Journal of Sound and Vibration (2000) **237**(3), 550–559
doi:10.1006/jsvi.2000.3008, available online at http://www.idealibrary.com on IDEAL®

MULTIPLE COMBINATIONS OF THE STOCHASTIC LINEARIZATION CRITERIA BY THE MOMENT APPROACH

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(*Received* 21 *February* 2000; *and in final form* 23 *March* 2000)

1. INTRODUCTION

Booton [1] and Kazakov [2] introduced the stochastic linearization method nearly simultaneously in 1953 and 1954 respectively. In nearly half a century since its inception, many review articles have been written, and its description appears in specialized monographs, that include the treatment of non-linear random vibration. In these two works, Booton [1]and Kazakov [2] replace the non-linear dynamic system by the linear system that is equivalent to the original, non-linear one is some probabilistic sense. The criterion that was suggested is that of the minimum mean-square deviation between the original non-linear expression of the force $\varphi(X)$, where X is the displacement, and the linear counterpart $k_{eq}X$, where k_{eq} is the spring constant of the equivalent linear system. The aim is formulated as follows: given the probabilistic properties of the excitation $F(t)$, find the mean-square values of the displacement $E(X^2)$ and the velocity $E(\dot{X}^2)$. Booton [1] and Kazakov [2], as well as numerous other investigators (in our estimate over 400)—utilize the following formula for the equivalent spring constant:

$$k_{eq}^{(1)} = E[X\varphi(X)]/E(X^2), \tag{1}$$

however, Kazakov's [2] classical work contains another criterion too. It demands that the mean-square values of the non-linear force $E[f^2(X)]$ and its replacement $E[(k_{eq}X)^2]$ be equal:

$$E[\varphi^2(X)] = E[(k_{eq}X)^2], \tag{2}$$

resulting in the following expression of k_{eq}:

$$k_{eq}^{(2)} = \sqrt{E[\varphi^2(X)]/E(X^2)}. \tag{3}$$

Unfortunately, investigators almost uniformly do not report this classic result of Kazakov's [2] work. This reminds us of Mark Twain's [3] definition of the classical work: "A classic is something that everybody wants to have but nobody wants to read." In this study, a short summary is given of some recent work on stochastic linearization criteria. In total, 81 criteria will be discussed, along with the possible implications and significance of such a multiplicity.

0022-460X/00/430550 + 10 $35.00/0

2. BASIC EQUATIONS

Consider a single-degree-of-freedom system described by the following non-linear differential equation:

$$m\ddot{X} + \varphi(X) + \psi(\dot{X}) = F(t), \tag{4}$$

where $\varphi(X)$ is the non-linear restoring force, while $\psi(\dot{X})$ is the non-linear damping force, m denotes the mass and $F(t)$ is the excitation with specified probabilistic properties. We will utilize the method of moments as suggested by Yamada [4–6] and Fujita [7] for the deterministic problems (see also reference [8]). As Finlayson [8] notes, "For the ordinary differential equations governing … phenomenon the weighting functions are x, x^2, x^3, … . Thus, successively higher moments of the residual are required to be zero." We replace the non-linear restoring force $\varphi(x)$ by the linear equivalent $k_{eq}X$. Since these two quantities are unequal, unless the restoring force is linear, we form the deficiency, or error

$$\varepsilon_\varphi(X) = \varphi(X) - k_{eq}X. \tag{5}$$

We demand that the deficiency be probabilistically orthogonal to x, i.e.,

$$\langle \varepsilon_\varphi(X), X \rangle = 0, \tag{6}$$

where the angle brackets denote the inner product

$$\langle f_1(X), f_2(X) \rangle = E[f_1(X) f_2(X)] = \int_{-\infty}^{\infty} f_1(x) f_2(x) p_x(x)\,\mathrm{d}x, \tag{7}$$

where $p_x(x)$ is the probability density function of x. Equation (6) leads to the requirement

$$\langle \varepsilon_\varphi(X), X \rangle = \langle \varphi(X) - k_{eq}X, X \rangle = 0 \tag{8}$$

from which we find k_{eq}:

$$k_{eq} = \frac{\langle \varphi(X), X \rangle}{\langle X, X \rangle}. \tag{9}$$

Bearing in mind the definition of the inner product in equation (7), we conclude that equation (9) coincides with equation (1); thus k_{eq} in Eq. (8) is $k_{eq}^{(1)}$:

$$\langle \varepsilon_\varphi(X), X \rangle = \langle \varphi(X) - k_{eq}^{(1)}X, X \rangle = 0. \tag{10}$$

The expression for c_{eq} can be obtained by forming the deficiency or error $\varepsilon_\psi(\dot{X})$, between the damping force $\psi(\dot{X})$ and its linear replacement $c_{eq}\dot{X}$:

$$\varepsilon_\psi(\dot{X}) = \psi(\dot{X}) - c_{eq}\dot{X}. \tag{11}$$

According to the method of moments [4–8], we require the following inner product to vanish:

$$\langle \varepsilon_\psi(\dot{X}), \dot{X} \rangle = 0 \tag{12}$$

or

$$\langle \psi(\dot{X}) - c_{eq}\dot{X}, \dot{X} \rangle = \langle \psi(\dot{X}), \dot{X} \rangle - c_{eq}\langle \dot{X}, \dot{X} \rangle = 0. \tag{13}$$

thus,

$$c_{eq} = \langle \psi(\dot{X}), \dot{X} \rangle / \langle \dot{X}, \dot{X} \rangle = c_{eq}^{(1)} \tag{14}$$

which coincides with the expression used by the pioneers of the stochastic linearization.

Thus, equations (1) and (10) can be derived without utilization of the so-called minimum mean-square difference criterion, as it was done in the literature. The derivation via the method of moments is *much simpler* than the explanation utilized in the literature. The demand that the mean-square values of the non-linear and linear dissipation forces to be equal, i.e., that

$$E[\psi^2(\dot{X})] = E[(c_{eq}\dot{X})^2] \tag{15}$$

yields the following expressions for c_{eq}:

$$c_{eq}^{(2)} = \sqrt{E[\psi^2(\dot{X})]/E(\dot{X}^2)}. \tag{16}$$

3. RE-DERIVATION OF ENERGY CRITERIA

Wang and Zhang [9], Elishakoff and Zhang [10], Zhang *et al.* [11] (see also the recent study of Murayov *et al.* [12]) utilized a potential energy of the system as a parameter with respect to which the linearization should be performed. This criterion is derived here in *much simpler* manner as follows. The potential energy of deformation stored in the non-linear spring equals

$$P(X) = \int_0^X \varphi(x)\,\mathrm{d}x, \tag{17}$$

the potential energy of the associated equivalent linear spring is

$$P_{eq}(X) = k_{eq}X^2/2. \tag{18}$$

We form the deficiency with respect to energy ε_p as follows:

$$\varepsilon_p(X) = P(X) - k_{eq}X^2/2 \tag{19}$$

and demand it to be orthogonal to X^2:

$$\langle \varepsilon_p(X), X^2 \rangle = 0 \tag{20}$$

or

$$\langle P(X), X^2 \rangle - k_{eq}\langle X^2/2, X^2 \rangle = 0, \tag{21}$$

leading to the following expression for $k_{eq} = k_{eq}^{(3)}$:

$$k_{eq}^{(3)} = 2E[X^2 P(X)]/E(X^4). \tag{22}$$

Another criterion based on the potential energy is somewhat analogous to that by Kazakov [2]. It demands that the mean-square value of the potential energy of the original system $E[P^2(X)]$ has to coincide with its counterpart in the linear system, namely

$$E[P^2(X)] = E[(k_{eq}X^2/2)^2]. \tag{23}$$

The attendant expression for the equivalent spring is

$$k_{eq}^{(4)} = 2\sqrt{E[P^2(X)]/E(X^4)}. \quad (24)$$

Elishakoff and Bert [13] recently proposed a criterion based on the concept of the complementary energy. Namely, the complementary energy $C(X) = X\varphi(X) - P(X)$ is used as the basis for linearization.

The deficiency between the complementary energy $C(X)$ of the original non-linear system and that of the linear system $k_{eq}X^2/2$,

$$\varepsilon_c(X) = C(X) - k_{eq}X^2/2, \quad (25)$$

is being first formed. Then it is made orthogonal to X^2 to yield

$$k_{eq}^{(5)} = 2E[X^2C(X)]/E(X^2). \quad (26)$$

An additional criterion in terms of the complementary energy can be suggested, that of equality of the mean-square values of $C(X)$ and its linear counterpart:

$$E[C^2(X)] = E(k_{eq}X^2/2)^2. \quad (27)$$

An attendant expression for the equivalent stiffness is

$$k_{eq}^{(6)} = 2\sqrt{E[C^2(X)]/E(X^4)}. \quad (28)$$

It appears instructive to turn now to the linearization in the damping element $\psi(\dot{X})$ in equations (4).

4. LINEARIZATION OF DAMPING

Equation (14) gave a derivation of the linearized damping coefficient c_{eq} in a manner different from its conventional derivation. However, other expressions can be obtained. In this context the energy dissipation function

$$D(\dot{X}) = \int_0^{\dot{X}} \psi(z)\,\mathrm{d}z \quad (29)$$

is instructive. This concept was utilized by Wang and Zhang [9] in their study. We will derive their expression by *other* means. We form the residual ε_D due to the replacement of the damping force $c_{eq}\dot{X}$. We demand the deficiency

$$\varepsilon_D = D(\dot{X}) - c_{eq}\dot{X}^2/2 \quad (30)$$

between the energy dissipation functions representing the non-linear and linear systems to be orthogonal to $\dot{X}^2$,

$$\langle \varepsilon_D, \dot{X}^2 \rangle = 0. \quad (31)$$

The associated expression for c_{eq} reads

$$c_{eq}^{(3)} = 2E[\dot{X}^2D(\dot{X})]/E(\dot{X}^4). \quad (32)$$

One can also demand equality of the mean-square values of the appropriate, linear and non-linear, energy dissipation functions (see reference [14])

$$E[D^2] = E[(c_{eq}\dot{X}^2/2)^2]. \tag{33}$$

From the condition we obtain

$$c_{eq}^{(4)} = 2\sqrt{E[D^2]/E(\dot{X}^4)}. \tag{34}$$

One can also *formally* introduce the "complementary energy dissipation function"

$$M(\dot{X}) = \dot{X}\psi(\dot{X}) - D(\dot{X}). \tag{35}$$

With this concept we can arrive at two criteria. One demands that the deficiency to be orthogonal to $\dot{X}^2$, yielding:

$$\varepsilon_M = M(\dot{X}) - c_{eq}\dot{X}^2/2, \qquad c_{eq}^{(5)} = 2E[\dot{X}^2 M(\dot{X})]/E(\dot{X}^4). \tag{36, 37}$$

On the other hand, demand that

$$E[M^2(\dot{X})] = E[(c_{eq}\dot{X}^2/^2)] \tag{38}$$

leads to

$$c_{eq}^{(6)} = 2\sqrt{E[M^2(\dot{X})]/E(\dot{X}^4)}. \tag{39}$$

Thus, we have six different expressions for the equivalent spring constant and six different expressions for the equivalent damping coefficients. Yet, these do not exhaust all possible linearization avenues.

In the process of evaluating the expressions for the equivalent stiffness and damping, it may appear that we have overlooked the goal of the analysis itself! This evaluation of the mathematical expectations involved in the expressions for k_{eq} and c_{eq} *presupposes* the knowledge of the probability density involved, namely $p_X(x)$. Yet, had we known the density we would not go through the trouble of utilizing the linearization technique, but would directly evaluate the desired response quantities

$$E(X^2) = \int_{-\infty}^{\infty} x p_X(x)\,\mathrm{d}x, \qquad E(\dot{X}^2) = \int_{-\infty}^{\infty} \dot{x}^2 p_{\dot{X}}(x)\,\mathrm{d}x. \tag{40}$$

Yet, we do not know $p_X(x)$ and $p_{\dot{X}}(\dot{x})$. If so, the formulas for k_{eq} and c_{eq} may appear to be void of sense. Indeed, to evaluate c_{eq} we need to know the density, which is not known to us. This situation is not entirely different from the use of the Rayleigh's quotient in vibration analysis. For the Bernoulli–Euler beams it reads

$$\omega^2 = \frac{\int_0^L EI(\mathrm{d}^2W/\mathrm{d}x^2)^2\,\mathrm{d}x}{\int_0^L \rho(x)A(x)W^2(x)\,\mathrm{d}x}, \tag{41}$$

where ω is the natural frequency to be determined, $E(x)$ the modulus of elasticity, $I(x)$ the moment of inertia of the beam's cross-section, $\rho(x)$ the material density, $A(x)$ the cross-sectional area, and $W(x)$ the mode shape. As Professor V. V. Bolotin remarked during the exposition of this subject, the Rayleigh's quotient maintains, in essence, that in order to obtain the natural frequency, one needs the knowledge of the mode shape; yet, if we were in possession of the mode shape, we would not need the Rayleigh quotient, but obtain the natural frequency from the governing differential equation itself,

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[E(x)I(x)\frac{\mathrm{d}^2W(x)}{\mathrm{d}x^2}\right] = \rho(x)A(x)\omega^2W(x), \tag{42}$$

as

$$\omega^2 = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[E(x)I(x)\frac{\mathrm{d}^2W(x)}{\mathrm{d}x^2}\right]\Big/\rho(x)A(x)W(x). \tag{43}$$

The main idea by Rayleigh was to use not an exact, but an *approximate* expression for the mode shape. We substitute the expression $\tilde{W}(x)$ that approximates the true (and unknown) mode shape. The quotient in equation (41) yields an approximate expression for the natural frequency.

An analogous idea is utilized to justify the expressions we obtained for the linearization coefficients. We approximate the true (and unknown) probability density $p_X(x)$ by density $\tilde{p}_X(x)$ that contains some unknown parameter(s). Consider as an example an approximation that has unspecified parameters $\tilde{\sigma}_X$ and $\tilde{\sigma}_{\dot{X}}$, these quantities being mean-square deviations of the approximating process and its derivative respectively. Mathematical expectation involved in determination of the stiffness and damping coefficients become dependent upon $\tilde{\sigma}_X$ and $\tilde{\sigma}_{\dot{X}}$. Thus,

$$k_{eq} = k_{eq}(\tilde{\sigma}_X), \qquad c_{eq} = c_{eq}(\tilde{\sigma}_{\dot{X}}). \tag{44}$$

At this juncture we pretend we know k_{eq} and c_{eq}. These characterize the replacing linear system governed by the differential equation

$$m\ddot{X} + k_{eq}(\tilde{\sigma}_X)X + c_{eq}(\tilde{\sigma}_{\dot{X}})\dot{X} = F(t) \tag{45}$$

well studied in the random vibration literature. For example, if $F(t)$ is a random process that is stationary, in the wide sense, than the spectral analysis yields the mean-square responses. If, for example, the excitation is the white noise, and mathematical expectation is zero, i.e., its spectral density is constant, so we get

$$E(X^2) = \frac{\pi S_0}{c_{eq}k_{eq}}, \qquad E(\dot{X}^2) = \frac{\pi S_0}{mc_{eq}}. \tag{46}$$

Yet, $E(X^2)$ due to utilized approximating $p_X(x) \approx \tilde{p}_X(x)$ equals $\tilde{\sigma}_X^2$, while, on the other hand k_{eq} is a function of $\tilde{\sigma}_X$. Likewise, $E(\dot{X}^2)$ equals $\tilde{\sigma}_{\dot{X}}^2$, whereas c_{eq} is a function of $\tilde{\sigma}_{\dot{X}}$. Thus, equations (45) and (46) can be rewritten as

$$\tilde{\sigma}_X^2 = \pi S_0/c_{eq}k_{eq}(\tilde{\sigma}_X), \qquad \tilde{\sigma}_{\dot{X}}^2 = \pi S_0/mc_{eq}(\tilde{\sigma}_{\dot{X}}), \tag{47}$$

yielding equations for $\tilde{\sigma}_X$ and $\tilde{\sigma}_{\dot{X}}$. At this stage the approximate density is fully *revealed*, for the control parameter is determined; likewise, our initial problem, which was more modest than that of finding the probability density, is solved, for $E(X^2)$ and $E(\dot{X}^2)$ are approximated by $\tilde{\sigma}_X^2$ and $\tilde{\sigma}_{\dot{X}}^2$, respectively, and determined from equation (47).

5. CRITERIA BASED UPON APPROXIMATING PROBABILITY DENSITY *AB INITIO*

In previous derivations, we first postulated the criteria for determining the parameters k_{eq} and c_{eq}, found their analytical expressions, and then made an approximation of the probability density that enters into these expressions. We assumed that the probability densities of X and $\dot{X}$ depend upon $\tilde{\sigma}_X$ and $\tilde{\sigma}_{\dot{X}}$ respectively. Presently, we will use the assumption of the approximate probability density from the very start. The following criteria appear to be relevant.

Minimum mean-square difference between the original and the replacing restoring forces is the first criterion to be considered. We demand that [15, 18–20]:

$$\frac{\mathrm{d}}{\mathrm{d}k_{eq}} E[\varphi(X|\tilde{\sigma}_X) - k_{eq}X]^2 = \frac{\mathrm{d}}{\mathrm{d}k_{eq}} \{E[\varphi^2(X|\tilde{\sigma}_X)] - 2k_{eq}E[X\varphi(X|\tilde{\sigma}_X)] + k_{eq}^2\tilde{\sigma}_X^2\} = 0. \tag{48}$$

Equation (48) yields an equation for determining $k_{eq}^{(7)}$. Note that criterion (48) differs from expression (1) for $k_{eq}^{(1)}$. In fact, the latter equation can be obtained from equation (48) if the dependence of quantities $E[\varphi^2(X|\tilde{\sigma}_X)]$ and $E[X\varphi(X|\tilde{\sigma}_X)]$ upon k_{eq} could be neglected.

Analogously, one can demand [15] that the mean-square difference between appropriately evaluated mean-square difference of potential energies

$$E[(\Delta P)^2] = E[P(X|\tilde{\sigma}_X) - k_{eq}X^2/2]^2 \tag{49}$$

to be minimal:

$$\frac{\mathrm{d}}{\mathrm{d}k_{eq}} E[(\Delta P)^2] = \frac{\mathrm{d}}{\mathrm{d}k_{eq}} \{E[P^2(X|\tilde{\sigma}_X)] - k_{eq}E[X^2P(X|\tilde{\sigma}_X)] + k_{eq}^2E(X^4)/4\} = 0, \tag{50}$$

resulting in an equation for $k_{eq}^{(8)}$.

Again it should be stressed that this criterion differs from equation (22) since $E[P^2(X|\tilde{\sigma}_X)]$ and $E[X^2P(X|\tilde{\sigma}_X)]$ depend upon $\tilde{\sigma}_X$. If one would neglect this dependence, one would reduce the resulting expression for the k_{eq} to equation (22).

In perfect analogy, we reproduce here a direct generalization of the complementary energy criterion by Elishakoff and Bert [13]. We form the mean-square difference between the complementary energies, evaluated by using the approximating density

$$E[(\Delta C)^2] = E[C(X|\sigma_{\tilde{X}}) - k_{eq}X^2/2]^2 \tag{51}$$

and require it to be minimal with respect to the parameter k_{eq}:

$$\frac{\mathrm{d}}{\mathrm{d}k_{eq}} E[(\Delta C)^2] = \frac{\mathrm{d}}{\mathrm{d}k_{eq}} \{E[C^2(X|\tilde{\sigma}_X)] - k_{eq}E[X^2C(X|\tilde{\sigma}_X)] + k_{eq}^2E(X^4)/4\} = 0 \tag{52}$$

leading to an evaluation of $k_{eq}^{(9)}$.

To complete our task we should provide formulas for the derivation of c_{eq}. We form the mean-square difference, between the original non-linear damping ΔF_D forces:

$$E[(\Delta F_D)^2] = E[\psi(\dot{X}) - c_{eq}\dot{X}]^2. \tag{53}$$

Hereinafter, we will emphasize the dependence of the statistical moments appearing on the right-hand side of equation (57); we demand $E[(\Delta F_D)^2]$ to attain a minimum:

$$\frac{\mathrm{d}}{\mathrm{d}k_{eq}}\{E[\psi^2(\dot{X}|\tilde{\sigma}_{\dot{X}})] - 2c_{eq}E[\dot{X}\psi(\dot{X}|\tilde{\sigma}_{\dot{X}})] + c_{eq}^2E(\dot{X}^2)/4\} = 0 \tag{54}$$

leading to the value of $c_{eq}^{(7)}$.

The requirement for the mean-square difference of different energy dissipation functions

$$E[(\Delta D)^2] = E[D(\dot{X}|\tilde{\sigma}_{\dot{X}}) - c_{eq}\dot{X}^2/2]^2 \tag{55}$$

to attain a minimum value with respect to c_{eq} yields the requirement

$$\frac{\mathrm{d}}{\mathrm{d}c_{eq}}E[(\Delta D)^2] = \frac{\mathrm{d}}{\mathrm{d}c_{eq}}\{E[D^2(\dot{X}|\tilde{\sigma}_{\dot{X}})] - c_{eq}E[\dot{X}^2D(\dot{X}|\tilde{\sigma}_{\dot{X}})] + c_{eq}^2E(\dot{X}^4)/4\} = 0, \tag{56}$$

from which we obtain the value of $c_{eq}^{(8)}$.

Finally, utilizing a somewhat artificial criterion of the minimum mean-square difference of the "complementary energy dissipation function"

$$E[(\Delta M(\dot{X}))^2] = E[M(\dot{X}|\tilde{\sigma}_{\dot{X}}) - c_{eq}\dot{X}^2/2]^2, \tag{57}$$

we get the condition

$$\frac{\mathrm{d}}{\mathrm{d}c_{eq}}E[(\Delta M(\dot{X}))^2] = \frac{\mathrm{d}}{\mathrm{d}c_{eq}}\{E[M^2(\dot{X}|\tilde{\sigma}_{\dot{X}})] - c_{eq}E[\dot{X}^2M(\dot{X}|\tilde{\sigma}_{\dot{X}})] + c_{eq}^2E(\dot{X}^4)/4\} = 0 \tag{58}$$

leading to the value of $c_{eq}^{(9)}$.

6. WHY DO WE NEED SO MANY CRITERIA?

We thus arrived at nine different criteria for evaluating the equivalent stiffness k_{eq} and nine different conditions for evaluating the stiffness coefficient c_{eq}. Since each criterion for computing k_{eq} can be combined with any criterion for calculating c_{eq}, we conclude that we arrive at $9^2 = 81$ different criteria for solving the non-linear stochastic problem at hand. The multiplicity of the methods to arrive at the sought solution may appear to be alarming in several respects.

The first question that begs itself to be asked is "*Why do we need so many criteria? Why not use a single criteria?*" The reply to this question may be a question itself: "*Why not?*" Indeed, for solving linear deterministic problems there are a multiplicity of methods. For example, for solving the linear eigenvalue problem one can resort to the methods of numerical integration method, method of successive approximations, Rayleigh–Ritz method, Bubnov–Galerkin method, Petrov–Galerkin method, finite difference method, finite element method, etc. An even more direct connection exists between the problem at hand and the failure criteria in the mechanics of solids. We have criteria of maximum stress, maximum strain, St-Venant's criterion, Tresca criterion, Goldenblat–Kopnov criterion, Tsai–Wu criterion etc. It appears that while the method of exact solution of non-linear stochastic differential equations is absent a tolerance may be exercised toward various stochastic linearization criteria.

How to choose the stochastic linearization criterion from their multiplicity? Not unlike the failure criteria, accumulation of the experience appears to be useful, for in various circumstances different criteria may prove to be advantageous over the others.

It appears instructive to classify 81 methods in two separate classes: like criteria and unlike criteria. Like criteria are the ones with analogous reasoning both for evaluating $k_{eq}^{(\alpha)}$ and $c_{eq}^{(\beta)}$. For like criteria, we have $\alpha = \beta$. Thus we have nine like criteria. For example probabilistic orthogonality criteria of the restoring force difference and the damping force difference constitute one pair of the like criteria with $\alpha = \beta = 1$. These are *classical* criteria. The rest 72 criteria constitute the unlike ones: for example, criterion of probabilistic orthogonality for the force difference for evaluating k_{eq}, and the criterion of equal mean-squares of the energy dissipation functions correspond to $\alpha = 1$, $\beta = 2$. The establishment of the class of problems in which any of the 80 *non-classical* criteria is advantageous over the classical one appears to be of interest. Selective review of the method is given in ref. [22].

7. WHAT TO DO WITH MULTIPLICITY OF CRITERIA?

A natural question arises: *What to do with these criteria?* Popov and Paltov [16], when discussing two criteria suggested by Kazakov [2], came up with the recommendation to use the arithmetic mean of result yielded by two methods, as a better approximation for the mean-square response, to *bracket* the exact result. Bolotin [17] showed that such an approach was not justified, since in the particular problem, studied by him, the exact solution was not bracked by the results obtained by two competitive criteria. We will refrain therefore to recommend to solve every problem by 81 competitive means and then to average the results. Rather, a study may be recommended to establish the regions when a specific criterion can be advantageous, over the other criteria. It appears that one can construct an arbitrary number of stochastic linearization criteria, for approximate solution of stochastic response problems. In such circumstances, the importance of closed-form benchmark solutions, on one hand, and fully numerical, Monte Carlo solutions, cannot be overestimated.

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