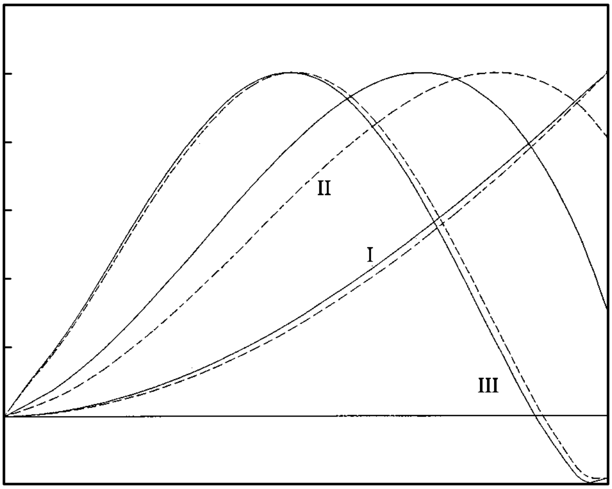

Figure 4. First three mode shapes: influence of tip mass inertia: —— $Z = 0{\cdot}25$; - - - $Z = 0{\cdot}5$.

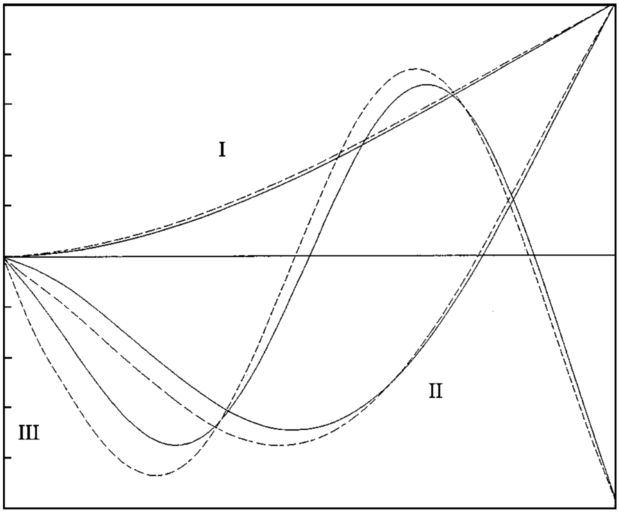

Figure 5. First three mode shapes: influence of r parameter: —— $r = 0{\cdot}04$; - - - -, $r = 0{\cdot}08$.

$Y^* \to \infty$ this displacement tends to zero, and the $(n + 1)$th vibration mode tends to the corresponding nth vibration mode of the clamped–pinned beam.

In Figure 4 are shown for $Y^* = 1$ and various Z. The results for $r = 0{\cdot}04$ and $0{\cdot}08$ are given in Figure 5. It is evident that for $r = 0$ the classical Euler–Bernoulli results are recovered, and consequently the beam becomes more flexible as r increases.

Finally, in Table 2, the p_i (i = 1–4) coefficients for $r = 0{\cdot}08$, $h^* = 0{\cdot}8$, $Y^* = 1$ and various values Z are given.

It seems intuitive that the rotation at L becomes smaller when Z increases. The proposed approach is particularly useful for beams with complex geometry and different boundary conditions. Moreover, together with the variational Ritz-like methods, it allows deduction of narrow lower–upper bounds to the true results.

TABLE 2

First four frequency coefficients p_i for $r = 0{\cdot}08$, $Y^* = l$ and various Z values

Z	p_1	p_2	p_3	p_4
0	1·3622	12·074	30·236	51·370
0·25	1·2845	5·446	16·340	33·690
0·5	1·0917	3·322	15·854	33·563
1	0·7188	2·545	15·740	33·532

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APPENDIX: NOMENCLATURE

h^*	taper ratio
E, G	Young's modulus, shear modulus
L	span of the beam
l_i	length of the ith rigid bar
I, A	moment of inertia, cross-section
m_i, m_t	ith mass, beam mass
M, I_M	mass at the tip; moment of inertia of the mass
J^*	radius of inertia of the mass
k	shear factor
$\mathbf{c}$	vector Lagrangian co-ordinates
v_i	displacements of the bars
$\mathbf{M}_M$	mass matrix
$\bar{\mathbf{M}}$	matrix of the rotatory inertia

$\tilde{Y}^*$, Z	non-dimensional parameters
t	number of rigid bars
$\Delta\varphi$, Δv	relative rotation, relative displacements
φ_M	rotation of the mass at the tip
ρ	mass density
ω, λ	free frequency, frequency coefficient