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GENERAL EQUATIONS OF ANISOTROPIC PLATES AND SHELLS INCLUDING TRANSVERSE SHEAR DEFORMATIONS, ROTARY INERTIA AND INITIAL CURVATURE EFFECTS

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The present work deals with a generalization of geometrically linear shear deformation theory for multilayered anisotropic shells of general shape. No assumptions are made other than to neglect the transverse normal strain. The results, which include the effects of shear deformations and rotary inertia as well as initial curvature (included in the stress resultants and assumed transverse shear stresses) are deduced by application of the virtual work principle, with displacements and transverse shear as independent variables. These equations are applied to different shell geometries, such as revolution, cylindrical, spherical and conical shells as well as rectangular and circular plates.

1. INTRODUCTION

Shells are widely used as structural elements in modern construction engineering, aircraft construction, ship building, rocket construction, the nuclear, aerospace and aeronautical industries, as well as the petroleum and petrochemical industries (pressure vessel, pipeline), etc. It is very important, therefore, that the static and dynamic behavior of these structures when subjected to different loads be clearly understood, in order that they may be used safely in industry. The analysis of thin elastic shells under static or dynamic loads has been the focus of a great deal of research. These shells have been studied in the light of such different factors as large displacements, thickness variation, residual stresses, rotary inertia, anisotropy, initial curvature and the effect of the surrounding medium (air, liquid), etc.

Many theories have been developed for thin elastic shells, in both linear and non-linear cases, and are based on the first approximation of Love–Kirchhoff theory which, because it does not take transverse shear deformations into account, can be grossly in error in predicting the transverse deflections, buckling loads and natural frequencies. In the case of plates and shells made of advanced laminated composite materials, the prediction errors are even more marked. The transverse shear effect on non-linear vibration and post-buckling behavior is significantly especially for laminates with moderately large thickness.

The present work presents the general equations of anisotropic shells (equilibrium, constitutive and kinematic relations) by considering the effects of shear deformation, rotary inertia and initial curvature. These relations are then applied to different shell geometries: shells of revolution, cylindrical, spherical and conical shells as well as the circular and rectangular plates.

2. LITERATURE REVIEW

The literature review covers three broad areas. In the first, both linear and non-linear theories on analysis of plates and shell structures are discussed. These theories were, in many instances, developed for isotropic materials before being extended to anisotropic material applications. The second part deals with the study of the effect of shear deformation on both the static and dynamic behavior of plates and shells; especially those made of advanced anisotropic materials. In the last part, we briefly discuss the effect of structure–fluid interaction on the vibrations of plates and shells. Special attention is given to cylindrical shells immersed in or filled with a liquid or subjected to a flowing fluid.

A shell structure may be defined as a body enclosed between two closely spaced and curved surfaces. In general, a shell has three fundamental identifying features; its reference surfaces, its thickness and its edges. Of these, the reference surface is the most significant because the behavior of the shell is governed by the behavior of its reference surface.

Many shell theories are derived from the equations of elasticity. The strain–displacement relations of shells can be derived from kinematics and the 3-D strain–displacement relations written in terms of arbitrary curvilinear co-ordinates [1]. In reality, the behavior of the top and bottom surfaces of a shell under load can vary widely.

The first attempt to formulate a bending theory of shells from the general equations of elasticity was made by Aron in 1874. A thin shell is one in which the thickness is small compared with the overall dimensions of the reference shell surface, and a two-dimensional (2-D) theory is used to approximate three-dimensional (3-D) phenomena. Many classical shell theories were developed originally for thin elastic shells, and are based on the Love–Kirchhoff assumptions which are: (1) the shell is thin; (2) the displacements and rotations are small; (3) normals to the shell reference surface before deformation remain normal after deformation; and (4) transverse normal stresses are negligible.

These assumptions led to a thin shell theory that can be viewed as an extension to Kirchhoff plate theory and is often called Kirchhoff–Love shell theory. The effects of the normal transverse strain are often neglected in the kinematics compared to the effects of the in-plane strains due to the thinness of the shell, and the shell is assumed to be in an approximate state of plane stress. The in-plane stresses became dominant because the transverse normal stress is, in general, of order h/R times the bending stresses, whereas the transverse shear stresses, obtained from equilibrium conditions, are of order h/L times the bending stresses. Therefore, for L/R less than 10, the transverse normal stress is negligible compared to transverse shear stresses.

On the other hand, the normal transverse strain can generally be included in the analysis through the constitutive relations. In deriving the equilibrium equations, statically equivalent forces and moments acting on the reference surface can be defined by integrating stresses through the thickness. In this way, the 3-D shell behavior can be fully described using a 2-D approximation [1–4]. The third assumption of the Love–Kirchhoff theory is that transverse shear strains may not be written in terms of displacements, which leads to their being completely ignored although transverse shear stresses should be included in equilibrium equations.

Surveys of various classical shell theories can be found in the works of Bert [5], Reissner [6] and Naghdi [7]. The last truncate, the Taylor's series expansion for tangential displacements after linear terms in the thickness co-ordinate, and many others followed him. An excellent collection of the research carried out on this topic has been produced by Leisssa [8]. Elegant representations, both linear and non-linear, of Love's shell theory can be derived strictly via definitions from surface theory without reference to 3-D relationships [3, 9].

One of the best known of these theories, Love's first approximation, yields sufficiently accurate results when (1) the lateral dimension to thickness ratio (L/h) is large; (2) the dynamic excitations are within the low-frequency range; (3) the material anisotropy is not severe. However, the application of such theories to layered anisotropic composites shells could lead to errors in the prediction of natural frequencies, deflections, stresses and buckling loads.

There is an inconsistency in the original version of Love's theory since all strains do not vanish for rigid-body motion. It was perhaps this inconsistency that encouraged many researchers to develop slightly different shell theories. Many shell theories based more or less on Love's assumptions have been developed, each different since each neglects or approximates small terms in its own way. Sanders [10] redefined the force and moment resultants in such a way that all strains vanish for any rigid-body motion.

The thin shell assumption in Love's theory have not been taken into account in the theories of Flügge *et al.* [3], which impose a less restrictive requirement on the thinness of the shell. Their theory also eliminates the rigid-body strains anomaly. Koiter [11] discussed the significance of Love's first theory and, based on an order magnitude study, states that refinements of Love first theory cannot consistently be made without including transverse deformation effects. Other prominent theories on this subject include those of Novozhilov [12].

Useful information about vibrations of shell-type structures can be found in the monograph by Soedel [13] dealing with different geometries of beams, plates and shells, isotropic and composite materials, computational methods and related advanced topics. Two types of basic equation, corresponding either to Flügge's or Donnell's equations for isotropic shells, have been formulated in the literature [2, 3]. Donnell's derivation is not easy to follow, since it completely neglects a number of terms both in the relationships between the changes of curvature and twist and the displacement, and in the relations of stress resultants and moment resultants in terms of displacement.

A small displacement Love theory has been used by Dong *et al.* [14] for the bending analysis of thin anisotropic plates and shells. These are specialized to give linear Donnell equations for anisotropic cylindrical shells. Bogner *et al.* [15] developed a linear cylindrical isotropic shell finite element based on the classical shell theory. Morley [16] extended the limits of the Donnell theory. Reissner [17] applied the Donnell's assumptions to a shallow spherical shell. The Donnell–Mushtari–Vlasov equations [8] are obtained from Donnell's assumptions are applied to a shallow shell of arbitrary geometry.

Cheng and He [18, 19] have developed an exact linear theory for circular cylindrical shells based on Love's assumptions. By retaining all the small terms which are neglected, in varying degrees, by other theories, the usual eighth order operator in the governing equilibrium equation of the transverse displacement can be separated into two complex conjugate operators, thereby reducing the solution complexity. A general theory for thin isotropic shells, which makes no simplifications for approximations beyond a fundamental hypothesis, was developed by Markov [20].

Padovan [21] used a complex multi-segment numerical integration procedure, which can handle the static analysis of mechanically, and thermally loaded branched laminated anisotropic shells of revolution with arbitrary meridional variation in thickness and material properties. The governing equations are based on the Love–Reissner theory (they did not consider the effects of shear deformation in their work).

Basar and Ding [22] used the finite rotation elements for the non-linear analysis of thin shell structures. Their work is based on the Kirchhoff–Love hypothesis. In the development of non-linear finite element using the Kirchhoff–Love hypothesis, the essential problem is the elimination of the rotation vector (the difference vector) without loss of accuracy. To do this, the Kirchhoff–Love hypothesis is expressed by two sets of equivalent conditions: one of them is used in the form of linear variational equations for elimination of the incremental rotational variables; the other, non-linear one, is needed for the exact calculation of the rotation vector of the fundamental state.

Most of the theories outlined above have been applied to a shell so thin that all transverse shear deformation effects, transverse stresses and strains can be neglected. These transverse effects become more pronounced as the shell becomes thicker relative to its in-plane dimensions and radius of curvature. This is particularly true of the transverse shear deformations [11] since classical theories can be grossly in error in predicting transverse deflections, buckling loads or natural frequencies. It is well known from experimental observations that the fact that classical plate theory neglects transverse shear strains leads to underestimations of deflections and overpredictions of natural frequencies and buckling loads.

These errors are even higher in the case of plates and shells made up of advanced anisotropic laminated composite materials such as graphite–epoxy and born–epoxy, where the ratio of elastic moduli to shear moduli are very great (i.e., of the order 25–40 instead of 2·6 for isotropic materials). As pointed out by Koiter [11], refinement of Love's approximation theory of thin elastics shells is meaningless unless the effects of transverse shear and normal stresses are taken into account. Transverse shear deformation plays a very important role in reducing the effective flexural stiffness of anisotropic laminated plates and shells because their in-plane elastic modulus to transverse shear modulus ratio is high.

The transverse shear effect on non-linear vibration and post-buckling behavior is significant, especially for laminates with moderately significant thickness, a high circumferential wave number and a greater number of layers. Study of the shear deformation shows that these effects can become quite meaningful for some geometrical parameters, such as small radius–thickness or length–thickness ratios, as well as for shorter wavelengths or longer shells.

In addition to the transverse shear deformation, the initial curvature effect should be considered for the analysis of thick shells as indicated by Voyiadjis and Shi [23] for isotropic materials. The initial curvature effect is very important in making accurate predictions of stresses even in the central region. In the shell structure, the curvature of each parallel surface through the thickness of the shell is different. To consider the initial curvature effects, the term $1 \pm z/R$ has to be included. The presence of curvature effectively increases the structural stiffness.

In the refined shell theories that take the transverse shear deformation effect into account, the normals to the reference surface of shells are permitted to rotate such that plane sections originally perpendicular to the middle surface remain planar, but, as a result of the deformation, are no longer perpendicular. The transverse shear is represented by inclusion of an independent degree of freedom (d.o.f.) in the kinematics. The shells is still fully described by the behavior of the reference surface and therefore these approaches represent 2-D theory [24].

Hildebrand *et al.* [25] were the first to make significant contributions by dispensing with Love's assumption and assuming instead a three-term Taylor's series expansion for the displacement vector for orthotropic and homogeneous shells. Naghdi [26] has employed Reissner's [27] mixed variational principle to develop a complete shell formulation similar to that of Hildebrand *et al.* [25], retaining two and three terms in the Taylor's series expansions for tangential and transverse displacement components respectively.

The first analysis to incorporate the bending and stretching coupling was carried out by Ambartsumyan [9]. He assumed that the individual orthotropic layers were oriented in such a way that the principal axes of material symmetry coincided with those of the principal co-ordinates of the shell reference surface. The effects of transverse shear deformation, transverse normal stresses and transverse normal strain on the behavior of laminated shells can be incorporated, on the basis of a mathematical model, through the inclusion of higher order terms in the power-series expansion of the assumed displacement field.

Dong and Tso [28] were perhaps the first to present a first order shear deformation theory, retaining one and two terms in the Taylor's series for transverse and tangential displacement components, respectively. The theory includes the effects of transverse shear deformation through the shell thickness, and hence they construct a laminated orthotropic shell theory. Hildebrand *et al.* [25] found that the effects of the additional terms in the transverse displacement that resulted in non-zero transverse normal strains are negligible. Reissner used these kinematic relations to analyze first plates [29] and then sandwich shells [30]. The rotary inertia terms have been included in the dynamic analysis of plates by Mindlin [31].

The above-mentioned first order shear theories result from the so-called Reissner–Mindlin (RM) kinematics do not satisfy the transverse shear boundary conditions on the top and bottom surfaces of the shell or plate, since a constant shear angle through the thickness is assumed, and plane sections remain plane. For this reason, the theories based on these kinematic relations usually require shear correction factors for equilibrium considerations. The shear correction factors are only functions of lamination parameters (number of layers, stacking sequence, degree of orthotropy and fiber orientation in each individual layer) [32, 33].

Levinson [34] and Reddy [35] have developed theories that include terms in-plane displacement kinematics. They used a parabolic shear strain distribution through the thickness for satisfying zero transverse shear stress on the top and bottom surfaces of the shell, thus producing closer agreement with linear elasticity. The parabolic shear strain distribution has been used to analyze the linear vibrational behavior of isotropic cylindrical shells by Bhimaraddi [36].

The effects of transverse shear deformation and transverse isotropy as well as thermal expansion through the thickness of cylindrical shells were considered by Gulati and Essenburg [37], Zukas and Vinson [38], Dong and his colleagues [14], Hsu and Wang [39], Chaudhuri and Abu-Arja [40] and Khdeir *et al.* [41].

Whitney and Sun [42, 43] developed a shear deformation theory for laminated cylindrical shells that includes both transverse shear deformation and transverse normal strain as well as expansional strains. The theory is based on a displacement field in which the displacements in the surface of the shell are expanded as linear functions of the thickness co-ordinate and the transverse displacement is expanded as a quadratic function of the thickness co-ordinate. They discussed some methods by which one can diagnose the mass matrix. They did not consider the product of the first order derivatives of the tangential displacement component with respect to x, y and z in the strain–displacement relations. These relations are based on von Karman's theory [12].

Reddy [44] extended Sanders' [10, 45] theory for simply supported cross-ply laminated shells assuming 5 d.o.f.s per node. The theory is based on a displacement field in which the displacements of the middle surface are expanded as cubic functions of the thickness co-ordinate, and the transverse displacement is assumed to be constant through the thickness. The Navier-type exact solutions for bending and natural vibration are presented for cylindrical and spherical shells under simply supported boundary conditions.

A generalization of geometrically linear shear deformation theories for small elastic stains was presented for multilayered axisymmetric shells of general shape by Touratier [46]. He proposed a general shear deformation theory for multilayered, moderately thick, axisymmetric shells. The theory, which is geometrically linear, is developed for small elastic strain and is restricted to axisymmetric shells under axisymmetric loading and classical boundary conditions. The principal advantage of this work is that it does not need shear correction factors.

Static analysis of laminated shells using a refined shear deformation theory was done by Ji-Fan He [47]. According to this theory, the thickness of the shell must be small compared to the principal radii of curvature. It can be expected that the present theory would tend to be fairly accurate for laminated shells with many layers. Hsu and Wang [39] and Di Sciuva [48] proposed a specially designed displacement field with traction continuity at the layer interface, and Reissner [49] proposed another type of general shell theory for transversely isotropic materials based on the Reissner mixed variational principle with independently assumed transverse stresses.

More recently, Jing and Tzeng [50] derived a mixed shear deformation theory for thick laminated shells of general shape based on proposed method of Jing and Liao [51]. The displacement field uses a zig–zag function in addition to the Reissner–Mindlin-type in-plane displacements and a constant transverse deflection. Kant and Ramesh [52] developed complete governing equations for a thick laminated composite shell. The theory is based on a three-term Taylor's series expansion of the displacement vector and generalized Hooke's law, as is the displacement model of Hildebrand *et al.* [25], and is applicable to orthotropic material layers having planes of symmetry coincident with shell co-ordinates.

Advanced composites materials are being used more and more in a variety of industries due to their high strength and stiffness-to-weight ratios; this has led to a rapid increase in the use of these materials in structural applications during the past decade. Structural elements made up of advanced fiber-reinforced composite materials offer unique advantages over those made of isotropic materials. They are being extensively used in high and low technology areas, e.g., the aerospace industry, where complex shell configurations are common structural elements.

The filament-winding techniques for manufacturing composite shells of revolution has recently been expanded in aircraft, shipbuilding, petroleum and other industries. In general, these materials are fiber-reinforced laminate, symmetric or antisymmetric cross- and angle-ply, which consist of numerous layers each with various fiber orientation. Although the total laminate may exhibit orthotropic-like properties, each layer of the laminate is usually anisotropic; thus the individual properties of each layer must be taken into account when attempting to gain insight into the actual stress and strain fields.

By optimizing the properties we can reduce the overall weight of a structure since stiffness and strength can be designed only where they are required. A lower weight structure translates into higher performance. Since optimized structural systems are often more sensitive to instabilities, it is necessary to exercise caution. The designer would be much better able to avoid any instabilities, if, when predicting a maximum load capacity, he either knew the equilibrium paths of structural elements or had accurate modelling of the load–displacement behavior of the structure.

Anisotropic laminated plates and shells have a further complication which must be considered during the design process: potentially large directional variations of stiffness properties in these structures due to tailoring mean that three-dimensional effects can become very important. The classical two-dimensional assumptions may lead to gross inaccuracies, although they may be valid for an identical shell structure made up of isotropic materials.

However, although they have properties that are superior to isotropic materials, advanced composite structures do present some technical problems in both manufacture and design. For computational reasons, the study of composite materials involves either their behaviors on the macroscopic level such as linear and non-linear loading responses, natural frequencies, buckling loads, etc., or their micro-mechanical properties like cracking, delamination, fiber-matrix debonding, etc.

A number of theories for layered anisotropic shells exist in the literature. Many of these theories were developed for thin shells and are based on the Kirchhoff–Love hypothesis. The first analysis that incorporated the bending–stretching coupling (due to asymmetric lamination in composites) was by Ambartsumayan [9]. In his analysis, he assumed that the individual orthotropic layers were oriented such that the principal axes of material symmetry coincided with the principal co-ordinates of the shell reference surface. He has written extensively on the matter, basing his work of Love's theory with some discussion of transverse stresses.

The simplifying assumption of laminated anisotropy is often used in applying a 2-D theory to plates and shells consisting of layers of composite materials [24]. In this approach, the individual properties of the composite constituents, the fibers and the matrix, are "smeared" and thus each lamina is treated as an orthotropic material.

A survey of the analysis of multilayered composite shells using Reissner's mixed variational principle was done by Grigolyuk and Kulikov [53]. They maintain that laminated anisotropy assumes perfect bonding between layers, and that the interply adhesive has infinitesimal thickness but infinite stiffness. This approach leads to classical laminated plate theory (CLPT) and the references by Jones [54] and Whitney and Pagano [55] to CLPT are based on the Kirchhoff–Love assumptions. However, both references point out that transverse shear deformation is more significant in laminated anisotropic than in similar isotropic constructions.

Bert [56] used Vlasov shell theory to formulate a linear laminated shell theory similar to CLPT. Pagano and Wang [57–60] and Srinivas and Rao [61] have developed some exact solutions of 3-D elasticity equations governing composite plates that have been used to validate the shear theory. They conclude that CLPT gives fairly good approximations for both the displacements and stresses if the plate is thin. Higher order shear theories do not give much better transverse stress results but displacements show a marked improvement over CLPT for the thicker plates. Transverse stresses are best calculated from equilibrium instead of from the constitutive relations [54]. Ren [62] similarly solved 3-D elasticity equations for a laminated cylindrical shell in cylindrical bending.

His work dealt with what is now known as laminated orthotropic shells rather than with laminated anisotropic shells. In laminated anisotropic shells, the individual layers are, in general, anisotropic, and the principal axes of material symmetry of the individual layers coincide with only one of the principal co-ordinates of the shell (the thickness-normal co-ordinate). Whitney and Pagano [55] applied the Reissner–Mindlin theory to composite plate analysis. The buckling of laminated cylindrical shells was studied by Hirano [63]. Reddy and Chao [64] applied the closed-form solution to the thick composite plate.

Reddy [24, 65] has extended the cubic kinematic approach to the analysis of laminated anisotropic plates and he has applied them to solving several linear static and buckling problems. Additionally, Soldatos applied the parabolic shear theory to examination of the stability of asymmetrically laminated cylindrical panels [66, 67]. Cheng and Ho [68] presented an analysis of laminated anisotropic cylindrical shells using Flügge's shell theory [2]. A first approximation theory for the asymmetric deformation of non-homogeneous, anisotropic, elastic cylindrical shells was derived by Widera and his colleagues [69, 70] by means of the asymptotic integration of the elasticity equations. For a homogeneous, isotropic material, the theory reduces to Donnell's equations.

Noor and Peters [71] presented the free vibration analysis of laminated anisotropic shells of revolution as well as the sensitivity of their response to anisotropic material coefficients. Their analytical formulation is based on a form of the Sanders–Budiansky shell theory, including the effects of both transverse shear deformation and the laminated anisotropic material response. Each of the shell variables is expressed in terms of trigonometric functions in the circumferential co-ordinate and a three-field mixed finite element model is used for the discretization in the meridional direction. They used a reduction method involving the successive use of the finite element method and classical Bubnov–Galerkin technique to substantially reduce the size of the eigenvalue problem.

Zienkiewicz [72] introduced a finite element approach with independent transverse displacement and rotational d.o.f.s such that a RM shear deformable shell element is obtained. A small rotation approach for anisotropic shell has been developed by Librescu and Schmidt [73].

Successive approximations, as steps towards an estimate of exact shell strain–displacement relations where displacements, large strains and rotations were all initially allowed, are presented for isotropic shells by Sanders [45] and anisotropic shells by Librescu [73].

Kant and Kommineni [74] presented higher order theories for general orthotropic as well as laminated shells. These theories were derived from the three-dimensional elasticity equations by expanding the displacement vector in a Taylor's series in the thickness co-ordinate. Reference [75] presented some elements, which can be successfully applied to the analysis of both thin and thick plate and shells. Kui *et al.* [76] applied the finite element method, displacement type, to analyses thin shells and to overcoming the shear-locking phenomena.

Pryor and Barker [77] developed a linear plate element based on the RM theory. They used a rectangular element with 28 d.o.f.s (8, 12, 8 for extension, bending and shear effects, respectively), to have continuity of transverse stress at any interface. Hinrichsen and Palazotto [78] applied a cubic spline function to the non-linear analysis of thick composite plates. Their theory is based on the usual Kirchhoff hypothesis. The theory was developed by considering the Lagrangian strains in conjunction with the second Piola–Kirchhoff stress hypothesis. This formulation leads to a quasi-three-dimensional element that combines large displacement with moderately large rotation but is restricted to small strains.

Schmit and Monforton [79] formulated an anisotropic cylindrical shell element, which allows them to predict the geometrically non-linear behavior of sandwich plate and cylindrical shell structures, based on accepted thin shell theory assumptions. Other recent papers by Meroueh [80] and Surana [81, 82] can be mentioned. Cylindrical shells are in general use in the aerospace, shipbuilding, structural and petroleum industries. They are the simplest shell structure to analyze yet have many of the characteristics of more complex shell geometries. The linear problem of composite cylindrical shells has been widely investigated by a number of researchers using different shell theories. Based on the Kirchhoff hypothesis, for example, Dong [83] studied the free vibration of laminated orthotropic cylindrical shells with homogeneous boundary conditions.

The governing equations of orthotropic cylindrical shells were solved via a pair of complex conjugate fourth order differential equations by Cheng and He [19]. Their work is based on the Kirchhoff hypothesis. For the static problem, Flügge and Kelkar [84] and Yao [85] obtained an exact solution for closed isotropic long cylinders under general two-dimensional surface traction. Using the Forbenius method, Srinivas [61] developed an exact three-dimensional solution for orthotropic finite cylinders with simply supported conditions. Varadan and Bhaskar [86] also performed the static stress analysis using the procedures proposed by Srinivas [61]. Pagano [87] obtained the stress field for a homogeneous, anisotropic closed cylinder under two-dimensional surface loads in which the problem are independent of the axial co-ordinate.

Ren [88] presented an exact solution for simply supported laminated cross-ply circular cylindrical panels of infinite and finite length in the axial direction. Leissa *et al.* [89] analyzed the vibration of cantilevered cylindrical panels by using the Ritz method, with algebraic polynomial functions for the displacements.

Widera and Logan [70] studied the non-homogeneous, anisotropic, circular cylindrical elastic shell, using the method of asymptotic expansion in terms of a small parameter in conjunction with Reissner's variational principle. In their work, the procedure used to derive the shell equation starts with substitution of non-dimensional shell co-ordinates in terms of a characteristic length scale for changes of stresses and displacements and Reissner functional direction. The employment of the formulation in terms of Reissner's principle allows one to obtain automatically all the equations necessary to formulate a complete boundary value problem for a first-approximation shell analysis. Non-dimensional stresses, displacements and Reissner functional direction are introduced and considered to be representable by asymptotic expansions in a power series in terms of a small shell parameter.

Recently, Bert and his colleagues [90, 91] and Hsu *et al.* [92] presented exact solutions for bending and vibration of cross-ply, thin cylindrical shells. These solutions are limited to cylindrical shells and sinusoidal distribution of the transverse load, and the procedure used is similar to that used by Whitney and Leissa [93], Whitney and Pagano [55], Bert and Chen [94], and Reddy and Chao [64] for laminated composite plates.

Tzeng [95] proposed a mixed shear deformation theory for the bending analysis of arbitrarily laminated, anisotropic panels and closed cylinders. The initial curvature effect is included in the strain–displacement relations, stress resultants and assumed transverse shear stresses. Two types of shell geometry, infinitely long cylindrical panels and closed cylinders of finite length, are employed in the numerical study. Suzuki and Leissa [96, 97] analyzed the free vibration of circular and non-circular cylindrical shells having circumferentially varying thickness.

The static response to the axisymmetric problem of arbitrarily laminated, anisotropic cylindrical shells of finite length using three-dimensional elasticity equations was studied by Jing and Zeng [98]. The closed cylinder is simply supported at both ends. The highly coupled partial differential equations are reduced to ordinary differential equations with variable coefficients by choosing the solution composed of trigonometric functions along the axial direction.

Kant *et al.* [52, 74] presented various higher order theories for laminated composite cylindrical shells using C_0 finite elements. Kant and co-workers did extensive numerical investigations on laminated plates and shells, both static and dynamic analysis, using C_0 finite elements and different higher order theories. They proved that the imposition of shear-free boundary conditions on the top and bottom bounding planes of the laminate gives stiffer solutions when compared to three-dimensional (3-D) elasticity solutions and various displacement models for flat laminates. The one having 9 d.o.f.s per node produces results very close to 3-D elasticity solution.

A higher order shear deformation theory of plates accounting for the von Karman strains was presented by Reddy [99]. This theory contains the same dependent unknowns as those in the Hencky–Mindlin-type first order shear deformation theory. The displacements are expanded in powers of the thickness of the plate, and accounts for parabolic distribution of the transverse shear strains through the thickness of plate. Hamilton's principle was used to derive the equations of motions and the Navier solution procedure was used to solve the equations of the simply supported plate.

Jing and Liao [51] proposed a mixed function with displacements and transverse shear stresses as independent variables and established the corresponding partial hybrid stress element for the analysis of thick laminated plates. Some comparison between the results obtained for plates by these two functions were made by Jing and Tzeng [100].

A refined laminated plate theory was developed by Whitney and Sun [42] and is applicable to fiber-reinforced composite materials under impact loading. The theory also includes the first symmetric thickness shear and thickness stretch motion, as well as the first antisymmetric thickness shear mode, by including higher order terms in the displacement expansion about the mid-plane of the laminate in a manner similar to that of Mindlin and Medick [101] for homogeneous isotropic plates.

Reddy and Phan [65] used a higher order shear deformation theory to determine the natural frequencies and buckling loads of elastic plates. The theory accounts for the transverse shear strain and rotary inertia. This work dealt with the exact solutions of the theory as applied to the free vibration and buckling of isotropic, orthotropic and laminated rectangular plates with simply supported edge conditions. Reddy [35] developed a higher order shear deformation theory for laminated composite plates. This theory uses a displacement approach similar to that in the Reissner–Mindlin-type theories. The in-plane displacements are expanded as cubic functions of the thickness co-ordinate and the transverse deflection is constant through the plate thickness.

The form is dictated by satisfying the conditions that the transverse shear stresses vanish on the plate surfaces and be non-zero elsewhere. This requires the use of a displacement field in which the in-plane displacements are expanded as cubic functions of the thickness co-ordinate and the transverse deflection is constant through the plate thickness. Ren and Hui [102] formulated a simple theory for non-linear bending of generally laminated composite rectangular plates, which accounts for the transverse shear stains by using the principle of virtual displacements. Moreover, because the total deflection of a plate is decomposed into a deflection due to bending and a deflection due to shear, the solution of the governing equations of the present theory becomes simpler.

The Jing and Liao's functional, modified from the Hellinger–Reissner principle by separating the stress field into a flexural part and a transverse shear part and leaving only displacements and transverse shear stresses as independent variables, has been used by Jing and Tzeng [50] to analyze laminated plates with satisfactory accuracy.

There are many situations in mechanics in which some simplifying assumptions have been considered to help the analyst in getting timely and accurate results. However, various air, water and land vehicles and structures such as aircraft, rocket, pressure vessel, petroleum and petrochemical units, etc., may be subjected to impacts, collisions, blasts and/or other intensive transient loads which can cause large transient structural deformation and damage.

Thin shells subjected to dynamic loads could encounter deflections of the order of the shell thickness or higher. Thin shells could also encounter a phenomenon of dynamic impacts or dynamic buckling and collapse, which are attributed to the change in the equilibrium state characterizing the load–response mode. Response of these kinds cannot be correctly predicted by using the small or intermediate displacement theory. In the intermediate non-linearity approach, the non-linear terms which represent in-plane rotations of the shell are neglected [103, 104]. This theory is often used in stability analysis.

The structural elements made up of the advanced composite materials undergo large deformations before they become inelastic, because of the high-modulus and high-strength properties of composite materials. Therefore, an accurate prediction of transient response is possible only when one accounts for the geometric non-linearity.

There are also cases where structural elements experience only small strains under load but may fail catastrophically due to their geometric configuration. It turns out that this class of structural system can be accurately analyzed on the basis of small strain, non-linear geometrical and linear elastic material behavior. The need for accurate and efficient methods for structural analysis and design, especially for this category of large-deflection (geometrically non-linear) and elastic–plastic (materially non-linear) dynamic response problems has recently become increasingly apparent.

In the proposed non-linear analysis methods, e.g., references [12, 45, 105], many of the non-linear displacement terms may be considered negligible depending, of course, on the specific situation. For example, an accurate load–displacement characterization of a flat plate is based on the von Karman equation where many non-linear rotational terms have been omitted. Similar assumptions for shell element result in equations of the type proposed by Donnel, Sanders and Novozhilov. These formulations are typically valid for the so-called intermediate non-linearity or theories that allow only moderate rotations.

The strain–displacement relations that include non-linear displacement terms are used to represent large displacements and rotations of differential elements of the shell. Non-linear vibrations of generally laminated circular cylindrical shells were examined using the Timoshenko–Mindlin kinematics hypothesis and an extension of Donnell's shell theory. The effects of the transverse shear deformation, rotary inertia and geometrically initial imperfection are included in the analysis. The Galerkin procedure furnishes an infinite systems of equations for time functions, which is solved by the method of harmonic balance [106].

It has been recognized that the non-linear behavior of composite cylindrical shells plays an important role in determining the stability and dynamic response of these shells. Chu [107] first presented an analysis for circular isotropic cylindrical shells with the hardening type of non-linearity for the amplitude–frequency response. Nowinski [108] confirmed the results of Chu [107] by investing the non-linear vibration of orthotropic cylindrical shells. Later, Evensen [109] pointed out that the mode shapes assumed by Chu do not satisfy the condition of continuity of the circumferential in-plane displacement. A more rigorous study of non-linear free flexural vibrations of circular cylindrical shells was conducted by Atluri [110], who compared his results with the available data and concluded by accepting the possibility of the softening type of non-linearity.

Chen and Babcock [111] adopted a perturbation technique in considering the large-amplitude vibration of a thin-walled cylindrical shell. Ramachandran [112] studied the non-linear vibration of cylindrical shells of varying thickness. Khot [113] studied the post-buckling behavior of a laminated cylindrical shell subjected to axial load torsion using the von Karman–Donnell equations. The results obtained by Khot [113] show that, in general, composite shells are less imperfection sensitive than isotropic shells.

Recently, Iu and Chia [114] discussed the non-linear vibration and post-buckling of antisymmetric cross-ply circular cylindrical shells on the basis of von Karman–Donnell kinematic assumptions and the effects of transverse shear on the non-linear behavior of these shells using the Timoshenko–Mindlin kinematic hypothesis. They neglected some terms (e.g., cross-product of displacement derivatives) in non-linear strain–displacement relations.

Neglecting the transverse rotational non-linear terms as well will result in a linear Love-type shell theory. These successive approximations to the shell strain–displacement relations are discussed in the paper by Librescu [115] and Sanders [45]. In the last work, the deformations are restricted by the Kirchhoff hypothesis (the transverse shear and normal strains were neglected), the middle surface strains were assumed small and the rotations were assumed to be moderately small. Most of the above approaches can include various degrees of non-linearity in the strain–displacement relations representing the displacements and rotations. Considerable simplification was achieved in the Donnel equations by use of assumption that the non-linear membrane strains derived only from out-of-plane rotations.

For example, Donnell's theory is not suitable for the analysis of shells in which the buckling mode involves fewer than three full waves around the circumference [105]. More accurate non-linear shell equations are given by Sanders and by Novozhilov, but these were somewhat more complex than the Donnell equations. More terms are retained because fewer assumptions are made about the relative magnitude of various terms in the non-linear strain–displacement. Reddy and Chandrashekhara [116] solved laminated shell problems, both cylindrical and spherical, assuming RM theory and an intermediate non-linearity. There are few such analytical closed-form solutions for shell geometries, especially those that govern non-linear behavior.

The formulation and computational procedure are presented for the geometrically non-linear analysis of laminated orthotropic and anisotropic composite shells based upon a modified incremental Hellinger–Reissner principal and the total Lagrangian description by Rothert and Di [117]. In this investigation, a computational model for a geometrically non-linear analysis has been studied on the basis of a rational approach for a hybrid stress model.

The through-thickness assumption used in the total Lagrangian formulation is introduced, incorporating the non-linear formulation for a large rotation assumption. Noor and Peters [118] analyzed the non-linear response of anisotropic cylindrical panel that included transverse shear deformation. Their formulations are based on the Rayleigh–Ritz technique and the Hu–Washizu mixed shallow shell finite element approach.

Stein [119] used truncated series expansions of exact non-linear strain–displacement relations in a shell approach that also included transverse shear deformation. The non-linear strain–displacement relations were expanded into a series that contains all first- and second-degree terms; only the first few terms have been retained for the displacements. Geometrically non-linear quasi-three-dimensional approaches for laminated composite plates and shells have been developed by Palazotto and Witt [120], Hinrichsen and Palazotto [78] and Dennis and Palazotto [121]. Their work is restricted to small strains; the exact Green's strain–displacement and linear strain displacement relations were assumed for the in-plane strains and the transverse strains, respectively, so the accuracy in rotation is limited by the linear assumption on the transverse shear strains.

Tsai and Palazotto [122] have developed a finite element formulation for the geometric non-linear vibration analysis of cylindrical shells, based upon a curved quadrilateral, 36 d.o.f.s, thin shell element. The equations of motion are based on a total Lagrangian frame of reference. A β method, which is a generalization of Newmark's time-matching integration scheme and the Newton–Raphson iterative method, are both applied in order to solve the set of non-linear equations of motion numerically.

The solution of a set of non-linear, second order differential equations which describe an anisotropic shell of revolution was presented by Martin and Drew [123]. Their analysis is based upon Sanders' non-linear shell theory without considering the shear deformation effects. The method for solving these equations follows the procedure used by Budiansky and Radkowski [124].

Kant and Kommineni [125] presented the geometrically non-linear transient analysis of laminated composite (transversely isotropic) and sandwich shells, based on von Karman's theory. In the time domain, the explicit central difference integrator is used in conjunction with the special mass matrix diagonalization scheme, which conserves the total mass of the element and includes effects due to rotary inertia terms.

Rotter and Jumikis [105] have presented a set of non-linear strain–displacement relations for axisymmetric thin shells subject to large displacements with moderate rotations, by retaining more terms. Their works is based on Kirchhoff's assumptions. They have shown that non-linear strains arising from products of inplane strain terms, which were omitted in previous theories, may be important in certain buckling problems. The new relations are particularly important when branched shells are being studied and when the buckling mode may involve a translation of the branching joint. Their work does not include any numerical result.

A modal approximation in deriving the equations of motion for the non-linear flexural vibrations of a cylindrical shell by using the Donnell's shallow shell theory was presented by Dowell and Ventres [126]. The purpose of their work was to satisfy more accurately the boundary and the continuity conditions and investigate their effects on the form of the modal equations.

Horrigmoe and Bergan [127] presented classical variational principles for non-linear problems by considering incremental deformations of a continuum. Wunderlich [128] and Stricklin *et al.* [129] have reviewed various principles of incremental analysis and solution procedures for geometrical non-linear problems respectively. Noor and Hartley [130] employed the shallow shell theory with transverse shear strains and geometric non-linearities to develop triangular and quadrilateral finite elements.

Chao and Reddy [131], Reddy and Chandrasekhara [116] have presented a first order shear deformation theory based on the kinematic and geometric assumption of Sanders' thin shell theory for the geometrically non-linear analysis of doubly curved composite shells. An analysis of the dynamic responses of cylindrical shells including geometric and material non-linearities was made by Wu and Witmer [132]. The methods of finite element analysis were applied to the problem of large deflection, elastic–plastic dynamic response of cylindrical shells to transient loading. The formulation is based upon the virtual work principle and D'Alembert's principle. Wu and Witmer used a bilinear polynomial for the axial displacement, and bicubic polynomials for both the circumferential displacement and the transverse displacement, and explicitly excluded rigid-body modes.

The analytical solution of the shell motion equations is generally considered to be difficult. Approximation methods can be suitably used (e.g., the finite difference, Galerkin, Rayleigh–Ritz, transfer matrix and finite element methods). All of these methods have advantages and disadvantages. One of the most important criteria in determining the versatility of the resolution is the capacity to predict, with precision, both high and low frequencies.

In the finite difference method, the initial values are given and this method requires a great deal of calculation time. The Galerkin approach loses precision in the higher frequencies of shells. The Rayleigh–Ritz method presents several drawbacks, among which are the displacement function choice, which has to take the boundary conditions into account, and the necessity to use a large number of terms of express displacement functions. Also in the Galerkin method, both geometric and force boundary conditions must be satisfied. On the other hand, the finite element method [72, 133–136] is satisfactory from these viewpoints

The accuracy of solutions reached by the finite element displacement formulation depends on whether the assumed functions accurately model the deformation modes of structures. To satisfy this criterion, Lakis and his group have developed a hybrid type of finite element, whereby the displacement functions in the finite element method are derived from Sanders' classical shell theory [10]. This method has been applied with satisfactory results to the dynamic linear and non-linear analysis of cylindrical shells, both closed and open [137–147], spherical [148], conical [149], isotropic and anisotropic, uniform and axially non-uniform shells, both empty and liquid filled. This method has also been applied to the dynamic analysis of circular and annular plates by Lakis and Selmane [150–152].

The effect of the surrounding medium (air, liquid, etc.) upon the vibration of plates and shells is of primary interest to scientists and engineers working in aerospace, marine and reactor technology. The effect of the fluid on the structural response is usually significant except in the case of extremely thick shells. The dynamic response of the shells when subjected to a flowing fluid, as well as the influence of fluid speed on the shell-free vibrations, were studied by many researchers. Lakis and Païdoussis [137–139], Païdoussis and Denis [153], Weaver and Unny [154], Cheng [155] and Jain [156]. Païdoussis and Li made an elaborate review in this field [157].

The fluid effect on the dynamic behavior of the structure can be taken into account by considering the hydrodynamic mass, which is added to the mass matrix of the structure. The effective mass is a function of the mode shape being considered, the shell and liquid geometrical parameters, plus the physical parameters. In addition, the forces exerted by free surface motion have to be considered; the pressure distribution due to surface motion during vibration could be neglected; however, since resonant sloshing frequencies of thin shells are considerably below the natural frequencies of the combined fluid–structure system.

The dynamics of coupled fluid shells were reviewed extensively by Yang [158] and Brown [159]. Dynamic analysis of the structure–fluid systems was studied by Brenneman and Yang [160], using the modal and hybrid methods. They obtained the structure and fluid modes by applying the stiffness and flexibility methods, following MacNeal's approach. Crouzet-Pascal and Garnet [161] studied a ring-reinforced cylindrical shell immersed in a fluid medium, and its dynamic response to an axisymmetric step pulse. MacNeal [162] presented another approach, which is based on a hybrid finite element formulation in which the structure is modelled with displacements as the unknown variables, and a fluid is modelled with pressure as the variables. To utilize existing mainframe structural analysis programs, MacNeal showed how to recover symmetry by manipulating the equations and adding auxiliary variables to the problem.

The free vibration of simply supported vertical cylindrical shells partially filled with or submerged in a fluid has been analyzed by Gonçalves and Batista [163]. The Rayleigh–Ritz method was used to obtain an approximate solution, which coincide with the exact solution for the cases of an empty shell or a shell completely in contact with fluid. Their work is based upon the consistent shell theory of Sanders. The fluid is taken as non-viscous and compressible and the coupling between the deformable shell and this acoustic medium is taken into account.

Since the lowest natural frequency of bending vibration of shells, immersed in or filled with a fluid, is much less than the corresponding natural frequency of the shell in air, they investigated the effects of variable height of fluid on the vibration response of vertical cylinders filled with or submerged in an acoustic fluid medium. In general, the lowest frequency is depending on liquid level, mode shapes and shell and liquid geometrical and physical parameters.

The free vibration analysis of cylindrical storage tanks with axial thickness variation and partially filled with liquid was studied by Han and Liu [164]. The tank is modelled using Flügge's thin shell theory (in the isotropic case) and the fluid in the tank, according to potential flow theory, is assumed to be inviscid and incompressible. In their work, the shear deformation effects have not been considered. They solved the partial differential equations by using the transfer matrix technique.

An analysis of the non-linear vibration of cylindrical shells of varying thickness in an incompressible fluid was made by Ramachandran [112]. The Rayleigh–Ritz procedure was used to analyze non-linear transverse vibrations of elastic, orthotropic cylindrical shells of linearly varying thickness, embedded in an incompressible fluid (there is no shear deformation effect in his work). There are several reasons for undertaking the development of this theory. First, developing a theory for either dynamic or stress analysis of anisotropic laminated plates and shells, with various geometry shapes. The accurate prediction of the dynamic response or failure characteristics of these structures made up from advanced composite materials requires the use of refined theory where the effect of transverse shear deformation and other factors such as rotary inertia and initial curvature effects are taken into account. This is because the transverse shear deformation plays a more important role in reducing the effective flexural stiffness of plates or shells made of these advanced materials than for corresponding isotropic materials; the present study focuses on this last effect.

The next step deals with the study of the free vibration characteristics of thin anisotropic laminated cylindrical shells based on the present theory. One of the criteria of success of a method may be considered to be its capability of yielding the high, as well as the low, natural frequencies and modal shapes with comparable high accuracy. The numerical method will be based on a combination of hybrid finite element analysis [139] and the refined shear deformation theory of shells. This allows us to use the thin shell equation in full for the determination of the displacement functions, and hence the mass, stiffness and stress-resultant matrices, instead of the more usual polynomial displacement functions.

This formulation yields the natural frequencies and mode shapes of shell defined by arbitrary conditions without changing the displacement functions in each case. Numerical results for fundamental frequencies will be presented for anisotropic laminated cylindrical shells.

At the same time, the flowing fluid effect on the natural frequencies of anisotropic, open cylindrical shells will be studied.

3. THEORETICAL DEVELOPMENT

This work is based on the following assumptions: (1) linear elastic behavior of laminated anisotropic materials; (2) use the strain–displacement relations expressed in arbitrary orthogonal curvilinear co-ordinate system; (3) the shell is thin and therefore we assume that the thickness-direction normal stress is negligible compared with stress tangential to the shell surface; (4) the transverse shear deformation, rotary inertia and initial curvature are considered to influence the governing equations.

3.1. STRAIN–DISPLACEMENT RELATIONS

The normal and shear strain components are related to the components of the displacement vector by [3]

$$
\begin{aligned}
\varepsilon_i &= \frac{\partial}{\partial \alpha_i}\left(\frac{\bar{u}_i}{\sqrt{g_i}}\right) + \frac{1}{2g_i}\sum_{k=1}^{3}\frac{\partial g_i}{\partial \alpha_k}\frac{\bar{u}_k}{\sqrt{g_k}}, \quad i = 1,2,3, \\
\gamma_{ij} &= \frac{1}{\sqrt{g_i\, g_j}}\left[g_i \frac{\partial}{\partial \alpha_j}\left(\frac{\bar{u}_i}{\sqrt{g_i}}\right) + g_j \frac{\partial}{\partial \alpha_i}\left(\frac{\bar{u}_j}{\sqrt{g_i}}\right)\right], \quad i = 1,2,3, \quad i \neq j,
\end{aligned} \tag{1}
$$

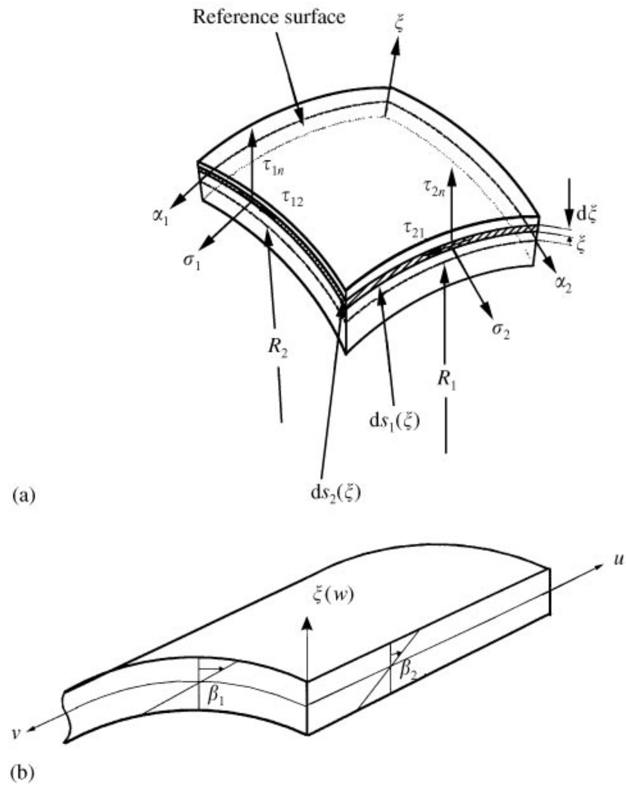

Figure 1. (a) Differential element of a shell, (b) definition of shell co-ordinate system.

where α_i, $\bar{u}_i$ and g_i are, respectively, the curvilinear co-ordinates of the surface, components of the displacement vector and geometrical scale factor quantities, and are defined below for application to shells (Figure 1):

$$\begin{gathered}\alpha_1 = \alpha_1, \quad \alpha_2 = \alpha_2, \quad \alpha_3 = \zeta, \qquad \bar{u}_1 = U_1, \quad \bar{u}_2 = U_2, \quad \bar{u}_3 = W, \\ g_1 = A_1^2\,(1 + \zeta/R_1)^2; \quad g_2 = A_2^2\,(1 + \zeta/R_2)^2, \quad g_3 = 1,\end{gathered} \tag{2}$$

where U_1, U_2, ,W, A_i, R_i and ζ are, respectively, the displacement vector components, Lamé's parameters, the curvature radius and the thickness co-ordinate. Substituting equation (2) into equation (1), we obtain the following strain–displacements equations in the shell space:

$$\begin{aligned}\varepsilon_1 &= \frac{1}{A_1(1 + \zeta/R_1)}\left(\frac{\partial U_1}{\partial \alpha_1} + \frac{U_2}{A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{A_1 W}{R_1}\right), \\ \varepsilon_2 &= \frac{1}{A_2(1 + \zeta/R_2)}\left(\frac{\partial U_2}{\partial \alpha_2} + \frac{U_1}{A_1}\frac{\partial A_2}{\partial \alpha_1} + \frac{A_2 W}{R_2}\right), \\ \varepsilon_n &= \frac{\partial W}{\zeta},\end{aligned} \tag{3a}$$

$$\gamma_{1n} = \frac{1}{A_1(1+\zeta/R_1)}\frac{\partial W}{\partial \alpha_1} + A_1(1+\zeta/R_1)\frac{\partial}{\partial \zeta}\left[\frac{U_1}{A_1(1+\zeta/R_1)}\right],$$

$$\gamma_{2n} = \frac{1}{A_2(1+\zeta/R_2)}\frac{\partial W}{\partial \alpha_2} + A_2(1+\zeta/R_2)\frac{\partial}{\partial \zeta}\left[\frac{U_2}{A_2(1+\zeta/R_2)}\right], \tag{3b}$$

$$\gamma_{12} = \frac{A_2(1+\zeta/R_2)}{A_1(1+\zeta/R_1)}\frac{\partial}{\partial \alpha_1}\left[\frac{U_2}{A_2(1+\zeta/R_2)}\right] + \frac{A_1(1+\zeta/R_1)}{A_2(1+\zeta/R_2)}\frac{\partial}{\partial \alpha_2}\left[\frac{U_1}{A_1(1+\zeta/R_1)}\right],$$

where ε_i and $(\gamma_{in}, \gamma_{12})$ are, respectively, the normal and shearing strain components. We can assume that the displacement components are presented by the following relationships:

$$U_1(\alpha_1, \alpha_2, \zeta) = u_1(\alpha_1, \alpha_2) + \zeta\beta_1(\alpha_1, \alpha_2),$$

$$U_2(\alpha_1, \alpha_2, \zeta) = u_2(\alpha_1, \alpha_2) + \zeta\beta_2(\alpha_1, \alpha_2), \tag{4}$$

$$W(\alpha_1, \alpha_2, \zeta) = w(\alpha_1, \alpha_2).$$

The β_1 and β_2 represent the rotation of tangents to the reference surface oriented along the parametric lines α_1 and α_2 respectively. Substituting equation (4) into equation (3),

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \\ \gamma_{1n} \\ \gamma_{2n} \end{Bmatrix} = \begin{bmatrix} \frac{1}{(1+\zeta/R_1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{(1+\zeta/R_2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{(1+\zeta/R_1)} & \frac{1}{(1+\zeta/R_2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{(1+\zeta/R_1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{(1+\zeta/R_2)} \end{bmatrix}$$

$$\times \left[\begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \gamma_1^0 \\ \gamma_2^0 \\ \mu_1^0 \\ \mu_2^0 \end{Bmatrix} + \zeta \begin{Bmatrix} \kappa_1 \\ \kappa_2 \\ \tau_1 \\ \tau_2 \\ 0 \\ 0 \end{Bmatrix} \right], \tag{5}$$

where

$$\varepsilon_1^0 = \frac{1}{A_1}\frac{\partial u_1}{\partial \alpha_1} + \frac{u_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{w}{R_1}; \quad \kappa_1 = \frac{1}{A_1}\frac{\partial \beta_1}{\partial \alpha_1} + \frac{\beta_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2},$$

$$\varepsilon_2^0 = \frac{1}{A_2}\frac{\partial u_2}{\partial \alpha_2} + \frac{u_1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{w}{R_2}; \quad \kappa_2 = \frac{1}{A_2}\frac{\partial \beta_2}{\partial \alpha_2} + \frac{\beta_1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1},$$

$$\gamma_1^0 = \frac{1}{A_1}\frac{\partial u_2}{\partial \alpha_1} - \frac{u_1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}, \quad \tau_1 = \frac{1}{A_1}\frac{\partial \beta_2}{\partial \alpha_1} - \frac{\beta_1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}, \tag{6}$$

$$\gamma_2^0 = \frac{1}{A_2}\frac{\partial u_1}{\partial \alpha_2} - \frac{u_2}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}, \quad \tau_2 = \frac{1}{A_2}\frac{\partial \beta_1}{\partial \alpha_2} - \frac{\beta_2}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1},$$

$$\mu_1^0 = \frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} - \frac{u_1}{R_1} + \beta_1, \quad \mu_2^0 = \frac{1}{A_2}\frac{\partial w}{\partial \alpha_2} - \frac{u_2}{R_2} + \beta_2,$$

where ε_i^0, γ_i^0, κ_i, τ_i and μ_i^0 are, respectively, the in-surface normal and in-surface shearing strain, the change in the curvature and torsion of the reference surface and the shearing strain components. The Coddazi conditions which were used for the above equations are

$$\frac{\partial}{\partial \alpha_2}[A_1(1+\zeta/R_1)] = \frac{\partial A_1}{\partial \alpha_2}(1+\zeta/R_2), \quad \frac{\partial}{\partial \alpha_1}[A_2(1+\zeta/R_2)] = \frac{\partial A_2}{\partial \alpha_1}(1+\zeta/R_1), \qquad (7)$$

where R_i, ζ, A_i and α_i were defined earlier by equations (1) and (2).

3.2. THE RELATIONSHIP BETWEEN THE STRESS AND STRAIN VECTORS (HOOKE'S LAW)

The relationship between the stress and strain vectors (Hooke's law) is

$$\{\sigma\} = [Q]\{\varepsilon\}. \qquad (8)$$

The constitutive equation of the Kth lamina (for a lamina of fibre-reinforced composite material) in the lamina reference axes (α, β, γ) can be written as follows, for only one lamina, (Figure 2):

$$\begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \sigma_\gamma \\ \tau_{\beta\gamma} \\ \tau_{\alpha\gamma} \\ \tau_{\alpha\beta} \end{Bmatrix} = \begin{bmatrix} Q_{\alpha\alpha} & Q_{\alpha\beta} & Q_{\alpha\gamma} & 0 & 0 & 0 \\ Q_{\beta\alpha} & Q_{\beta\beta} & Q_{\beta\gamma} & 0 & 0 & 0 \\ Q_{\gamma\alpha} & Q_{\gamma\beta} & Q_{\gamma\gamma} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2Q_{44} & & \\ 0 & 0 & 0 & & 2Q_{55} & \\ 0 & 0 & 0 & & & 2Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\gamma \\ \gamma_{\beta\gamma} \\ \gamma_{\alpha\gamma} \\ \gamma_{\alpha\beta} \end{Bmatrix}. \qquad (9)$$

The $[Q]$ matrix denotes the elastic stiffness in the material co-ordinates (local axes). It is useful to mention that the shear strains used in this work are tensor shear strains, not engineering shear strains. Q_{ij}'s elements are defined as follows:

$$Q_{\alpha\alpha} = E_{\alpha\alpha}(1 - \nu_{\beta\gamma}\nu_{\gamma\beta})/\Delta, \quad Q_{\alpha\beta} = (\nu_{\beta\alpha} + \nu_{\gamma\alpha}\nu_{\beta\gamma})E_{\alpha\alpha}/\Delta = (\nu_{\alpha\beta} + \nu_{\gamma\beta}\nu_{\alpha\gamma})E_{\beta\beta}/\Delta,$$

$$Q_{\beta\beta} = E_{\beta\beta}(1 - \nu_{\gamma\alpha}\nu_{\alpha\gamma})/\Delta, \quad Q_{\alpha\gamma} = (\nu_{\gamma\alpha} + \nu_{\beta\alpha}\nu_{\gamma\beta})E_{\alpha\alpha}/\Delta = (\nu_{\alpha\gamma} + \nu_{\alpha\beta}\nu_{\beta\gamma})E_{\gamma\gamma}/\Delta,$$

$$Q_{\gamma\gamma} = E_{\alpha\alpha}(1 - \nu_{\alpha\beta}\nu_{\beta\alpha})/\Delta, \quad Q_{\beta\gamma} = (\nu_{\gamma\beta} + \nu_{\alpha\beta}\nu_{\gamma\alpha})E_{\beta\beta}/\Delta = (\nu_{\beta\gamma} + \nu_{\beta\alpha}\nu_{\alpha\gamma})E_{\gamma\gamma}/\Delta, \qquad (10)$$

$$Q_{44} = G_{\beta\gamma}, \quad Q_{55} = G_{\alpha\gamma}, \quad Q_{66} = G_{\alpha\beta},$$

$$\Delta = 1 - \nu_{\alpha\beta}\nu_{\beta\alpha} - \nu_{\beta\gamma}\nu_{\gamma\beta} - \nu_{\gamma\alpha}\nu_{\alpha\gamma} - 2\nu_{\beta\alpha}\nu_{\gamma\beta}\nu_{\alpha\gamma},$$

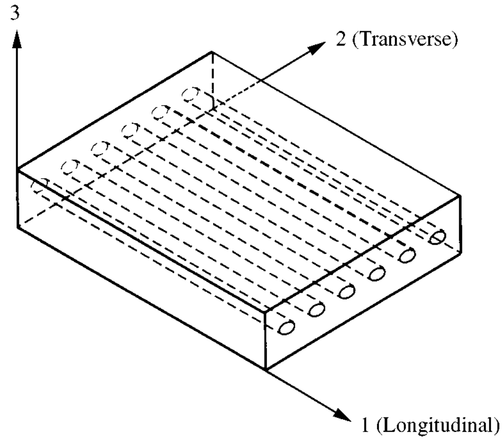

Figure 2. Uni-directional lamina and principal co-ordinate axes.

where $E_{\alpha\beta}$, $G_{\alpha\beta}$ and $\nu_{\alpha\beta}$ are, respectively, Young's moduli of elasticity in the principal directions, rigidity moduli characterizing the change of angles between the principal directions, and the Poisson ratios characterizing the transverse contraction (expansion) under tension (compression) in the directions of the co-ordinate axes. The stress–strain relations of the Kth lamina in the laminate co-ordinate axes (1, 2, 3, global co-ordinates) can be written as (Figure 3)

$$\{\bar{\sigma}\} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_n \\ \tau_{2n} \\ \tau_{1n} \\ \tau_{12} \end{Bmatrix} = \begin{Bmatrix} \overline{Q_{11}} & \overline{Q_{12}} & \overline{Q_{13}} & 0 & 0 & \overline{2Q_{16}} \\ \overline{Q_{21}} & \overline{Q_{22}} & \overline{Q_{23}} & 0 & 0 & \overline{2Q_{26}} \\ \overline{Q_{31}} & \overline{Q_{32}} & \overline{Q_{33}} & 0 & 0 & \overline{2Q_{36}} \\ 0 & 0 & 0 & \overline{2Q_{44}} & \overline{2Q_{45}} & 0 \\ 0 & 0 & 0 & \overline{2Q_{54}} & \overline{2Q_{55}} & 0 \\ \overline{Q_{16}} & \overline{Q_{26}} & \overline{Q_{36}} & 0 & 0 & \overline{2Q_{66}} \end{Bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_n \\ \gamma_{2n} \\ \gamma_{1n} \\ \gamma_{12} \end{Bmatrix}, \tag{11}$$

where

$$[\bar{Q}] = [T]^{-1}\,[Q][T]. \tag{12}$$

The transformation matrix $[\mathbf{T}]$ is defined by

$$[\mathbf{T}] = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & 2mn \\ n^2 & m^2 & 0 & 0 & 0 & -2mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -mn & mn & 0 & 0 & 0 & (m^2-n^2) \end{bmatrix}, \tag{13}$$

where: $m \equiv \cos\alpha$, $n \equiv \sin\alpha$. The orientation angle $\boldsymbol{\alpha}$ is measured counter-clockwise from the 1-axis to the x-axis (Figure 3). $[\bar{Q}]$ elements are defined as follows

$$\overline{Q_{11}} = Q_{\alpha\alpha}m^4 + 2(Q_{\alpha\beta} + 2Q_{66})m^2n^2 + Q_{\beta\beta}n^4, \quad \overline{Q_{12}} = (Q_{\alpha\alpha} + Q_{\beta\beta} - 4Q_{66})m^2n^2 + Q_{\alpha\beta}(m^4 + n^4),$$

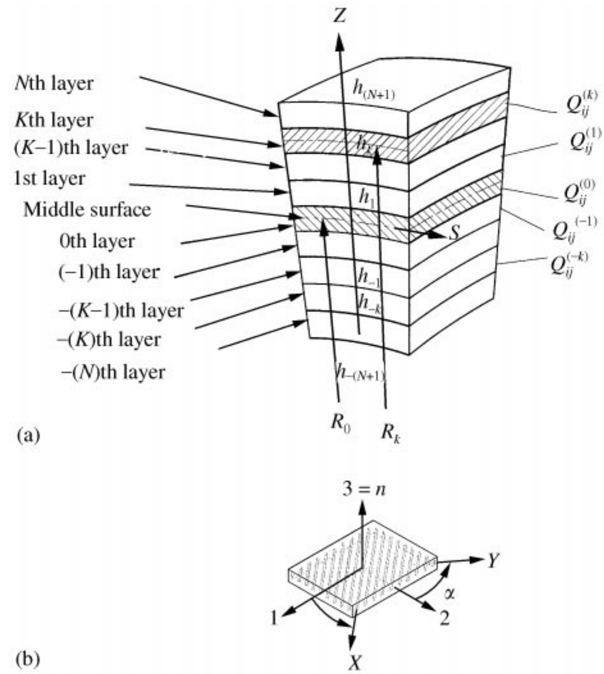

Figure 3. (a) Multi-directional laminate with co-ordinate notation of individual plies, (b) a fiber-reinforced lamina with global and material co-ordinate systems.

$$
\begin{aligned}
\overline{Q_{13}} &= Q_{\alpha\gamma}m^2 + Q_{\beta\gamma}n^2, \quad \overline{Q_{16}} = -mn^3 Q_{\beta\beta} + m^3 n Q_{\alpha\alpha} - mn(m^2 - n^2)(Q_{\alpha\beta} + 2Q_{66}), \\
\overline{Q_{22}} &= Q_{\alpha\alpha}n^4 + 2(Q_{\alpha\beta} + 2Q_{66})m^2n^2 + Q_{\beta\beta}m^4, \quad \overline{Q_{23}} = Q_{\alpha\gamma}n^2 + Q_{\beta\gamma}m^2, \\
\overline{Q_{26}} &= -m^3 n Q_{\beta\beta} + mn^3 Q_{\alpha\alpha} + mn(m^2 - n^2)(Q_{\alpha\beta} + 2Q_{66}), \quad \overline{Q_{33}} = Q_{\gamma\gamma}; \\
\overline{Q_{36}} &= (Q_{\alpha\gamma} - Q_{\beta\gamma})mn, \quad \overline{Q_{66}} = (Q_{\alpha\alpha} + Q_{\beta\beta} - 2Q_{\alpha\beta})m^2n^2 + Q_{66}(m^2 - n^2)^2, \\
\overline{Q_{44}} &= Q_{44}m^2 + Q_{55}n^2, \quad \overline{Q_{45}} = (Q_{55} - Q_{44})mn, \quad \overline{Q_{55}} = Q_{55}m^2 + Q_{44}n^2.
\end{aligned} \tag{14}
$$

3.3. THE EQUATIONS OF MOTION

Using the virtual work principle for the present case yields

$$
\begin{aligned}
&\frac{\partial A_2 N_1}{\partial \alpha_1} + \frac{\partial A_1 N_{21}}{\partial \alpha_2} + N_{12}\frac{\partial A_1}{\partial \alpha_2} - N_2 \frac{\partial A_2}{\partial \alpha_1} + \frac{Q_1 A_1 A_2}{R_1} + A_1 A_2 q_1 = I_1 \ddot{u}_1 + I_2 \ddot{\beta}_1, \\
&\frac{\partial A_1 N_2}{\partial \alpha_2} + \frac{\partial A_2 N_{12}}{\partial \alpha_1} + N_{21}\frac{\partial A_2}{\partial \alpha_1} - N_1 \frac{\partial A_1}{\partial \alpha_2} + \frac{Q_2 A_1 A_2}{R_2} + A_1 A_2 q_2 = I_1 \ddot{u}_2 + I_2 \ddot{\beta}_2, \\
&\frac{\partial A_2 Q_1}{\partial \alpha_1} + \frac{\partial A_1 Q_2}{\partial \alpha_2} - A_1 A_2 \left(\frac{N_1}{R_1} + \frac{N_2}{R_2}\right) - A_1 A_2 q_n = I_1 \ddot{w},
\end{aligned} \tag{15}
$$

$$\frac{\partial A_2 M_1}{\partial \alpha_1} + \frac{\partial A_1 M_{21}}{\partial \alpha_2} + M_{12}\frac{\partial A_1}{\partial \alpha_2} - M_2\frac{\partial A_2}{\partial \alpha_1} - Q_1 A_1 A_2 = I_2 \ddot{u}_1 + I_3 \ddot{\beta}_1,$$

$$\frac{\partial A_1 M_2}{\partial \alpha_2} + \frac{\partial A_2 M_{12}}{\partial \alpha_1} + M_{21}\frac{\partial A_2}{\partial \alpha_1} - M_1\frac{\partial A_1}{\partial \alpha_2} - Q_2 A_1 A_2 = I_2 \ddot{u}_2 + I_3 \ddot{\beta}_2,$$

$$N_{12} - N_{21} + \frac{M_{12}}{R_1} - \frac{M_{21}}{R_2} = 0,$$

where

$$I_1, I_2, I_3 = \sum_{k=1}^{N} \int_{h_k}^{h_{k-1}} \rho^{(k)} (1, \zeta, \zeta^2)\, \mathrm{d}\zeta \tag{16}$$

and where I_i, $\rho^{(k)}$ and ζ are, respectively, inertia moments, density of the Kth's lamina material and the thickness co-ordinate.

The sixth equation of equilibrium is identically satisfied by the integral definitions of the shearing-stress resultants in terms of the shearing stress components. Two basic approaches that enable us to eliminate *exactly* this sixth equation of equilibrium have been developed:

1. Derivation of constitutive equations which fulfill identically this extra equation of equilibrium. For the classical theory of isotropic shell, such constitutive equations are known as Flügge–Lure–Byrne constitutive equations. In the case of composite shells, the reader is referred to the monographs [165–167] by Librescu.
2. Formulation of modified stress resultants and stress couple measures, satisfying identically this sixth equation of equilibrium. For the classical theory of isotropic shells, such modified stress resultants and stress couples have been defined by Sanders [45], Koiter [11] and Novozhilov [12]. For orthotropic and anisotropic shells, a similar stress resultants and stress couples were considered by Lakis and Laveau [142] and Librescu [165–167] respectively.

Now, we see that there are five independent boundary conditions to be applied at given edges. The transverse shear deformations do not vanish in the present theory and, therefore, the β_i cannot be expressed in terms of U_i and W. The transverse shear theory recommended here leads to no strains during rigid-body motion.

3.4. THE STRESS RESULTANTS AND STRESS COUPLES

The stress resultants and stress coupled are given by [3]

$$\begin{Bmatrix} N_1 \\ N_{12} \\ Q_1 \\ M_1 \\ M_{12} \end{Bmatrix} = \int_\zeta \begin{Bmatrix} \sigma_1 \\ \tau_{12} \\ \tau_{1n} \\ \sigma_1 \\ \tau_{12} \end{Bmatrix} (1 + \zeta/R_2)\, \mathrm{d}\zeta, \quad \begin{Bmatrix} N_2 \\ N_{21} \\ Q_2 \\ M_2 \\ M_{21} \end{Bmatrix} = \int_\zeta \begin{Bmatrix} \sigma_2 \\ \tau_{21} \\ \tau_{2n} \\ \sigma_2 \\ \tau_{21} \end{Bmatrix} (1 + \zeta/R_1)\, \mathrm{d}\zeta. \tag{17}$$

The quantities (N_{11}, N_{22}, N_{12}, N_{21}) are called the *in-plane stress resultants*, and (M_{11}, M_{22}, M_{12}, M_{21}) are called the *stress couples resultants*; (Q_{11}, Q_{22}) denote the *transverse force* *resultants*. We notice, in equation (17), that the symmetry of the stress tensor ($\tau_{12} = \tau_{21}$) does not necessarily imply that N_{12} and N_{21} are equal or that M_{12} and M_{21} are equal except in the case of a spherical shell, a plate or a thin shell of any shape.

3.5. THE CONSTITUTIVE EQUATIONS

The stress resultants and stress couples, which correspond to the stress components given by equation (17), have been, therefore, obtained by using equations (5), (11) and (17):

$$\begin{Bmatrix} N_1 \\ N_{12} \\ N_2 \\ N_{21} \end{Bmatrix} = \begin{bmatrix} G_{ij} & A_{ij} \\ A_{ij} & G'_{ij} \end{bmatrix}_{(4\times 4)} \begin{Bmatrix} \varepsilon_1^0 \\ \gamma_1^0 \\ \varepsilon_2^0 \\ \gamma_2^0 \end{Bmatrix} + \begin{bmatrix} H_{ij} & B_{ij} \\ B_{ij} & H'_{ij} \end{bmatrix}_{(4\times 4)} \begin{Bmatrix} \kappa_1 \\ \tau_1 \\ \kappa_2 \\ \tau_2 \end{Bmatrix}, \quad i,j = 1, 6, 2, 6,$$

$$\begin{Bmatrix} M_1 \\ M_{12} \\ M_{22} \\ M_{21} \end{Bmatrix} = \begin{bmatrix} H_{ij} & B_{ij} \\ B_{ij} & H'_{ij} \end{bmatrix}_{(4\times 4)} \begin{Bmatrix} \varepsilon_1^0 \\ \gamma_1^0 \\ \varepsilon_2^0 \\ \gamma_2^0 \end{Bmatrix} + \begin{bmatrix} J_{ij} & D_{ij} \\ D_{ij} & J'_{ij} \end{bmatrix}_{(4\times 4)} \begin{Bmatrix} \kappa_1 \\ \tau_1 \\ \kappa_2 \\ \tau_2 \end{Bmatrix}, \quad i,j = 1, 6, 2, 6, \tag{18}$$

where

$$\begin{gathered} G_{ij} = A_{ij} + a_1 B_{ij} + a_2 D_{ij} + a_3 E_{ij}, \quad H_{ij} = B_{ij} + a_1 D_{ij} + a_2 E_{ij} + a_3 F_{ij}, \\ G'_{ij} = A_{ij} + b_1 B_{ij} + b_2 D_{ij} + b_3 E_{ij}, \quad H'_{ij} = B_{ij} + b_1 D_{ij} + b_2 E_{ij} + b_3 F_{ij}, \\ J_{ij} = D_{ij} + a_1 E_{ij} + a_2 F_{ij} + a_3 C_{ij}, \\ J'_{ij} = D_{ij} + b_1 E_{ij} + b_2 F_{ij} + b_3 C_{ij}. \end{gathered} \tag{19}$$

$$\begin{gathered} a_1 = \frac{1}{R_2} - \frac{1}{R_1}, \quad a_2 = \frac{1}{R_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right), \quad a_3 = \frac{1}{R_1^2 R_2}, \\ b_1 = \frac{1}{R_1} - \frac{1}{R_2}, \quad b_2 = \frac{1}{R_2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right), \quad b_3 = \frac{1}{R_2^2 R_1}, \end{gathered} \tag{20}$$

and

$$\begin{gathered} A_{ij} = \sum_{k=1}^{N} (\overline{Q_{ij}})_k\ (h_k - h_{k-1}), \quad E_{ij} = \frac{1}{4}\sum_{k=1}^{N} (\overline{Q_{ij}})_k\ (h_k^4 - h_{k-1}^4), \\ B_{ij} = \frac{1}{2}\sum_{k=1}^{N} (\overline{Q_{ij}})_k\ (h_k^2 - h_{k-1}^2), \quad F_{ij} = \frac{1}{5}\sum_{k=1}^{N} (\overline{Q_{ij}})_k\ (h_k^5 - h_{k-1}^5), \quad i,j = 1, 6, 2, 6, \\ D_{ij} = \frac{1}{3}\sum_{k=1}^{N} (\overline{Q_{ij}})_k\ (h_k^3 - h_{k-1}^3), \quad C_{ij} = \frac{1}{6}\sum_{k=1}^{N} (\overline{Q_{ij}})_k\ (h_k^6 - h_{k-1}^6), \end{gathered} \tag{21}$$

where N is the number of lamina.

Note: $N_{12} \neq N_{21}$ and $M_{12} \neq M_{21}$ and in the case of a sphere $a_3 = b_3$ and in the case of conical and cylindrical shells as well as the case of circular and rectangular plates, $a_3 = b_3 = 0$. Next,

$$\frac{1}{(1+\zeta/R_1)} = 1 - \zeta/R + (\zeta/R)^2 \mp \cdots. \tag{22}$$

This expansion requires only that $(\zeta/R)^2 < 1$. So

$$\int_{-h/2}^{h/2} \frac{1+\zeta/R_2}{1+\zeta/R_1}\mathrm{d}\zeta = h\left[1+\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{h^2}{12R_1}\right], \quad \int_{-h/2}^{h/2} \frac{1+\zeta/R_2}{1+\zeta/R_1}\zeta\mathrm{d}\zeta = -\frac{h^3}{12}\left(\frac{1}{R_1}-\frac{1}{R_2}\right). \tag{23}$$

We also have

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = K_s \begin{Bmatrix} \int \tau_{1n}(1+\zeta/R_2)\mathrm{d}\zeta \\ \int \tau_{2n}(1+\zeta/R_1)\mathrm{d}\zeta \end{Bmatrix} = K_s \begin{bmatrix} AA_{55} & A_{54} \\ A_{45} & BB_{44} \end{bmatrix} \begin{Bmatrix} \mu_1^0 \\ \mu_2^0 \end{Bmatrix}, \tag{24}$$

where

$$AA_{55} = A_{55} + a_1 B_{55} + a_2 D_{55} + a_3 E_{55}, \quad BB_{44} = A_{44} + b_1 B_{44} + b_2 D_{44} + b_3 E_{44},$$

$$A_{\alpha\beta} = \sum_{k=1}^{N} (\overline{Q_{\alpha\beta}})_k (h_k - h_{k-1}), \quad B_{\alpha\beta} = \frac{1}{2}\sum_{k=1}^{N} (\overline{Q_{\alpha\beta}})_k (h_k^2 - h_{k-1}^2), \quad \alpha, \beta = 4, 5, \tag{25}$$

$$D_{\alpha\beta} = \frac{1}{3}\sum_{k=1}^{N} (\overline{Q_{\alpha\beta}})_k (h_k^3 - h_{k-1}^3), \quad E_{\alpha\beta} = \frac{1}{4}\sum_{k=1}^{N} (\overline{Q_{\alpha\beta}})_k (h_k^4 - h_{k-1}^4).$$

In composite laminated plates and shells, the transverse shear stresses vary through the layer thickness. This discrepancy between the actual stress state and the constant stress state is often corrected in computing the transverse shear force resultants (Q_1 and Q_2) by multiplying equation (24) with a parameter $\mathbf{K}_s$, called shear correction factor.

This factor is computed such that the strain energy due to transverse shear stresses equals the strain energy due to the true transverse stresses predicted by the three-dimensional elasticity theory. Values of $\mathbf{K}_s$ for various special cases are available in the literature such as the shear correction factor for a homogeneous case obtained separatly by Reissner [27] and Mindlin [31].

The determination of the shear correction factor, $\mathbf{K}_s$, for composite laminated structures is still an unresolved issue. This factor depends, in general, on the lamination parameters such as number of layers, stacking sequence, degree of orthotropy and fiber orientation in each individual layer.

Finally,

$$\{N_{11}\, N_{12}\, Q_{11}\, N_{22}\, N_{21}\, Q_{22}\, M_{11}\, M_{12}\, M_{22}\, M_{21}\}^{\mathrm{T}} = [P]_{(10\times 10)} \{\varepsilon_1^0\, \gamma_1^0\, \mu_1^0\, \varepsilon_2^0\, \gamma_2^0\, \mu_2^0\, \kappa_1\, \tau_1\, \kappa_2\, \tau_2\}^{\mathrm{T}}. \tag{26}$$

The $\varepsilon_1^0; \gamma_1^0, \ldots$ and τ_2 were given earlier in equations (5) whereas the P_{ij} elements are given in Appendix A and are defined by equations (19)–(21) and equation (25).

The formulations of the governing equations will be developed hereafter in terms of displacement measures only. There are other formulations in terms of stress resultants and stress couples, in terms of strain measures, as well, mixed formulation in terms of stress resultants (or stress potential functions like the Airy functions) and displacement quantities. In order to obtain such formulation, it is necessary to first develop the compatibility equations associated with the strain–displacement relationships. In order to obtain such equations, the reader may follow the procedures described by Sanders [10] and Brull and Librescu [168].

The five remaining equations of motion are implicit relations among the 10 resultant forces and moments. Therefore, the five necessary boundary conditions must be specified on each edge of the shell. It should be noted that these governing equations will be expressed in terms of displacement measures only, named Dirichlet (essential) boundary value problem. Therefore, five necessary boundary conditions must be specified on each edge of the shell:

$$u_1 = u_1^*, \quad u_2 = u_2^*, \quad w = w^*, \quad \beta_1 = \beta_1^* \quad \text{and} \quad \beta_2 = \beta_2^*.$$

For instance, in the case of a dynamic investigation of a clamped shell, we may write $u_1 = u_2 = w = 0$ and $\beta_1 = \beta_2 = 0$ at each edge. In the case of a simply supported edge, we obtain $u_2 = w = 0$, $\beta_2 = 0$ and $u_1 = C_1$, $\beta_1 = C_2$ at each edge, etc.

Finally, it can be seen that there are three different stiffness matrices in connection with the constructive equation (18) considered in conjunction with equations (19)–(21). The first one relates the in-plane stress resultants (N's) to the mid-surface strain (ε's) and is called the extensional stiffness matrix. The flexural stiffness matrix relates the stress couples (M's) to the curvature (κ's). The last one is called the bending– stretching coupling matrix which relates (M's) to (ε's) and (N's) to (κ's). In fact, the bending–stretching matrix components equal to zero, only when the structure is exactly symmetric about its middle surface and this requires symmetry in lamina properties, orientation and location from the middle surface. With this regard, the stiffness quantities B_{ij}, E_{ij}, C_{ij}, $B_{\alpha\beta}$ and $E_{\alpha\beta}$, given in relations (21) and (25), become immaterial in the case of symmetrically laminated structures. So, the governing equations of rectangular and circular plates, as shown in Appendices B and E, can be decoupled in two groups associated with stretching and bending. In the same connection, the constitutive equations of symmetrically laminated composite shells can be also decoupled, only if Love's first approximation is adopted.

Now, we develop (1) equilibrium equations, (2) constitutive equations, (3) kinematic relations (strain–displacement relations) for shells of revolution, cylindrical shells, rectangular plates, spherical shells, conical shells and circular plates.

4. SHELL OF REVOLUTION

4.1. THE EQUILIBRIUM EQUATIONS

Substituting the geometry definitions of shells of revolution (Figure 4) into equations (15)

$$\frac{1}{R_\phi R_\theta \sin\phi}\left[N_\phi R_\theta \cos\phi + N_{\phi\phi} R_\theta \sin\phi + N_{\theta\phi,\theta} R_\phi - N_\theta R_\theta \cos\phi\right] + \frac{Q_\phi}{R_\phi} + q_\phi = I_1 \ddot{u}_\phi + I_2 \ddot{\beta}_\phi,$$

$$\frac{1}{R_\phi R_\theta \sin\phi}\left[N_{\theta,\theta} R_\phi + N_{\theta\phi} R_\theta \cos\phi + N_{\phi\theta,\phi} R_\theta \sin\phi + N_{\phi\theta} R_\theta \cos\phi\right] + \frac{Q_\theta}{R_\theta} + q_\theta = I_1 \ddot{u}_\theta + I_2 \ddot{\beta}_\theta,$$

$$\frac{1}{R_\phi R_\theta \sin\phi}\left[Q_\phi R_\theta \cos\phi + Q_{\phi,\phi} R_\theta \sin\phi + Q_{\theta,\theta} R_\phi\right] - \frac{N_\phi}{R_\phi} - \frac{N_\theta}{R_\theta} + q_n = I_1 \ddot{w}, \qquad (27)$$

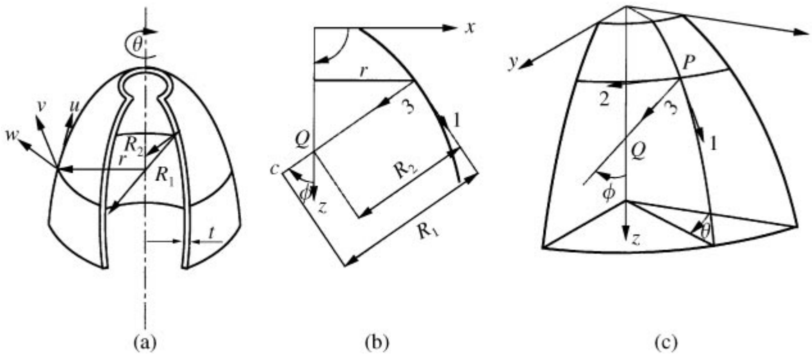

Figure 4. (a) Revolution shell, (b) principal curvatures, (c) curvilinear co-ordinates on the revolution surface. $\theta = cte.$: $ds = r_\phi \, d\phi$; $\phi = cte.$: $ds = r\,d\theta$; $\alpha_1 = \phi$; $\alpha_2 = \theta$; $A_1 = R_\phi$, $A_2 = R_\theta \sin\phi$; $R_1 = R_\phi$, $R_2 = R_\theta$; $\partial A_1/\partial\alpha_2 = 0$, $\partial A_2/\partial\alpha_1 = R_\theta \cos\phi$.

$$\frac{1}{R_\phi R_\theta \sin\phi}\left[M_{\phi,\phi} R_\theta \sin\phi + M_\phi R_\theta \cos\phi + M_{\theta\phi,\theta} R_\phi - M_\theta R_\theta \cos\phi\right] - Q_\phi = I_2 \ddot{u}_\phi + I_3 \ddot{\beta}_\phi,$$

$$\frac{1}{R_\phi R_\theta \sin\phi}\left[M_{\theta,\theta} R_\phi + M_{\phi\theta,\phi} R_\theta \sin\phi + M_{\phi\theta} R_\theta + M_{\theta\phi} R_\theta \cos\phi\right] - Q_\theta = I_2 \ddot{u}_\theta + I_3 \ddot{\beta}_\theta,$$

where the (φ, θ) and (R_φ, R_θ) are curvilinear co-ordinates and curvature radius of the revolution surface respectively (Figure 4).

4.2. CONSTITUTIVE EQUATIONS

We have the same equations as those of equations (26), but the definitions given in equation (20) must be changed. The constitutive equation is given in Appendix A.

$$\begin{aligned} a_1 &= \frac{1}{R_\theta} - \frac{1}{R_\phi}, \quad a_2 = \frac{1}{R_\phi}\left(\frac{1}{R_\phi} - \frac{1}{R_\theta}\right), \quad a_3 = \frac{1}{R_\phi^2 R_\theta}, \\ b_1 &= \frac{1}{R_\phi} - \frac{1}{R_\theta}, \quad b_2 = \frac{1}{R_\theta}\left(\frac{1}{R_\theta} - \frac{1}{R_\phi}\right), \quad b_3 = \frac{1}{R_\phi R_\theta^2}. \end{aligned} \tag{28}$$

4.3. KINEMATIC RELATIONS (LINEAR STRAIN–DISPLACEMENT RELATIONS)

Using geometrical parameters given in (Figure 4), equations (5) can be defined as

$$\begin{Bmatrix} \varepsilon_\phi \\ \varepsilon_\theta \\ \gamma_{\phi\theta} \\ \gamma_{\phi n} \\ \gamma_{\theta n} \end{Bmatrix} = \begin{bmatrix} \frac{1}{(1+\zeta/R_\phi)} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{(1+\zeta/R_\theta)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{(1+\zeta/R_\phi)} & \frac{1}{(1+\zeta/R_\theta)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{(1+\zeta/R_\phi)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{(1+\zeta/R_\theta)} \end{bmatrix}$$

$$\times\left[\begin{Bmatrix}\varepsilon_\phi^0\\ \varepsilon_\theta^0\\ \gamma_\phi^0\\ \gamma_\theta^0\\ \mu_\phi^0\\ \mu_\theta^0\end{Bmatrix}+\zeta\begin{Bmatrix}\kappa_\phi\\ \kappa_\theta\\ \tau_\phi\\ \tau_\theta\\ 0\\ 0\end{Bmatrix}\right], \tag{29}$$

where

$$\varepsilon_\phi^0=\frac{1}{R_\phi}\left(W+\frac{\partial U_\phi}{\partial\phi}\right),\quad \kappa_\phi=\frac{1}{R_\phi}\frac{\partial\beta_\phi}{\partial\phi},$$

$$\varepsilon_\theta^0=\frac{1}{R_\theta\sin\phi}\frac{\partial U_\theta}{\partial\theta}+\frac{1}{R_\phi}\operatorname{cotg}\phi U_\phi+\frac{W}{R_\theta},\quad \kappa_\theta=\frac{1}{R_\theta\sin\phi}\frac{\partial\beta_\theta}{\partial\theta}+\frac{\beta_\phi}{R_\phi}\operatorname{cotg}\phi,$$

$$\gamma_\phi^0=\frac{1}{R_\phi}\frac{\partial U_\theta}{\partial\phi},\quad \tau_\phi=\frac{1}{R_\phi}\frac{\partial\beta_\theta}{\partial\phi}, \tag{30}$$

$$\gamma_\theta^0=\frac{1}{R_\theta\sin\phi}\frac{\partial U_\phi}{\partial\theta}-\frac{U_\theta}{R_\phi}\operatorname{cotg}\phi,\quad \tau_\theta=\frac{1}{R_\theta\sin\phi}\frac{\partial\beta_\phi}{\partial\theta}+\frac{\beta_\theta}{R_\phi}\operatorname{cotg}\phi,$$

$$\mu_\phi^0=\frac{1}{R_\phi}\frac{\partial W}{\partial\phi}-\frac{U_\phi}{R_\phi}+\beta_\phi,\quad \mu_\theta^0=\frac{1}{R_\theta\sin\phi}\frac{\partial W}{\partial\theta}-\frac{U_\theta}{R_\theta}+\beta_\theta.$$

5. CYLINDRICAL SHELLS

5.1. THE EQUILIBRIUM EQUATIONS

Using the geometry definitions of circular cylindrical shells given in (Figure 5), equations (27) will become

$$\frac{\partial N_{xx}}{\partial x}+\frac{1}{R}\frac{\partial N_{\theta x}}{\partial\theta}+q_x=I_1\ddot u_x+I_2\ddot\beta_x,$$

$$\frac{\partial N_{x\theta}}{\partial x}+\frac{1}{R}\frac{\partial N_{\theta\theta}}{\partial\theta}+\frac{Q_{\theta\theta}}{R}+q_\theta=I_1\ddot u_\theta+I_2\ddot\beta_\theta,$$

$$\frac{\partial Q_{xx}}{\partial x}+\frac{1}{R}\frac{\partial Q_{\theta\theta}}{\partial\theta}-\frac{N_{\theta\theta}}{R}+q_n=I_1\ddot w, \tag{31}$$

$$\frac{\partial M_{xx}}{\partial x}+\frac{1}{R}\frac{\partial M_{\theta x}}{\partial\theta}-Q_{xx}=I_2\ddot u_x+I_3\ddot\beta_x,$$

$$\frac{1}{R}\frac{\partial M_{\theta\theta}}{\partial\theta}+\frac{\partial M_{x\theta}}{\partial x}-Q_{\theta\theta}=I_2\ddot u_\theta+I_3\ddot\beta_\theta,$$

where x and θ are curvilinear co-ordinates of the cylindrical shells (Figure 5).

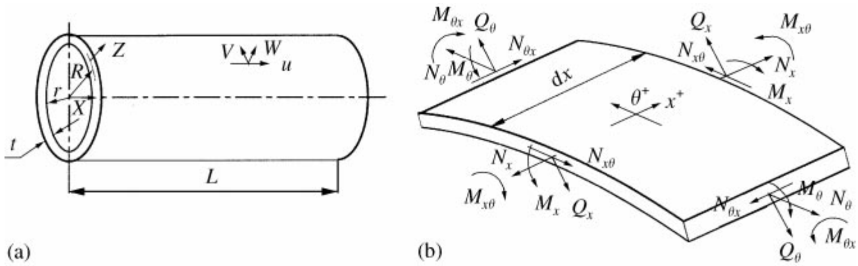

Figure 5. (a) Circular cylindrical shell geometry, (b) positive direction of integrated stress quantities. $\theta = \theta$, $\phi = x$; $R_\phi \, d\phi = dx$, $R_\phi = \infty$, $R_\theta = R$.

5.2. CONSTITUTIVE EQUATIONS

Equation (26) can be used by changing the definitions given in Figure 5, this equation is given in Appendix A:

$$a_1 = \frac{1}{R}, \quad a_2 = 0, \quad a_3 = 0, \quad b_1 = \frac{1}{R}, \quad b_2 = \frac{1}{R^2}, \quad b_3 = 0. \tag{32}$$

5.3. KINEMATIC RELATIONS (LINEAR STRAIN–DISPLACEMENT RELATIONS)

The kinematic relations are obtained by using equation (30) and shell geometry definitions:

$$\varepsilon_x^0 = \frac{\partial U_x}{\partial x}, \quad \varepsilon_\theta^0 = \frac{1}{R}\frac{\partial U_\theta}{\partial \theta} + \frac{W}{R}, \quad \gamma_x^0 = \frac{\partial U_\theta}{\partial x}, \quad \gamma_\theta^0 = \frac{1}{R}\frac{\partial U_x}{\partial \theta},$$

$$\kappa_x = \frac{\partial \beta_x}{\partial x}, \quad \kappa_\theta = \frac{1}{R}\frac{\partial \beta_\theta}{\partial \theta}, \quad \tau_x = \frac{\partial \beta_\theta}{\partial x}, \quad \tau_\theta = \frac{1}{R}\frac{\partial \beta_x}{\partial \theta}, \tag{33}$$

$$\mu_x^0 = \frac{\partial W}{\partial x} + \beta_x, \quad \mu_\theta^0 = \frac{1}{R}\frac{\partial W}{\partial \theta} - \frac{U_\theta}{R} + \beta_\theta.$$

Substituting the above equations into the constitutive equations (taking into account the coefficients which were given in equations (32)) and then into equations (31), we obtain

$$L_k = (U_x, U_\theta, W, \beta_x, \beta_\theta, \overline{P_{ij}}) = 0, \quad (k = 1, 2, \ldots, 5). \tag{34}$$

These relations are defined fully by the equations given in Appendix A. In order to compare them with *classical shell theory*, the three equations of motion for cylindrical shells are also given in Appendix A [147].

6. RECTANGULAR PLATES

6.1. THE EQUILIBRIUM EQUATIONS

The same cylindrical shell equations are used, taking into account the rectangular plate geometry definitions (Figure 6), so equations (31) become

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{yx}}{\partial y} + q_x = I_1 \ddot{u}_x + I_2 \ddot{\beta}_x,$$

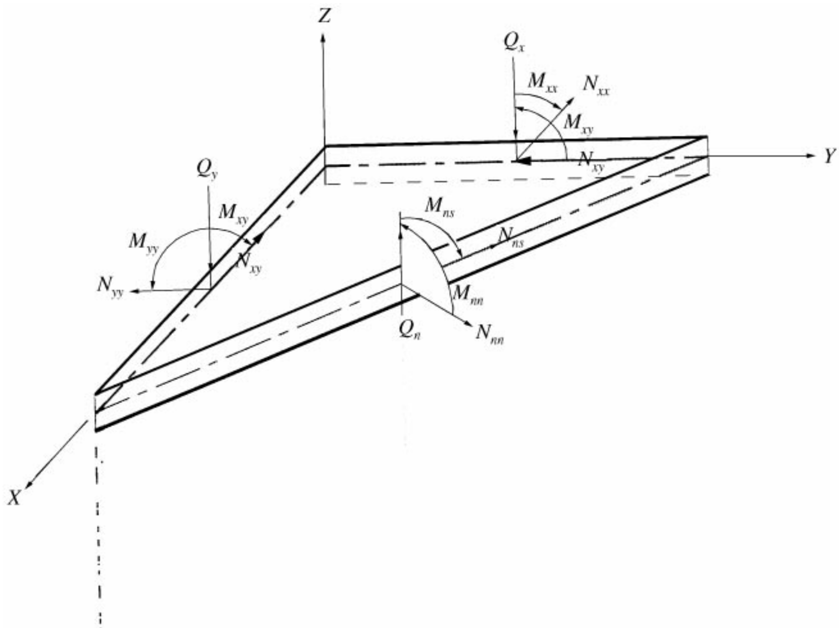

Figure 6. Force and moment resultant on a plate element: $r \to \infty$, $\theta \to \infty$, $r\,\mathrm{d}\theta \to \mathrm{d}y$.

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} + q_y = I_1 \ddot{u}_y + I_2 \ddot{\beta}_y,$$

$$\frac{\partial Q_{xx}}{\partial x} + \frac{\partial Q_y}{\partial y} + q_n = I_1 \ddot{w}, \tag{35}$$

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{yx}}{\partial y} + Q_{xx} = I_2 \ddot{u}_x + I_3 \ddot{\beta}_x,$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_{yy} = I_2 \ddot{u}_y + I_3 \ddot{\beta}_y.$$

6.2. CONSTITUTIVE EQUATIONS

We have the same equations as those of equation (26), but definition (20) must be changed, this equation is defined in Appendix A:

$$a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 0. \tag{36}$$

6.3. KINEMATIC RELATIONS (LINEAR STRAIN–DISPLACEMENT RELATIONS)

These relations can be obtained by substituting the structural geometry definitions into the kinematic relations of the cylindrical shell (33):

$$\varepsilon_x^0 = \frac{\partial U_x}{\partial x}, \quad \varepsilon_y^0 = \frac{\partial U_y}{\partial y}, \quad \gamma_x^0 = \frac{\partial U_y}{\partial x}, \quad \gamma_y^0 = \frac{\partial U_x}{\partial y},$$

$$\kappa_x = \frac{\partial \beta_x}{\partial x}, \quad \kappa_y = \frac{\partial \beta_y}{\partial y}, \quad \tau_x = \frac{\partial \beta_y}{\partial x}, \quad \tau_y = \frac{\partial \beta_x}{\partial y}, \tag{37}$$

$$\mu_x^0 = \frac{\partial W}{\partial x} + \beta_x, \quad \mu_y^0 = \frac{\partial W}{\partial y} + \beta_y.$$

Now, we can substitute the constitutive equations into equation (35) in the same way that we obtained the five differential equations for the case of cylindrical shells, and can obtain the implicit equations of the form (34). These equations are given fully in Appendix B.

7. SPHERICAL SHELLS

7.1. THE EQUILIBRIUM EQUATIONS

The equilibrium equations for the spherical shells can be derived by using equation (27) and following definitions (Figure 7):

$$\frac{\operatorname{cosec}\phi}{R}[N_\phi \cos\phi + N_{\phi,\phi}\sin\phi + N_{\theta\phi,\theta} - N_\theta\cos\phi] + \frac{Q_\phi}{R_\phi} + q_\phi = I_1\ddot{u}_\phi + I_2\ddot{\beta}_\phi,$$

$$\frac{1}{R}[N_{\theta,\theta}\operatorname{cosec}\phi + N_{\theta\phi}\operatorname{cotg}\phi + N_{\phi\theta,\phi} + N_{\phi\theta}\operatorname{cotg}\phi] + \frac{Q_\theta}{R} + q_\theta = I_1\ddot{u}_\theta + I_2\ddot{\beta}_\theta,$$

$$\frac{1}{R}[Q_\phi\operatorname{cotg}\phi + Q_{\phi,\phi} + Q_{\theta,\theta}\operatorname{cosec}\phi] - \frac{N_\phi}{R} - \frac{N_\theta}{R} + q_n = I_1\ddot{w}, \tag{38}$$

$$\frac{1}{R}[M_{\phi,\phi} + M_\phi\operatorname{cotg}\phi + M_{\theta\phi,\theta}\operatorname{cosec}\phi - M_\theta\operatorname{cotg}\phi] - Q_\phi = I_2\ddot{u}_\phi + I_3\ddot{\beta}_\phi,$$

$$\frac{1}{R}[M_{\theta,\theta}\operatorname{cosec}\phi + M_{\phi\theta,\phi} + M_{\phi\theta}\operatorname{cotg}\phi + M_{\theta\phi}\operatorname{cotg}\phi] - Q_\theta = I_2\ddot{u}_\theta + I_3\ddot{\beta}_\theta,$$

7.2. CONSTITUTIVE EQUATIONS

We have the same equations as in equations (26), but the definitions given in equation (20) must be changed; these relations are given fully in Appendix A:

$$a_1 = a_2 = b_1 = b_2 = 0, \qquad a_3 = b_3 = \frac{1}{R^3}. \tag{39}$$

7.3. KINEMATIC RELATIONS

Substituting $r_\theta = r_\phi = R$ into the definitions of equations (30), equations (5) are defined:

$$\varepsilon_\phi^0 = \frac{1}{R}\left(\frac{\partial U_\phi}{\partial\phi} + W\right), \quad \varepsilon_\theta^0 = \frac{1}{R\sin\phi}\frac{\partial U_\theta}{\partial\theta} + \frac{1}{R}\operatorname{cotg}\phi\, U_\phi + \frac{W}{R},$$

$$\gamma_\phi^0 = \frac{1}{R}\frac{\partial U_\theta}{\partial\phi}, \quad \gamma_\theta^0 = \frac{1}{R\sin\phi}\frac{\partial U_\phi}{\partial\theta} - \frac{1}{R}\operatorname{cotg}\phi\, U_\theta,$$

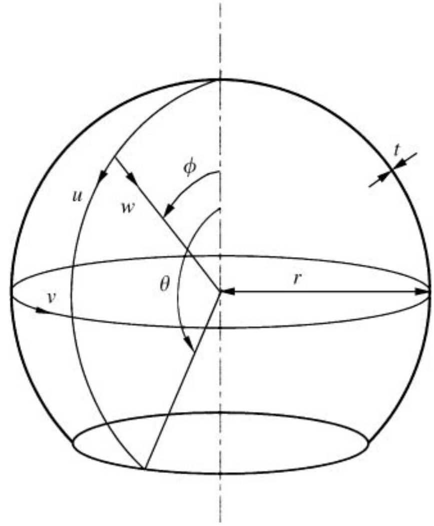

Figure 7. Geometry of spherical shell: $R_\theta = R_\phi = R$.

$$\kappa_\phi = \frac{1}{R}\frac{\partial \beta_\phi}{\partial \phi}, \quad \kappa_\theta = \frac{1}{R\sin\phi}\frac{\partial \beta_\theta}{\partial \theta} + \frac{1}{R}\operatorname{cotg}\phi\,\beta_\phi, \tag{40}$$

$$\tau_\phi = \frac{1}{R}\frac{\partial \beta_\theta}{\partial \phi}, \quad \tau_\theta = \frac{1}{R\sin\phi}\frac{\partial \beta_\phi}{\partial \theta} - \frac{1}{R}\operatorname{cotg}\phi\,\beta_\theta,$$

$$\mu_\phi^0 = \frac{1}{R}\frac{\partial W}{\partial \phi} - \frac{U_\phi}{R} + \beta_\phi, \quad \mu_\theta^0 = \frac{1}{R\sin\phi}\frac{\partial W}{\partial \theta} - \frac{U_\theta}{R} + \beta_\theta.$$

Now, we substitute relation (40) into the constitutive equations and then into equations (38), giving five differential equations which describe the equations of motion in terms of the displacement field and mechanical properties of the shell, so that we have the same implicit equations as in equations (34). Li's is equations are given in Appendix C.

8. CONICAL SHELLS

8.1. THE EQUILIBRIUM EQUATIONS

We substitute the geometry definitions of conical shells (Figure 8) into equation (27):

$$\frac{\operatorname{cosec}\alpha}{x} N_{\theta x,\theta} + N_{x,x} + q_x = I_1 \ddot{u}_x + I_2 \ddot{\beta}_x,$$

$$\frac{\operatorname{cosec}\alpha}{x} N_{\theta,\theta} + N_{x\theta,x} + \frac{1}{x\tan\alpha} Q_\theta + q_\theta = I_1 \ddot{u}_\theta + I_2 \ddot{\beta}_\theta,$$

$$\frac{\operatorname{cosec}\alpha}{x} Q_{\theta,\theta} + Q_{x,x} - \frac{1}{x\tan\alpha} N_\theta + q_n = I_1 \ddot{w} \tag{41}$$

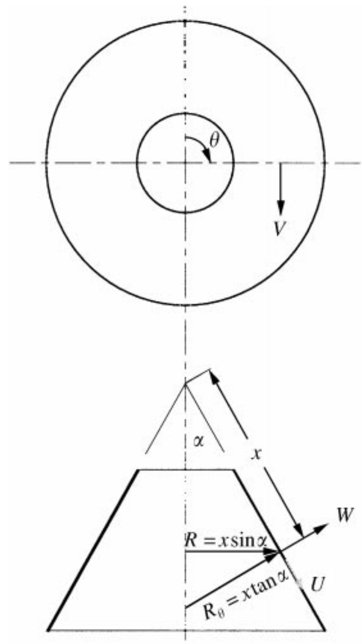

Figure 8. Geometry of conical shell: $R_\phi = \infty$, $R_\theta = x\tan\alpha$, $\phi = \pi/2 - \alpha$, $\sin\phi = \cos\alpha$, $\cos\phi = \sin\alpha$, $r_\phi\,\mathrm{d}\phi \to \mathrm{d}x$.

$$\frac{\operatorname{cosec}\alpha}{x} M_{\theta x,\theta} + M_{x,x} - Q_x = I_2 \ddot{u}_x + I_3 \ddot{\beta}_x,$$

$$\frac{\operatorname{cosec}\alpha}{x} M_{\theta,\theta} + M_{x\theta,x} - Q_\theta = I_2 \ddot{u}_\theta + I_3 \ddot{\beta}_\theta.$$

8.2. CONSTITUTIVE EQUATIONS

Equation (26) has to be modified by changing the definitions given in equations (20) to obtain the constitutive equation of the conical shells; this equation is defined in Appendix A:

$$a_1 = \frac{1}{x\tan\alpha}, \quad a_2 = 0, \quad a_3 = 0,$$

$$b_1 = -\frac{1}{x\tan\alpha}, \quad b_2 = \frac{1}{x^2\tan^2\alpha}, \quad b_3 = 0. \tag{42}$$

8.3. KINEMATIC RELATIONS (LINEAR STRAIN–DISPLACEMENT RELATIONS)

These relations can be obtained by using the strain–displacement relations of shells of revolution (30) and conical shell geometry definitions given in (Figure 8)

$$\varepsilon_x^0 = \frac{\partial U_x}{\partial x}, \quad \varepsilon_\theta^0 = \frac{1}{x\sin\alpha}\frac{\partial U_\theta}{\partial \theta} + \frac{W}{x\tan\alpha}, \quad \gamma_x^0 = \frac{\partial U_\theta}{\partial x}, \quad \gamma_\theta^0 = \frac{1}{x\sin\alpha}\frac{\partial U_x}{\partial \theta},$$

$$\kappa_x = \frac{\partial \beta_x}{\partial x}, \quad \kappa_\theta = \frac{1}{x \sin \alpha} \frac{\partial \beta_\theta}{\partial \theta}, \quad \tau_x = \frac{\partial \beta_\theta}{\partial x}, \quad \tau_\theta = \frac{1}{x \sin \alpha} \frac{\partial \beta_x}{\partial \theta}, \tag{43}$$

$$\mu_x^0 = \frac{\partial W}{\partial x} + \beta_x, \quad \mu_\theta^0 = \frac{1}{x \sin \alpha} \frac{\partial W}{\partial \theta} - \frac{U\theta}{x \tan \alpha} + \beta_\theta .$$

The five differential equations of motion for conical shells, in terms of the displacement field and mechanical properties of shells, can be obtained by substituting the kinematic relations first into the constitutive equations, and then into the equilibrium equations. These implicit equations L_i are given fully in Appendix D.

9. CIRCULAR PLATES

9.1. THE EQUILIBRIUM EQUATIONS

These equations are obtained by using circular plate geometry definitions (Figure 9) and the same equations as we used for the conical shell (41)

$$\frac{1}{R} \frac{\partial N_{\theta r}}{\partial \theta} + \frac{\partial N_{rr}}{\partial r} + q_r = I_1 \ddot{u}_r + I_2 \ddot{\beta}_r,$$

$$\frac{1}{R} \frac{\partial N_{\theta\theta}}{\partial \theta} + \frac{\partial N_{r\theta}}{\partial r} + q_\theta = I_1 \ddot{u}_\theta + I_2 \ddot{\beta}_\theta,$$

$$\frac{1}{R} \frac{\partial Q_{\theta\theta}}{\partial \theta} + \frac{\partial Q_{rr}}{\partial r} + q_n = I_1 \ddot{w}, \tag{44}$$

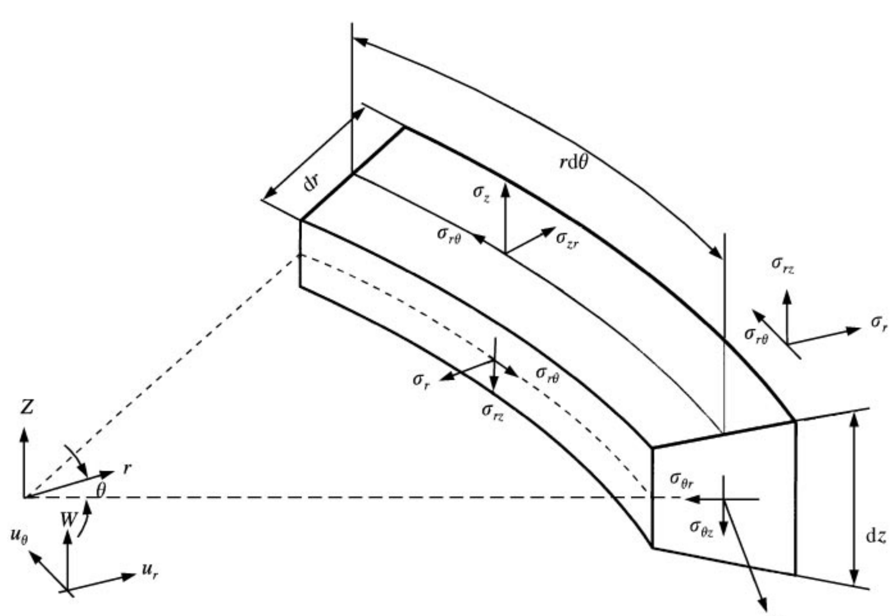

Figure 9. Circular plate element: $\alpha = \pi/2$, $x = r$.

$$\frac{1}{R}\frac{\partial M_{\theta r}}{\partial \theta}+\frac{\partial M_{rr}}{\partial r}-Q_{rr}=I_2\ddot{u}_r+I_3\ddot{\beta}_r,$$

$$\frac{1}{R}\frac{\partial M_{\theta\theta}}{\partial \theta}+\frac{\partial M_{r\theta}}{\partial r}-Q_{\theta\theta}=I_2\ddot{u}_\theta+I_3\ddot{\beta}_\theta.$$

9.2. CONSTITUTIVE EQUATIONS

Changing the relations defined in equations (20) and substituting into equation (26), the constitutive equation for a circular plate can be obtained and is given in Appendix A:

$$a_1=a_2=a_3=b_1=b_2=b_3=0. \tag{45}$$

9.3. KINEMATIC RELATIONS (LINEAR STRAIN–DISPLACEMENT RELATIONS)

These equations are obtained by substituting the geometry definitions of circular plates into the conical shell kinematic relations:

$$\varepsilon_r^0=\frac{\partial U_r}{\partial r},\quad \varepsilon_\theta^0=\frac{1}{R}\frac{\partial U_\theta}{\partial \theta},\quad \gamma_r^0=\frac{\partial U_\theta}{\partial r},\quad \gamma_\theta^0=\frac{1}{R}\frac{\partial U_r}{\partial \theta},$$

$$\kappa_r=\frac{\partial \beta_r}{\partial r},\quad \kappa_\theta=\frac{1}{R}\frac{\partial \beta_\theta}{\partial \theta},\quad \tau_r=\frac{\partial \beta_\theta}{\partial r},\quad \tau_\theta=\frac{1}{R}\frac{\partial \beta_r}{\partial \theta}, \tag{46}$$

$$\mu_r^0=\frac{\partial W}{\partial r}+\beta_r,\quad \mu_\theta^0=\frac{1}{R}\frac{\partial W}{\partial \theta}+\beta_\theta.$$

We substitute relation (46) first into the constitutive equations and then into equations (44), and obtain five differential equations which are defined in Appendix E.

10. CHARACTERISTIC EQUATION

In the present theory, β_1 and β_2 which represent the rotation of tangents to the reference surface oriented along parametric lines α_1 and α_2, cannot be expressed in terms of U_i and W. Therefore, the five differential equations of motion cannot be reduced to 3 as in classical shell theory. In the case of cylindrical shells, we obtain five differential equations of motion as shown in equations (A.2)–(A.6) in Appendix A. Also listed in Appendix A are the three differential equations (A.7)–(A.9) of Sanders' cylindrical shell theory. The accuracy of the finite element method depends primarily on the number and size of the finite element into which the structure is divided. Good accuracy can generally be obtained with a sufficiently large number of small elements. The optimum degree of approximation in the element stiffness and mass matrices will depend upon many factors, the most important perhaps being the choice of the displacement functions and the degree to which they satisfy the convergence criteria of the finite element method; here we do not mean numerical convergence but absolute convergence to the continuum.

The characteristic equations of vibration analysis of anisotropic laminated open circular cylindrical shells, formulated on the basis of the present theory, have been compared to that of Sanders' shell theory [147]. Assuming the displacement functions for the dynamic analysis of anisotropic circular cylindrical shells to be

$$\begin{Bmatrix} U(x,\theta) \\ V(x,\theta) \\ W(x,\theta) \\ \beta_x(x,\theta) \\ \beta_\theta(x,\theta) \end{Bmatrix} = \sum_{i=1}^{10} \begin{bmatrix} \cos\bar{m}x & 0 & 0 & 0 & 0 \\ 0 & \sin\bar{m}x & 0 & 0 & 0 \\ 0 & 0 & \sin\bar{m}x & 0 & 0 \\ 0 & 0 & 0 & \cos\bar{m}x & 0 \\ 0 & 0 & 0 & 0 & \sin\bar{m}x \end{bmatrix} \begin{Bmatrix} u_i(\theta) \\ v_i(\theta) \\ w_i(\theta) \\ \beta_{x_i}(\theta) \\ \beta_{\theta_i}(\theta) \end{Bmatrix} = \sum_{i=1}^{10} [T_1] \begin{Bmatrix} A_i e^{i\theta} \\ B_i e^{i\theta} \\ C_i e^{i\theta} \\ D_i e^{i\theta} \\ E_i e^{i\theta} \end{Bmatrix}, \tag{47}$$

we substitute these definitions into the equations of motion for cylindrical shells (34). We then take into account that the non-trivial solution leads to a tenth order polynomial equation (48) (characteristic equation) due to 5 d.o.f.s per node, instead of an eigth order equation (49) [147, equation (10)]:

$$f_{10}\eta^{10} + f_8\eta^8 + f_6\eta^6 + f_4\eta^4 + f_2\eta^2 + f_0 = 0, \tag{48}$$

where $f_i (i = 0–10)$ are the coefficients of the determinant of the matrix $[H]$ given in Appendix A. For the case of isotropic cylindrical shells based on classical shell theory, we obtain

$$h_8\eta^8 + h_6\eta^6 + h_4\eta^4 + h_2\eta^2 + h_0 = 0, \tag{49}$$

where h_i $(i = 0–8)$. The coefficients of the characteristic equation of cylindrical shells based on Sanders' shell theory are given in reference [147]. Each root of the characteristic equation (48) yields a solution to the equations of motion (34). The complete solution is obtained by finding the sum of all 10 solutions independently with the constants A_i, B_i, C_i, D_i and E_i. The fundamental unknowns consist of the 10 strain components, 10 stress resultants and the five generalized displacements of plates or shells.

It is necessary to formulate 10 boundary conditions for the finite elements; the axial, tangential and radial displacements as well as the rotations will be specified for each node. The displacement functions for this theory are derived and mass and stiffness matrices of each element are obtained by exact analytical integration.

The roots of the characteristic equation for equations (48, 49) obtained by the computer program are given for isotropic and anisotropic materials. One such set of calculation is shown in Table 1, where the computed values based on Sanders' theory, made by authors of reference [139], were compared with those from other theories, given in reference [139]. Tables 2 and 3 show the characteristic equation values of equation (49), reference [147], and those of equation (48) obtained by the present theory.

A cross-ply layered (0°/90°/90°/0°) cylindrical shell with the following material properties were used as an anisotropic material example. All layers are assumed to have the same geometric and material parameters and the individual layer is assumed to be orthotropic: $E_1 = 25E_2$, $G_{23} = 0{\cdot}2E_2$, $G_{13} = G_{12} = 0{\cdot}5E_2$, $v_{12} = 0{\cdot}25$, $\rho = 1$.

11. DISCUSSION AND CONCLUSION

General equations of multilayered laminated anisotropic shells were developed by taking into account the shear deformation and rotary inertia effects as well as the initial curvature.

TABLE 1

Roots of characteristic equations for $12\ R^2(1-\nu^2)/t^2 = 4\times 10^4$ *and* $\nu = 0{\cdot}3$

	$n=2$		$n=3$		$n=10$	
	λ_1	λ_2	λ_1	λ_2	λ_1	λ_2
Sanders*	10·2020 $\pm$ 9·8026**i**	0·1757 $\pm$ 0·17051**i**	10·4650 $\pm$ 9·5682**i**	0·43961 $\pm$ 0·40598**i**	15·2860 $\pm$ 7·3965**i**	5·2613 $\pm$ 2·5783**i**
Flügge†	10·1952 $\pm$ 9·8104**i**	0·1758 $\pm$ 0·17040**i**	10·4581 $\pm$ 9·5761**i**	0·43990 $\pm$ 0·40570**i**	15·2533 $\pm$ 7·4851**i**	5·2610 $\pm$ 2·5719**i**
Vlasov†	10·1955 $\pm$ 9·8107**i**	0·1756 $\pm$ 0·17060**i**	10·4591 $\pm$ 9·5771**i**	0·43960 $\pm$ 0·40590**i**	15·2881 $\pm$ 7·4183**i**	5·2759 $\pm$ 2·5766**i**
Timoshenko†	10·2025 $\pm$ 9·8027**i**	0·1758 $\pm$ 0·17040**i**	10·4652 $\pm$ 9·5632**i**	0·44000 $\pm$ 0·40560**i**	15·2840 $\pm$ 7·3951**i**	5·2645 $\pm$ 2·5741**i**
Novozhilov†	10·2022 $\pm$ 9·8024**i**	0·1757 $\pm$ 0·17050**i**	10·4645 $\pm$ 9·5674**i**	0·43960 $\pm$ 0·40601**i**	15·2796 $\pm$ 7·3859**i**	5·2657 $\pm$ 2·5779**i**
Naghdi and Berry†	10·2027 $\pm$ 9·8030**i**	0·1760 $\pm$ 0·17020**i**	10·4660 $\pm$ 9·5690**i**	0·44030 $\pm$ 0·40520**i**	15·2737 $\pm$ 7·4030**i**	5·2860 $\pm$ 2·5342**i**

* Data from computer program of authors [139].
† Data given in reference [139].

TABLE 2

Roots of characteristic equations (48, 49) *for isotropic materials* ($m = 1$)

	$\eta_1, \ldots, \eta_8$, reference [147]	$\eta_1, \ldots, \eta_{10}$, present
$R/t = 10$, $L/R = 1$	$\pm$ 2·2097 $\pm$ 2·9127**i** $\pm$ 4·8899 $\pm$ 1·2551**i**	$\pm$ 1·6124 $\pm$ 3·0289**i**, $\pm$ 34·0672 $\pm$ 5·3039 $\pm$ 1·3630**i**
$R/t = 20$, $L/R = 1$	$\pm$ 2·3750 $\pm$ 3·8041**i** $\pm$ 5·4940 $\pm$ 1·6173**i**	$\pm$ 1·2718 $\pm$ 3·7971**i**, $\pm$ 69·0023 $\pm$ 6·0694 $\pm$ 2·1717**i**

TABLE 3

Roots of characteristic equations (48, 49) *for anisotropic materials* ($0°/90°/90°/0°$)

	$\eta_1, \ldots, \eta_8$, reference [147]	$\eta_1, \ldots, \eta_{10}$, present
$R/t = 10$, $L/R = 1$; $m = 1$	$\pm$ 2·7864, $\pm$ 21·9869 $\pm$ 5·4093 $\pm$ 4·3960**i**	$\pm$ 1·7076 $\pm$ 3·1030**i**, $\pm$ 16·2057 $\pm$ 5·4873 $\pm$ 1·2258**i**
$R/t = 10$, $L/R = 1$; $m = 2$	$\pm$ 4·5689, $\pm$ 43·9733 $\pm$ 11·0662 $\pm$ 8·6369**i**	$\pm$ 4·5045 $\pm$ 3·8386**i**, $\pm$ 6·3269 $\pm$ 12·7532 $\pm$ 4·7411**i**

We believe that these effects will be more important to the dynamic behavior of anisotropic shells than of isotropic materials. The derivation used geometrically linear theory for small elastic strains and strains expressed in orthogonal curvilinear co-ordinates for general shells. The virtual work principle was applied in order to derive the equilibrium equations. The work of several researchers on this particular subject has been reviewed and summarized.

The theory used yields five coupled linear second order differential equations with constant coefficients, instead of three equations, as in the case of other theories. The reason for this is that transverse shear strains do not vanish in the present theory and, therefore, the β_i cannot be expressed in terms of displacement components. This theory leads to no strain during rigid-body motions.

A paper currently under preparation will deal with the dynamic analysis of open and closed nonuniform anisotropic laminated circular cylindrical shells with arbitrary boundary conditions. The effects of transverse shear deformations and rotary inertia on the vibration characteristics of cylindrical shells of different geometrical (R/t, L/R and L/t) and material (isotropic, symmetric and antisymmetric cross-ply laminated shells) parameters, as well as axial and circumferential wave number (m, n) are handled through several numerical examples with reasonable agreement with other theories. The computational method used is a combination of hybrid finite element analysis based on the method of reference [139] and refined shell theory. The displacement functions are obtained using the new shell equations developed in this paper. This method has been successfully tested; because of the use of classical shell theory in the framework of the finite element method, we can obtain the high as well as the low frequencies with good accuracy.

The first preliminary results indicate that the presence of the transverse shear deformation effects is very significant and tends to reduce the frequency parameters specially for laminated anisotropic shells. It has been suggested that the reason for the difference is a change in shear angle from layer to layer and the insensitivity of the classical shell theory (CST) to this change. In the case of rotary inertia, our preliminary results indicate that the effects of rotary inertia are practically limited. We are presently investigating these effects on the lowest branch as well as the highest branch of the frequency spectrum. It is important to note that Librescu [165] found, in the case of an anisotropic plate, that the rotary inertia effect was practically inexistant at the lowest branch of the frequency spectrum. Further work is under way to apply this theory to the dynamic analysis of open and closed anisotropic cylindrical shells filled with or subjected to a flowing fluid.

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APPENDIX A: CONSTITUTIVE AND GOVERNING EQUATIONS

This appendix contains the constitutive equations and governing equations for thin anisotropic plates and shells which were referred to this paper. The appendix is divided into five parts, covering, respectively, cylindrical shells, rectangular plates, spherical and conical shells, and circular plates.

The P_{ij}'s elements (A_{ij}, B_{ij}, D_{ij}, G_{ij}, G'_{ij}, H_{ij}, H'_{ij}, J_{ij} and J'_{ij}) have been defined by equations (19)–(21) and equation (25):

$$\begin{Bmatrix} N_{11} \\ N_{12} \\ Q_{11} \\ N_{22} \\ N_{21} \\ Q_{22} \\ M_{11} \\ M_{12} \\ M_{22} \\ M_{21} \end{Bmatrix} = \begin{bmatrix} G_{11} & G_{16} & 0 & A_{12} & A_{16} & 0 & H_{11} & H_{16} & B_{12} & B_{16} \\ G_{61} & G_{66} & 0 & A_{62} & A_{66} & 0 & H_{61} & H_{66} & B_{62} & B_{66} \\ 0 & 0 & AA_{55} & 0 & 0 & A_{54} & 0 & 0 & 0 & 0 \\ A_{21} & A_{26} & 0 & G'_{22} & G'_{26} & 0 & B_{21} & B_{26} & H'_{22} & H'_{26} \\ A_{61} & A_{66} & 0 & G'_{62} & G'_{66} & 0 & B_{61} & B_{66} & H_{62} & H'_{66} \\ 0 & 0 & A_{45} & 0 & 0 & BB_{44} & 0 & 0 & 0 & 0 \\ H_{11} & H_{16} & 0 & B_{12} & B_{16} & 0 & J_{11} & J_{16} & D_{12} & D_{16} \\ H_{61} & H_{66} & 0 & B_{62} & B_{66} & 0 & J_{61} & J_{66} & D_{62} & D_{66} \\ B_{21} & B_{26} & 0 & H'_{22} & H'_{26} & 0 & D_{21} & D_{26} & J'_{22} & J'_{26} \\ B_{61} & B_{66} & 0 & H'_{62} & H'_{66} & 0 & D_{61} & D_{66} & J'_{62} & J'_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1^0 \\ \gamma_1^0 \\ \mu_1^0 \\ \varepsilon_2^0 \\ \gamma_2^0 \\ \mu_2^0 \\ \kappa_1 \\ \tau_1 \\ \kappa_2 \\ \tau_2 \end{Bmatrix}. \tag{A.1}$$

A.1. CYLINDRICAL SHELLS

The governing equations are defined by the following equations:

$$
\begin{aligned}
&L_1(U_x, U_\theta, W, \beta_x, \beta_\theta, \overline{P_{ij}}) \\
&\quad = P_{11}\frac{\partial^2 U_x}{\partial x^2} + \frac{1}{R}(P_{15} + P_{51})\frac{\partial^2 U_x}{\partial x\,\partial\theta} + \frac{P_{55}}{R^2}\frac{\partial^2 U_x}{\partial\theta^2} - I_1\frac{\partial^2 U_x}{\partial t^2} + P_{12}\frac{\partial^2 U_\theta}{\partial x^2} \\
&\qquad + \frac{1}{R}(P_{14} + P_{52})\frac{\partial^2 U_\theta}{\partial x\,\partial\theta} + \frac{P_{54}}{R^2}\frac{\partial^2 U_\theta}{\partial\theta^2} + \frac{P_{14}}{R}\frac{\partial W}{\partial x} + \frac{P_{54}}{R^2}\frac{\partial W}{\partial\theta} + P_{17}\frac{\partial^2\beta_x}{\partial x^2} \\
&\qquad + \frac{1}{R}(P_{1,10} + P_{57})\frac{\partial^2\beta_x}{\partial x\,\partial\theta} + P_{5,10}\frac{\partial^2\beta_x}{\partial\theta^2} - I_2\frac{\partial^2\beta_x}{\partial t^2} + P_{18}\frac{\partial^2\beta_\theta}{\partial x^2} + \frac{1}{R}(P_{19} + P_{58})\times \\
&\qquad \times\frac{\partial^2\beta_\theta}{\partial x\,\partial\theta} + \frac{P_{59}}{R^2}\frac{\partial^2\beta_\theta}{\partial\theta^2},
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
&L_2(U_x, U_\theta, W, \beta_x, \beta_\theta, \overline{P_{ij}}) \\
&\quad = P_{21}\frac{\partial^2 U_x}{\partial x^2} + \frac{1}{R}(P_{25} + P_{41})\frac{\partial^2 U_x}{\partial x\,\partial\theta} + \frac{P_{45}}{R^2}\frac{\partial^2 U_x}{\partial\theta^2} + P_{22}\frac{\partial^2 U_\theta}{\partial x^2} \\
&\qquad + \frac{1}{R}(P_{24} + P_{42})\frac{\partial^2 U_\theta}{\partial x\,\partial\theta} + \frac{P_{44}}{R^2}\frac{\partial^2 U_\theta}{\partial\theta^2} - \frac{P_{66}}{R^2}U_\theta - I_1\frac{\partial^2 U_\theta}{\partial t^2} + \frac{1}{R}(P_{24} + P_{63})\times \\
&\qquad \times\frac{\partial W}{\partial x} + \frac{1}{R^2}(P_{44} + P_{66})\frac{\partial W}{\partial\theta} + P_{27}\frac{\partial^2\beta_x}{\partial x^2} + \frac{1}{R}(P_{2,10} + P_{47})\frac{\partial^2\beta_x}{\partial x\,\partial\theta} + \frac{P_{4,10}}{R^2}\frac{\partial^2\beta_x}{\partial\theta^2} \\
&\qquad + \frac{P_{36}}{R}\beta_x + P_{28}\frac{\partial^2\beta_\theta}{\partial x^2} + \frac{1}{R}(P_{29} + P_{48})\frac{\partial^2\beta_\theta}{\partial x\,\partial\theta} + \frac{P_{49}}{R^2}\frac{\partial^2\beta_\theta}{\partial\theta^2} + \frac{P_{66}}{R}\beta_\theta - I_2\frac{\partial^2 U_\theta}{\partial t^2},
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
&L_3(U_x, U_\theta, W, \beta_x, \beta_\theta, \overline{P_{ij}}) \\
&\quad = -\frac{P_{41}}{R}\frac{\partial U_x}{\partial x} - \frac{P_{45}}{R^2}\frac{\partial U_x}{\partial\theta} - \frac{1}{R}(P_{36} + P_{42})\frac{\partial U_\theta}{\partial x} - \frac{1}{R^2}(P_{66} + P_{44})\frac{\partial U_\theta}{\partial\theta} + P_{33}\frac{\partial^2 W}{\partial x^2} \\
&\qquad + \frac{1}{R}(P_{36} + P_{63})\frac{\partial^2 W}{\partial x\,\partial\theta} + \frac{P_{66}}{R^2}\frac{\partial^2 W}{\partial\theta^2} - \frac{P_{44}}{R^2}W - I_1\frac{\partial^2 W}{\partial t^2} + \left(P_{33} - \frac{P_{47}}{R}\right)\frac{\partial\beta_x}{\partial x} \\
&\qquad + \frac{1}{R}\left(P_{63} - \frac{P_{4,10}}{R}\right)\frac{\partial\beta_x}{\partial\theta} + \left(P_{36} - \frac{P_{48}}{R}\right)\frac{\partial\beta_\theta}{\partial x} + \frac{1}{R}\left(P_{66} - \frac{P_{49}}{R}\right)\frac{\partial\beta_\theta}{\partial\theta},
\end{aligned}
\tag{A.4}
$$

$$
\begin{aligned}
&L_4(U_x, U_\theta, W, \beta_x, \beta_\theta, \overline{P_{ij}}) \\
&\quad = P_{71}\frac{\partial^2 U_x}{\partial x^2} + \frac{1}{R}(P_{75} + P_{10,1})\frac{\partial^2 U_x}{\partial x\,\partial\theta} - I_2\frac{\partial^2 U_x}{\partial t^2} + \frac{P_{10,5}}{R^2}\frac{\partial^2 U_x}{\partial\theta^2} - I_2\frac{\partial^2 U_x}{\partial t^2} + P_{72}\frac{\partial^2 U_\theta}{\partial x^2} \\
&\qquad + \frac{1}{R}(P_{74} + P_{10,2})\frac{\partial^2 U_\theta}{\partial x\,\partial\theta} + \frac{P_{10,4}}{R^2}\frac{\partial^2 U_\theta}{\partial\theta^2} + \frac{P_{36}}{R^2}U_\theta + \left(\frac{P_{74}}{R} - P_{33}\right)\frac{\partial W}{\partial x} \\
&\qquad + \frac{1}{R}\left(\frac{P_{10,4}}{R} - P_{36}\right)\frac{\partial W}{\partial\theta} + P_{77}\frac{\partial^2\beta_x}{\partial x^2} + \frac{1}{R}(P_{7,10} + P_{10,7})\frac{\partial^2\beta_x}{\partial x\,\partial\theta} + \frac{P_{10,10}}{R^2}\frac{\partial^2\beta_x}{\partial\theta^2}
\end{aligned}
\tag{A.5}
$$

$$- P_{33}\beta_x - I_3\frac{\partial^2\beta_x}{\partial t^2} + P_{78}\frac{\partial^2\beta_\theta}{\partial x^2} + \frac{1}{R}(P_{79} + P_{10,8})\frac{\partial^2\beta_\theta}{\partial x\,\partial\theta} + \frac{P_{10,9}}{R^2}\frac{\partial^2\beta_\theta}{\partial\theta^2} - P_{36}\beta_\theta,$$

$$\begin{aligned} L_5(U_x, U_\theta, W, \beta_x, \beta_\theta, \overline{P_{ij}}) &= P_{81}\frac{\partial^2 U_x}{\partial x^2} + \frac{1}{R}(P_{91} + P_{85})\frac{\partial^2 U_x}{\partial x\,\partial\theta} + \frac{P_{95}}{R^2}\frac{\partial^2 U_x}{\partial\theta^2} + P_{82}\frac{\partial^2 U_\theta}{\partial x^2} \\ &+ \frac{1}{R}(P_{84} + P_{92})\frac{\partial^2 U_\theta}{\partial x\,\partial\theta} + \frac{P_{94}}{R^2}\frac{\partial^2 U_\theta}{\partial\theta^2} + \frac{P_{66}}{R}U_\theta - I_2\frac{\partial^2 U_\theta}{\partial t^2} + \left(\frac{P_{84}}{R} - P_{63}\right)\frac{\partial W}{\partial x} \\ &+ \frac{1}{R}\left(\frac{P_{94}}{R} - P_{66}\right)\frac{\partial W}{\partial\theta} + P_{87}\frac{\partial^2\beta_x}{\partial x^2} + \frac{1}{R}(P_{8,10} + P_{97})\frac{\partial^2\beta_x}{\partial x\,\partial\theta} + \frac{P_{9,10}}{R^2}\frac{\partial^2\beta_x}{\partial\theta^2} - P_{63}\beta_x \\ &+ P_{88}\frac{\partial^2\beta_\theta}{\partial x^2} + \frac{1}{R}(P_{89} + P_{98})\frac{\partial^2\beta_\theta}{\partial x\,\partial\theta} + \frac{P_{99}}{R^2}\frac{\partial^2\beta_\theta}{\partial\theta^2} - P_{66}\beta_\theta - I_3\frac{\partial^2\beta_\theta}{\partial t^2}. \end{aligned} \tag{A.6}$$

The equations of equilibrium for a thin cylindrical shell (hybrid finite element method based on Sanders' shell theory) are defined as [147]

$$\begin{aligned} L_1(U, V, W, P_{ij}) = P_{11}\frac{\partial^2 U}{\partial x^2} &+ \frac{P_{12}}{R}\left(\frac{\partial^2 V}{\partial x\,\partial\theta} + \frac{\partial W}{\partial x}\right) - P_{14}\frac{\partial^3 W}{\partial x^3} + \frac{P_{15}}{R^2}\left(\frac{\partial^3 W}{\partial x\,\partial\theta^2} + \frac{\partial^2 V}{\partial x\,\partial\theta}\right) \\ &+ \left(\frac{P_{33}}{R} - \frac{P_{63}}{2R^2}\right)\left(\frac{\partial^2 V}{\partial x\,\partial\theta} + \frac{1}{R}\frac{\partial^2 U}{\partial\theta^2}\right) + \left(\frac{P_{36}}{R^2} - \frac{P_{66}}{2R^3}\right) \\ &\times\left(-\frac{2\partial^3 W}{\partial x\,\partial\theta^2} + \frac{3}{2}\frac{\partial^2 V}{\partial x\,\partial\theta} - \frac{1}{2R}\frac{\partial^2 U}{\partial\theta^2}\right), \end{aligned} \tag{A.7}$$

$$\begin{aligned} L_2(U, V, W, P_{ij}) = &\left(\frac{P_{21}}{R} + \frac{P_{51}}{R^2}\right)\left(\frac{\partial^2 U}{\partial x\,\partial\theta}\right) + \frac{1}{R}\left(\frac{P_{22}}{R} + \frac{P_{52}}{R^2}\right)\left(\frac{\partial^2 V}{\partial\theta^2} + \frac{\partial W}{\partial\theta}\right) \\ &- \left(\frac{P_{24}}{R} + \frac{P_{54}}{R^2}\right)\left(\frac{\partial^3 W}{\partial x\,\partial\theta}\right) + \frac{1}{R^2}\left(\frac{P_{25}}{R} + \frac{P_{55}}{R^2}\right)\left(-\frac{\partial^3 w}{\partial\theta^3} + \frac{\partial^2 V}{\partial\theta^2}\right) \\ &+ \left(P_{33} + \frac{3P_{63}}{2R}\right)\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 U}{R\partial x\,\partial\theta}\right) + \frac{1}{R}\left(P_{36} + \frac{3P_{66}}{2R}\right) \\ &\times\left(-2\frac{\partial^3 W}{\partial x^2\,\partial\theta} + \frac{3}{2}\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 U}{2R\partial x\,\partial\theta}\right), \end{aligned} \tag{A.8}$$

$$\begin{aligned} L_3(U, V, W, P_{ij}) = P_{41}\frac{\partial^2 U}{\partial x^3} &+ \frac{P_{42}}{R}\left(\frac{\partial^3 V}{\partial x^2\,\partial\theta} + \frac{\partial^2 W}{\partial x^2}\right) - P_{44}\frac{\partial^4 W}{\partial x^4} + \frac{P_{45}}{R^2}\left(-\frac{\partial^4 W}{\partial x^2\,\partial\theta^2} + \frac{\partial^3 V}{\partial x^2\,\partial\theta}\right) \\ &+ \frac{2P_{63}}{R}\left(\frac{\partial^3 U}{R\,\partial x\,\partial\theta^2} + \frac{\partial^3 V}{\partial x^2\partial\theta}\right) + \left(\frac{2P_{66}}{R^2}\right)\left(-2\frac{\partial^4 W}{\partial x^2\partial\theta^2} + \frac{3}{2}\frac{\partial^3 V}{\partial x^2\partial\theta} - \frac{\partial^3 U}{2R\,\partial x\,\partial\theta^2}\right) \\ &+ \frac{P_{51}}{R^2}\frac{\partial^3 U}{\partial x\,\partial\theta^2} + \frac{P_{52}}{R^3}\left(\frac{\partial^3 V}{\partial\theta^3}\frac{\partial^2 W}{\partial\theta^2}\right) + \frac{P_{55}}{R^4}\left(-\frac{\partial^4 W}{\partial\theta^4} + \frac{\partial^3 V}{\partial\theta^3}\right) \end{aligned} \tag{A.9}$$

$$-\frac{P_{21}}{R}\frac{\partial U}{\partial x}-\frac{P_{54}}{R^2}\frac{\partial^4 W}{\partial x^2\,\partial\theta^2}-\frac{P_{22}}{R^2}\left(\frac{\partial V}{\partial\theta}+W\right)+\frac{P_{24}}{R}\frac{\partial^2 W}{\partial\theta^2}$$
$$-\frac{P_{25}}{R^3}\left(-\frac{\partial^2 W}{\partial\theta^2}+\frac{\partial V}{\partial\theta}\right).$$

The P_{ij}'s elements are defined (only for one lamina) [147]:

$$P_{11}=C_{11},\quad P_{12}=C_{12},\quad P_{21}=P_{12},\quad P_{22}=C_{22},\quad P_{33}=C_{33},$$
$$P_{44}=D_{11},\quad P_{45}=D_{12},\quad P_{54}=P_{45},\quad P_{55}=D_{22},\quad P_{66}=D_{33},$$

where

$$\begin{gathered}C_{11}=E_x t/\Delta,\ C_{22}=E_\theta t/\Delta,\ C_{12}=\nu_x E_\theta t/\Delta,\ C_{33}=G_x\theta t,\\ D_{11}=E_x t^3/12\Delta,\ D_{22}=E_\theta t^3/12\Delta,\ D_{12}=\nu_x E_\theta t^3/12\Delta,\ D_{33}=G_x\theta\, t^3/12,\end{gathered}\tag{A.10}$$

where

$$\Delta=(1-\nu_x\nu_\theta).$$

Matrix $[H]$:

$$\begin{bmatrix}H_{11}&H_{12}&H_{13}&H_{14}&H_{15}\\H_{21}&H_{22}&H_{23}&H_{24}&H_{25}\\H_{31}&H_{32}&H_{33}&H_{34}&H_{35}\\H_{41}&H_{42}&H_{43}&H_{44}&H_{45}\\H_{51}&H_{52}&H_{53}&H_{54}&H_{55}\end{bmatrix}\begin{Bmatrix}A\\B\\C\\D\\E\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\\0\end{Bmatrix},$$

where

$$H_{11}=P_{11}(-\bar m^2)-\left[\frac{P_{5,10}+P_{10,5}}{2R^3}-\frac{P_{10,10}}{4R^4}-\frac{P_{55}}{R^2}\right]\eta^2-\left[\frac{P_{15}+P_{51}}{R}-\frac{P_{10,1}+P_{1,10}}{2R^2}\right](\bar m\eta),$$

$$H_{12}=\left(P_{12}+\frac{P_{18}}{2R}\right)(-\bar m^2)+\left[\frac{P_{14}+P_{52}}{R}+\frac{P_{58}-P_{10,2}}{2R^2}-\frac{P_{10,8}}{4R^3}\right]\bar m\eta+\left(\frac{P_{54}}{R^2}-\frac{P_{10,4}}{2R^3}\right)\eta^2,$$

$$H_{13}=\frac{P_{14}}{R}\bar m+\left(\frac{P_{54}}{R^2}-\frac{P_{10,4}}{2R^3}\right)\eta,$$

$$H_{14}=P_{17}(-\bar m^2)-\left[\frac{P_{1,10}+P_{57}}{R}-\frac{P_{10,7}}{2R^2}\right]\bar m\eta-\left(\frac{P_{10,10}}{2R^3}-\frac{P_{5,10}}{R^2}\right)\eta^2,$$

$$H_{15}=P_{18}(-\bar m^2)+\left[\frac{P_{19}+P_{58}}{R}-\frac{P_{10,8}}{2R^2}\right]\bar m\eta-\left(\frac{P_{10,9}}{2R^3}-\frac{P_{59}}{R^2}\right)\eta^2,$$

$$H_{22}=\left[P_{22}+\frac{P_{28}+P_{82}}{2R}+\frac{P_{88}}{4R^2}\right](-\bar m^2)+\left[\frac{P_{24}+P_{42}}{R}+\frac{P_{48}+P_{84}}{2R^2}\right]\bar m\eta-\frac{1}{R^2}(P_{66}-P_{44}\eta^2),$$

$$H_{23}=\left(\frac{P_{24}+P_{63}}{R}+\frac{P_{84}}{2R^2}\right)\bar m+(P_{44}+P_{66})\,\frac{\eta}{R^2},$$

$$H_{24} = \left(P_{27} + \frac{P_{87}}{2R}\right)(-\bar{m}^2) - \left[\frac{P_{2,10} + P_{47}}{R} + \frac{P_{8,10}}{2R^2}\right]\bar{m}\eta + \frac{\eta^2}{R^2}P_{4,10} + \frac{P_{63}}{R},$$

$$H_{25} = \left(P_{28} + \frac{P_{88}}{2R}\right)(-\bar{m}^2) + \left(\frac{P_{29} + P_{48}}{R} + \frac{P_{89}}{2R^2}\right)\bar{m}\eta + \frac{P_{49}}{R^2}\eta^2 + \frac{P_{66}}{R},$$

$$H_{33} = P_{33}(-\bar{m}^2) + \left(\frac{P_{36} + P_{63}}{R}\right)\bar{m}\eta + \frac{P_{66}}{R^2}\eta^2 - \frac{P_{44}}{R^2},$$

$$H_{34} = \left(\frac{P_{47}}{R} - P_{33}\right)(\bar{m}) + \left(\frac{P_{63}}{R} - \frac{P_{4,10}}{R^2}\right)\eta,$$

$$H_{35} = \left(P_{36} - \frac{P_{48}}{R}\right)(\bar{m}) + \left(P_{66} - \frac{P_{49}}{R}\right)\frac{\eta}{R},$$

$$H_{44} = P_{77}(-\bar{m}^2) - (P_{7,10} + P_{10,7})\frac{\bar{m}\eta}{R} + \frac{P_{10,10}}{R^2}\eta^2 - P_{33},$$

$$H_{45} = P_{78}(-\bar{m}^2) + (P_{79} + P_{10,8})\frac{\bar{m}\eta}{R} + P_{10,9}\frac{\eta^2}{R^2} - P_{36},$$

$$H_{55} = P_{88}(-\bar{m}^2) + (P_{98} + P_{89})\frac{\bar{m}\eta}{R} + \frac{P_{99}}{R^2}\eta^2 - P_{66},$$

where

$$\bar{m} = \frac{m\pi}{L}. \tag{A.11}$$

APPENDIX B: RECTANGULAR PLATES

The L_i equations are given below:

$$L_1(U_x, U_y, W, \beta_x, \beta_y, \overline{P_{ij}})$$

$$= P_{11}\frac{\partial^2 U_x}{\partial x^2} + (P_{15} + P_{51})\frac{\partial^2 U_x}{\partial x\,\partial y} + P_{55}\frac{\partial^2 U_x}{\partial y^2} - I_1\frac{\partial^2 U_x}{\partial t^2} + P_{12}\frac{\partial^2 U_y}{\partial x^2}$$

$$+ (P_{14} + P_{52})\frac{\partial^2 U_y}{\partial x\,\partial y} + P_{54}\frac{\partial^2 U_y}{\partial y^2} + P_{17}\frac{\partial^2 \beta_x}{\partial x^2} + (P_{1,10} + P_{57})\frac{\partial^2 \beta_x}{\partial x\,\partial y} + P_{5,10}\frac{\partial^2 \beta_x}{\partial y^2} - I_2\frac{\partial^2 \beta_x}{\partial t^2}$$

$$+ P_{18}\frac{\partial^2 \beta_y}{\partial x^2} + (P_{19} + P_{58})\frac{\partial^2 \beta_y}{\partial x\,\partial y} + P_{59}\frac{\partial^2 \beta_y}{\partial y^2}, \tag{B.1}$$

$$L_2(U_x, U_y, W, \beta_x, \beta_y, \overline{P_{ij}})$$

$$= P_{21}\frac{\partial^2 U_x}{\partial x^2} + (P_{25} + P_{41})\frac{\partial^2 U_x}{\partial x\partial y} + P_{45}\frac{\partial^2 U_x}{\partial y^2} + P_{22}\frac{\partial^2 U_y}{\partial x^2} + (P_{24} + P_{42})\frac{\partial^2 U_y}{\partial x\,\partial y} + P_{44}\frac{\partial^2 U_y}{\partial y^2}$$

$$- I_1\frac{\partial^2 U_y}{\partial t^2} + P_{27}\frac{\partial^2 \beta_x}{\partial x^2} + (P_{2,10} + P_{47})\frac{\partial^2 \beta_x}{\partial x\,\partial y} + P_{4,10}\frac{\partial^2 \beta_x}{\partial y^2} \tag{B.2}$$

$$+ P_{28}\frac{\partial^2 \beta_y}{\partial x^2} + (P_{29} + P_{48})\frac{\partial^2 \beta_y}{\partial x\,\partial y} + P_{49}\frac{\partial^2 \beta_y}{\partial y^2} - I_2\frac{\partial^2 \beta_y}{\partial t^2},$$

$$L_3(U_x, U_y, W, \beta_x, \beta_y, \overline{P_{ij}})$$

$$= P_{33}\frac{\partial^2 W}{\partial x^2} + (P_{36} + P_{63})\frac{\partial^2 W}{\partial x\,\partial y} + P_{66}\frac{\partial^2 W}{\partial y^2} - I_1\frac{\partial^2 W}{\partial t^2} + P_{33}\frac{\partial \beta_x}{\partial x}$$

$$+ P_{63}\frac{\partial \beta_x}{\partial y} + P_{36}\frac{\partial \beta_y}{\partial x} + P_{66}\frac{\partial \beta_y}{\partial y}, \tag{B.3}$$

$$L_4(U_x, U_y, W, \beta_x, \beta_y, \overline{P_{ij}})$$

$$= P_{71}\frac{\partial^2 U_x}{\partial x^2} + (P_{75} + P_{10,1})\frac{\partial^2 U_x}{\partial x\,\partial y} + P_{10,5}\frac{\partial^2 U_x}{\partial y^2} - I_2\frac{\partial^2 U_x}{\partial t^2} + P_{72}\frac{\partial^2 U_y}{\partial x^2}$$

$$+ (P_{74} + P_{10,2})\frac{\partial^2 U_y}{\partial x\,\partial y} + P_{10,4}\frac{\partial^2 U_y}{\partial y^2} - P_{33}\frac{\partial W}{\partial x} - P_{36}\frac{\partial W}{\partial y} + P_{77}\frac{\partial^2 \beta_x}{\partial x^2}$$

$$+ (P_{7,10} + P_{10,7})\frac{\partial^2 \beta_x}{\partial x\,\partial y} + P_{10,10}\frac{\partial^2 \beta_x}{\partial y^2} - P_{33}\beta_x - I_3\frac{\partial^2 \beta_x}{\partial t^2} + P_{78}\frac{\partial^2 \beta_y}{\partial x^2} + (P_{79} + P_{10,8})$$

$$\times\frac{\partial^2 \beta_y}{\partial x\,\partial y} + P_{10,9}\frac{\partial^2 \beta_y}{\partial y^2} - P_{36}\beta_y, \tag{B.4}$$

$$L_5(U_x, U_y, W, \beta_x, \beta_y, \overline{P_{ij}})$$

$$= P_{81}\frac{\partial^2 U_x}{\partial x^2} + (P_{85} + P_{91})\frac{\partial^2 U_x}{\partial x\,\partial y} + P_{95}\frac{\partial^2 U_x}{\partial y^2} + P_{82}\frac{\partial^2 U_y}{\partial x^2}$$

$$+ (P_{84} + P_{92})\frac{\partial^2 U_y}{\partial x\,\partial y} + P_{94}\frac{\partial^2 U_y}{\partial y^2} - I_2\frac{\partial^2 U_y}{\partial t^2} - P_{63}\frac{\partial W}{\partial x} - P_{66}\frac{\partial W}{\partial y} + P_{87}\frac{\partial^2 \beta_x}{\partial x^2}$$

$$+ (P_{8,10} + P_{97})\frac{\partial^2 \beta_x}{\partial x\,\partial y} + P_{9,10}\frac{\partial^2 \beta_x}{\partial y^2} - P_{63}\beta_x + P_{88}\frac{\partial^2 \beta_y}{\partial x^2} + (P_{89} + P_{98})\frac{\partial^2 \beta_y}{\partial x\,\partial y}$$

$$+ P_{99}\frac{\partial^2 \beta_y}{\partial y^2} - P_{66}\beta_y - I_3\frac{\partial^2 \beta_y}{\partial t^2}. \tag{B.5}$$

APPENDIX C: SPHERICAL SHELLS

The L_i's equations (equations of motion) are given below:

$$L_1(U_\phi, U_\theta, W, \beta_\phi, \beta_\theta, \overline{P_{ij}})$$

$$= \frac{P_{11}}{r^2}\frac{\partial^2 U_\phi}{\partial \phi^2} + \frac{(P_{15} + P_{51})}{r^2 \sin\phi}\frac{\partial^2 U_\phi}{\partial \phi\,\partial \theta} + \frac{P_{55}}{r^2 \sin^2\phi}\frac{\partial^2 U\phi}{\partial \theta^2}$$

$$+ \frac{P_{11}}{r^2}\operatorname{cotg}\phi\frac{\partial U_\phi}{\partial \phi} + \frac{P_{54}}{r^2}\frac{\cos\phi}{\sin^2\phi}\frac{\partial U_\phi}{\partial \theta} - \frac{1}{r^2}(P_{14} + P_{33} + P_{44}\operatorname{cotg}^2\phi)U_\phi - I_1\frac{\partial^2 U_\phi}{\partial t^2}$$

$$
\begin{aligned}
&+\frac{P_{12}}{r^2}\frac{\partial^2 U_\theta}{\partial\phi^2}+\frac{(P_{52}+P_{14})}{r^2\sin\phi}\frac{\partial^2 U_\theta}{\partial\phi\,\partial\theta}+\frac{P_{54}}{r^2\sin^2\phi}\frac{\partial^2 U\theta}{\partial\theta^2}\\
&+\frac{(P_{12}-P_{15}-P_{42})}{r^2}\operatorname{cotg}\phi\frac{\partial U_\theta}{\partial\phi}-\frac{(P_{55}+P_{44})}{r^2}\frac{\cos\phi}{\sin^2\phi}\frac{\partial U_\theta}{\partial\theta}\\
&+\frac{1}{r^2}((P_{15}+P_{45})\operatorname{cotg}^2\phi-P_{36})U_\theta+\frac{(P_{11}+P_{14}+P_{33})}{r^2}\frac{\partial W}{\partial\phi}+\frac{(P_{51}+P_{54}+P_{36})}{r^2\sin\phi}\frac{\partial W}{\partial\theta}\\
&+\left(\frac{P_{11}}{\sin\phi}-P_{44}\right)\frac{\operatorname{cotg}\phi}{r^2}W+\frac{P_{17}}{r^2}\frac{\partial^2\beta\phi}{\partial\phi^2}+\frac{1}{r^2\sin\phi}(P_{1,10}+P_{57})\frac{\partial^2\beta_\phi}{\partial\phi\partial\theta}\\
&+\frac{P_{5,10}}{r^2\sin^2\phi}\frac{\partial^2\beta_\phi}{\partial\theta^2}+\frac{(P_{17}+P_{19}-P_{47})}{r^2}\operatorname{cotg}\phi\frac{\partial\beta_\phi}{\partial\phi}\\
&+\frac{(P_{59}-P_{4,10})}{r^2}\frac{\cos\phi}{\sin^2\phi}\frac{\partial\beta_\phi}{\partial\theta}-\left(\frac{1}{r^2}(P_{19}+P_{49})\operatorname{cotg}^2\phi-\frac{P_{33}}{r}\right)\beta_\phi-I_2\frac{\partial^2\beta_\phi}{\partial t^2}\\
&+\frac{P_{18}}{r^2}\frac{\partial^2\beta_\theta}{\partial\phi^2}+\frac{(P_{19}+P_{58})}{r^2\sin\phi}\frac{\partial^2\beta_\theta}{\partial\phi\,\partial\theta}+\frac{P_{59}}{r^2\sin^2\phi}\frac{\partial^2\beta_\theta}{\partial\theta^2}\\
&+\frac{(P_{18}-P_{48}-P_{1,10})}{r^2}\operatorname{cotg}\phi\frac{\partial\beta_\theta}{\partial\phi}\\
&-\frac{(P_{49}+P_{5,10})}{r^2}\frac{\cos\phi}{\sin^2\phi}\frac{\partial\beta_\theta}{\partial\theta}+\frac{1}{r^2}(rP_{36}+(P_{1,10}+P_{4,10}\operatorname{cotg}^2\phi))\beta_\theta, \qquad \text{(C.1)}
\end{aligned}
$$

$$
\begin{aligned}
&L_2(U_\phi, U_\theta, W, \beta_\phi, \beta_\theta, \overline{P_{ij}})\\
&=\frac{P_{21}}{r^2}\frac{\partial^2 U_\phi}{\partial\phi^2}+\frac{(P_{41}+P_{25})}{r^2\sin\phi}\frac{\partial^2 U_\phi}{\partial\phi\,\partial\theta}+\frac{P_{45}}{r^2\sin^2\phi}\frac{\partial^2 U\phi}{\partial\theta^2}+\frac{(P_{51}+P_{24}+P_{21})}{r^2}\operatorname{cotg}\phi\frac{\partial U_\phi}{\partial\phi}\\
&+\frac{(P_{44}+P_{55})\cos\phi}{r^2\sin^2\phi}\frac{\partial U_\phi}{\partial\theta}+\frac{1}{r^2}(P_{54}\operatorname{cotg}^2\phi-P_{24}-P_{63})U_\phi\\
&+\frac{P_{22}}{r^2}\frac{\partial^2 U_\theta}{\partial\phi^2}+\frac{(P_{42}+P_{24})}{r^2\sin\phi}\frac{\partial^2 U_\theta}{\partial\phi\,\partial\theta}+\frac{P_{44}}{r^2\sin^2\phi}\frac{\partial^2 U\theta}{\partial\theta^2}+\frac{P_{22}}{r^2}\operatorname{cotg}\phi\frac{\partial U_\theta}{\partial\phi}\\
&-\frac{P_{54}\cos\phi}{r^2\sin^2\phi}\frac{\partial U_\theta}{\partial\theta}+\frac{1}{r^2}(P_{25}-P_{55}\operatorname{cotg}^2\phi-P_{66})U_\theta-I_1\frac{\partial^2 U_\theta}{\partial t^2}\\
&+\frac{(P_{21}+P_{24}+P_{63})}{r^2}\frac{\partial W}{\partial\phi}+\frac{(P_{41}+P_{44}+P_{66})}{r^2\sin\phi}\frac{\partial W}{\partial\theta}+\frac{1}{r^2}(p_{51}+P_{54}+P_{21}+P_{24})\operatorname{cotg}\phi W\\
&+\frac{P_{27}}{r^2}\frac{\partial^2\beta_\phi}{\partial\phi^2}+\frac{(P_{47}+P_{2,10})}{r^2\sin\phi}\frac{\partial^2\beta_\phi}{\partial\phi\partial\theta}+\frac{p_{4,10}}{r^2\sin^2\phi}\frac{\partial^2\beta_\phi}{\partial\theta^2}+\frac{(P_{57}+P_{29}+P_{27})}{r^2}\operatorname{cotg}\phi\frac{\partial^2\beta_\phi}{\partial\phi}\\
&+\frac{(P_{49}+P_{5,10})\cos\phi}{r^2\sin^2\phi}\frac{\partial\beta_\phi}{\partial\theta}+\left[\frac{1}{r^2}(P_{59}\operatorname{cotg}^2\phi-P_{29})+\frac{P_{63}}{r}\right]\beta_\phi
\end{aligned}
$$

$$+\frac{P_{28}}{r^2}\frac{\partial^2\beta_\theta}{\partial\phi^2}+\frac{(P_{48}+P_{29})}{r^2\sin\phi}\frac{\partial^2\beta_\theta}{\partial\phi\,\partial\theta}+\frac{p_{49}}{r^2\sin^2\phi}\frac{\partial^2\beta_\theta}{\partial\theta^2}$$

$$+\frac{(P_{58}+P_{28}-P_{2,10})}{r^2}\operatorname{cotg}\phi\frac{\partial\beta_\theta}{\partial\phi}$$

$$+\frac{(P_{59}-P_{4,10})\cos\phi}{r^2\sin^2\phi}\frac{\partial\beta_\theta}{\partial\theta}+\frac{1}{r^2}(rP_{66}+P_{2,10}-P_{5,10}\operatorname{cotg}^2\phi)\beta_\theta-I_2\frac{\partial^2U_\theta}{\partial t^2}, \qquad \text{(C.2)}$$

$$L_3(U_\phi,U_\theta,W,\beta_\phi,\beta_\theta,\overline{P_{ij}})$$

$$=-\frac{1}{r^2}(P_{33}+P_{11}+P_{41})\frac{\partial U_\phi}{\partial\phi}-\frac{1}{r^2\sin\phi}(P_{63}+P_{15}+P_{45})\frac{\partial U_\phi}{\partial\theta}$$

$$-\frac{1}{r^2}(P_{33}+P_{14}+P_{44})\operatorname{cotg}\phi\,U_\phi-\frac{1}{r^2}\ (P_{36}+P_{12}+P_{42})\frac{\partial U_\phi}{\partial\phi}$$

$$-\frac{1}{r^2\sin\phi}(P_{66}+P_{14}+P_{44})\frac{\partial U_\phi}{\partial\theta}+\frac{1}{r^2}(P_{15}+P_{45}-P_{36})\operatorname{cotg}\phi\,U_\theta$$

$$+\frac{P_{33}}{r^2}\frac{\partial^2W}{\partial\phi^2}+\frac{(P_{36}+P_{63})}{r^2\sin\phi}\frac{\partial^2W}{\partial\phi\,\partial\theta}+\frac{P_{66}}{r^2\sin^2\phi}\frac{\partial^2W}{\partial\theta^2}$$

$$+\frac{P_{33}}{r^2}\operatorname{cotg}\phi\frac{\partial W}{\partial\phi}-\frac{1}{r^2}(P_{11}+P_{14}+P_{41}+P_{44})W-I_1\frac{\partial^2W}{\partial t^2}$$

$$+\frac{1}{r}\left(P_{33}-\frac{1}{r}(P_{17}+P_{47})\right)\frac{\partial\beta_\phi}{\partial\phi}+\frac{1}{r\sin\phi}\left(P_{63}-\frac{1}{r}(P_{1,10}+P_{4,10})\right)\frac{\partial\beta_\phi}{\partial\theta}$$

$$+\frac{1}{r^2}(rP_{33}-P_{19}-P_{49})\operatorname{cotg}\phi\,\beta_\phi$$

$$+\frac{1}{r^2}(r\,P_{36}-P_{18}-P_{48})\frac{\partial\beta_\theta}{\partial\phi}$$

$$+\frac{1}{r^2\sin\phi}(rP_{66}-P_{19}-P_{49})\frac{\partial\beta_\theta}{\partial\theta}+\frac{1}{r}\left[P_{36}+\frac{1}{r}(P_{1,10}+P_{4,10})\right]\operatorname{cotg}\phi\,\beta_\theta, \qquad \text{(C.3)}$$

$$L_4(U_\phi,U_\theta,W,\beta_\phi,\beta_\theta,\overline{P_{ij}})$$

$$=\frac{P_{71}}{r^2}\frac{\partial^2U_\phi}{\partial\phi^2}+\frac{1}{r^2\sin\phi}(P_{75}+P_{10,1})\ \frac{\partial^2U_\phi}{\partial\phi\,\partial\theta}+\frac{1}{r^2\sin^2\phi}P_{10,5}\frac{\partial^2U_\phi}{\partial\theta^2}$$

$$+\frac{1}{r^2}(P_{74}+P_{71}-P_{91})\operatorname{cotg}\phi\frac{\partial U_\phi}{\partial\phi}+\frac{(P_{10,4}-P_{95})}{r^2\sin^2\phi}\cos\phi\frac{\partial U_\phi}{\partial\theta}$$

$$+\frac{1}{r}\left(P_{33}-\frac{P_{74}}{r\sin^2\phi}-\frac{P_{94}}{r}\operatorname{cotg}^2\phi\right)U_\phi-I_2\frac{\partial^2U_\phi}{\partial t^2}$$

$$
\begin{aligned}
&+\frac{P_{72}}{r^2}\frac{\partial^2 U_\theta}{\partial\phi^2}+\frac{(P_{74}+P_{10,2})}{r^2\sin\phi}\frac{\partial^2 U_\theta}{\partial\phi\,\partial\theta}+\frac{P_{10,4}}{r^2\sin^2\phi}\frac{\partial^2 U\theta}{\partial\theta^2}\\
&+\frac{(P_{72}-P_{75}-P_{92})}{r^2}\operatorname{cotg}\phi\frac{\partial U_\theta}{\partial\phi}-\frac{1}{r^2\sin^2\phi}(P_{10,5}+P_{94})\ \cos\phi\frac{\partial U_\theta}{\partial\theta}\\
&+\frac{1}{r^2}(rP_{36}+P_{75}+P_{95}\operatorname{cotg}^2\phi)\,U_\theta\\
&+\frac{(P_{71}+P_{74}-rP_{33})}{r^2}\frac{\partial W}{\partial\phi}+\frac{1}{r^2\sin\phi}(P_{10,4}+P_{10,1}-rP_{36})\frac{\partial W}{\partial\theta}\\
&+\frac{1}{r^2}(P_{71}+P_{74}-P_{91}-P_{94})\operatorname{cotg}\phi\,W\\
&+\frac{P_{77}}{r^2}\frac{\partial^2\beta_\phi}{\partial\phi^2}+\frac{1}{r^2\sin\phi}(P_{10,7}+P_{7,10})\frac{\partial^2\beta_\phi}{\partial\phi\partial\theta}+\frac{1}{r^2\sin^2\phi}P_{10,10}\frac{\partial^2\beta_\phi}{\partial\theta^2}\\
&+\frac{1}{r^2}(P_{79}+P_{77}-P_{97})\operatorname{cotg}\phi\frac{\partial\beta_\phi}{\partial\phi}+\frac{1}{r^2\sin^2\phi}(P_{10,9}-P_{9,10})\cos\phi\frac{\partial\beta_\phi}{\partial\theta}\\
&-\frac{1}{r^2}(P_{79}+r^2P_{33}+P_{99}\operatorname{cotg}^2\phi)\,\beta_\phi-I_3\frac{\partial^2\beta_\phi}{\partial t^2}\\
&+\frac{P_{78}}{r^2}\frac{\partial^2\beta_\theta}{\partial\phi^2}+\frac{1}{r^2\sin\phi}(P_{79}+P_{10,8})\frac{\partial^2\beta_\theta}{\partial\phi\,\partial\theta}\\
&+\frac{1}{r^2\sin\phi}P_{10,9}\frac{\partial^2\beta_\theta}{\partial\theta^2}+\frac{1}{r^2}[P_{78}-(P_{98}+P_{7,10})]\operatorname{cotg}\phi\frac{\partial\beta_\theta}{\partial\phi}\\
&-\frac{1}{r^2\sin^2\phi}(P_{99}+P_{10,10})\cos\phi\frac{\partial\beta_\theta}{\partial\theta}\\
&-\frac{1}{r^2}(r^2P_{36}-P_{7,10}-P_{9,10}\operatorname{cotg}^2\phi)\,\beta_\theta
\end{aligned}
\tag{C.4}
$$

$$
\begin{aligned}
&L_5(U_\phi,U_\theta,W,\beta_\phi,\beta_\theta,\overline{P_{ij}})\\
&\quad=\frac{P_{81}}{r^2}\frac{\partial^2 U_\phi}{\partial\phi^2}+\frac{1}{r^2\sin\phi}(P_{85}+P_{91})\ \frac{\partial^2 U_\phi}{\partial\phi\,\partial\theta}+\frac{1}{r^2\sin^2\phi}P_{95}\frac{\partial^2 U_\phi}{\partial\theta^2}\\
&\quad+\frac{1}{r^2}(P_{84}+P_{81}+P_{10,1})\operatorname{cotg}\phi\frac{\partial U_\phi}{\partial\phi}+\frac{1}{r^2\sin^2\phi}(P_{94}+P_{10,5})\cos\phi\frac{\partial U_\phi}{\partial\theta}\\
&\quad+\frac{1}{r^2}(rP_{63}-P_{84}+P_{10,4}\operatorname{cotg}^2\phi)U_\phi\\
&\quad+\frac{P_{82}}{r^2}\frac{\partial^2 U_\theta}{\partial\phi^2}+\frac{1}{r^2\sin\phi}(P_{84}+P_{92})\ \frac{\partial^2 U_\theta}{\partial\phi\,\partial\theta}+\frac{1}{r^2\sin^2\phi}\ P_{94}\frac{\partial^2 U_\theta}{\partial\theta^2}
\end{aligned}
$$

$$+\frac{1}{r^2}(P_{82}-P_{85}+P_{10,2})\operatorname{cotg}\phi\ \frac{\partial U_\theta}{\partial\phi}+\frac{1}{r^2\sin^2\phi}(P_{10,4}-P_{95})\ \cos\phi\frac{\partial U_\theta}{\partial\theta}$$

$$+\frac{1}{r^2}(rP_{66}+P_{85}-P_{10,5}\operatorname{cotg}^2\phi)\ U_\theta-I_2\frac{\partial^2 U_\theta}{\partial t^2}$$

$$+\frac{1}{r^2}(P_{81}+P_{84}-rP_{63})\frac{\partial W}{\partial\phi}+\frac{1}{r^2\sin\phi}(P_{91}+P_{94}-rP_{66})\frac{\partial W}{\partial\theta}$$

$$+\frac{1}{r^2}(P_{81}+P_{84}+P_{10,1}+P_{10,4})\operatorname{cotg}\phi W$$

$$+\frac{P_{87}}{r^2}\frac{\partial^2\beta_\phi}{\partial\phi^2}+\frac{1}{r^2\sin\phi}(P_{8,10}+P_{97})\frac{\partial^2\beta_\phi}{\partial\phi\,\partial\theta}+\frac{1}{r^2\sin^2\phi}P_{9,10}\frac{\partial^2\beta_\phi}{\partial\theta^2}$$

$$+\frac{1}{r^2}(P_{89}+P_{87}+P_{10,7})\operatorname{cotg}\phi\frac{\partial\beta_\phi}{\partial\phi}+\frac{1}{r^2\sin^2\phi}(P_{99}+P_{10,10})\cos\phi\frac{\partial\beta_\phi}{\partial\theta}$$

$$-\frac{1}{r^2}[P_{89}-P_{10,9}\operatorname{cotg}^2\phi+r^2P_{63}]\,\beta_\phi$$

$$+\frac{P_{88}}{r^2}\frac{\partial^2\beta_\phi}{\partial\phi^2}+\frac{1}{r^2\sin\phi}(P_{89}+P_{98})\frac{\partial^2\beta_\phi}{\partial\phi\,\partial\theta}+\frac{1}{r^2\sin^2\phi}P_{99}\frac{\partial^2\beta_\theta}{\partial\theta^2}$$

$$+\frac{1}{r^2}(P_{88}+P_{10,8}+P_{8.10})\operatorname{cotg}\phi\frac{\partial\beta_\theta}{\partial\phi}+\frac{1}{r^2\sin^2\phi}\ P_{10,9}\cos\phi\frac{\partial\beta_\theta}{\partial\theta}$$

$$+\frac{1}{r^2}(P_{8,10}-P_{10,10}\operatorname{cotg}^2\phi-r^2P_{66})\ \beta_\theta-I_3\frac{\partial^2 U_\theta}{\partial t^2}. \tag{C.5}$$

APPENDIX D: CONICAL SHELLS

The L_i's equations are given as follows:

$$L_1(U_x, U_\theta, W, \beta_x, \beta_\theta, \overline{P_{ij}})$$

$$=P_{11}\frac{\partial^2 U_x}{\partial x^2}+\frac{(P_{15}+P_{51})}{x\sin\alpha}\frac{\partial^2 U_x}{\partial x\,\partial\theta}+\frac{P_{55}}{x^2\sin^2\alpha}\frac{\partial^2 U_x}{\partial\theta^2}-\frac{P_{15}}{x^2\sin\alpha}\frac{\partial U_x}{\partial\theta}-I_1\frac{\partial^2 U_x}{\partial t^2}$$

$$+P_{12}\frac{\partial^2 U_\theta}{\partial x^2}+\frac{(P_{14}+P_{52})}{x\sin\alpha}\frac{\partial^2 U_\theta}{\partial x\,\partial\theta}\rightarrow\frac{P_{54}}{x^2\sin^2\alpha}\frac{\partial^2 U_\theta}{\partial\theta^2}-\frac{P_{14}}{x^2\sin\alpha}\frac{\partial U_\theta}{\partial\theta}$$

$$+\frac{P_{14}}{x\tan\alpha}\frac{\partial W}{\partial x}+\frac{P_{54}}{x^2\sin^2\alpha}\cos\alpha\frac{\partial W}{\partial\theta}-\frac{P_{14}}{x^2\tan\alpha}W$$

$$+P_{17}\frac{\partial^2\beta_x}{\partial x^2}+\frac{(P_{1,10}+P_{57})}{x\sin\alpha}\frac{\partial^2\beta_x}{\partial x\,\partial\theta}+\frac{P_{5,10}}{x^2\sin^2\alpha}\frac{\partial^2\beta_x}{\partial\theta^2}-\frac{P_{1,10}}{x^2\sin\alpha}\frac{\partial\beta_x}{\partial\theta}-I_2\frac{\partial^2\beta_x}{\partial t^2}$$

$$+P_{18}\frac{\partial^2\beta_\theta}{\partial x^2}+\frac{(P_{19}+P_{58})}{x\sin\alpha}\frac{\partial^2\beta_\theta}{\partial x\,\partial\theta}+\frac{P_{59}}{x^2\sin^2\alpha}\frac{\partial^2\beta_\theta}{\partial\theta^2}-\frac{P_{19}}{x^2\sin\alpha}\frac{\partial\beta_\theta}{\partial\theta}$$

$L_2(U_x, U_\theta, W, \beta_x, \beta_\theta, \overline{P_{ij}})$

$$
\begin{aligned}
&= P_{21}\frac{\partial^2 U_x}{\partial x^2} + \frac{(P_{25}+P_{41})}{x\sin\alpha}\frac{\partial^2 U_x}{\partial x\,\partial\theta} + \frac{P_{45}}{x^2\sin^2\alpha}\frac{\partial^2 U_x}{\partial\theta^2} - \frac{P_{25}}{x^2\sin\alpha}\frac{\partial U_x}{\partial\theta} \\
&\quad + P_{22}\frac{\partial^2 U_\theta}{\partial x^2} + \frac{(P_{24}+P_{42})}{x\sin\alpha}\frac{\partial^2 U_\theta}{\partial x\,\partial\theta} \rightarrow \frac{P_{44}}{x^2\sin^2\alpha}\frac{\partial^2 U_\theta}{\partial\theta^2} - \frac{P_{24}}{x^2\sin\alpha}\frac{\partial U_\theta}{\partial\theta} - \frac{P_{66}}{x^2\tan^2\alpha}U_\theta - I_1\frac{\partial^2 U_\theta}{\partial t^2} \\
&\quad + \frac{P_{24}+P_{63}}{x\tan\alpha}\frac{\partial W}{\partial x} + \frac{P_{44}+P_{66}}{x^2\sin^2\alpha}\cos\alpha\frac{\partial W}{\partial\theta} - \frac{P_{24}}{x^2\tan\alpha}W \\
&\quad + P_{27}\frac{\partial^2\beta_x}{\partial x^2} + \frac{(P_{2,10}+P_{47})}{x\sin\alpha}\frac{\partial^2\beta_x}{\partial x\,\partial\theta} + \frac{P_{4.10}}{x^2\sin^2\alpha}\frac{\partial^2\beta_x}{\partial\theta^2} \\
&\quad - \frac{P_{2,10}}{x^2\sin\alpha}\frac{\partial\beta_x}{\partial\theta} + \frac{P_{63}}{x\tan\alpha}\beta_x + P_{28}\frac{\partial^2\beta_\theta}{\partial x^2} + \frac{(P_{29}+P_{48})}{x\sin\alpha}\frac{\partial^2\beta_\theta}{\partial x\,\partial\theta} \\
&\quad + \frac{P_{49}}{x^2\sin^2\alpha}\frac{\partial^2\beta_\theta}{\partial\theta^2} - \frac{P_{29}}{x^2\sin\alpha}\frac{\partial\beta_\theta}{\partial\theta} + \frac{P_{66}}{x\tan\alpha}\beta_\theta - I_2\frac{\partial^2\beta_\theta}{\partial t^2}, \qquad \text{(D.1, 2)}
\end{aligned}
$$

$L_3(U_x, U_\theta, W, \beta_x, \beta_\theta, \overline{P_{ij}})$

$$
\begin{aligned}
&= -\frac{P_{41}}{x\tan\alpha}\frac{\partial U_x}{\partial x} - \frac{P_{45}}{x^2\sin^2\alpha}\cos\alpha\frac{\partial U_x}{\partial\theta} \\
&\quad - \frac{(P_{36}+P_{42})}{x\tan\alpha}\frac{\partial U_\theta}{\partial x} - \frac{(P_{44}+P_{66})}{x^2\sin^2\alpha}\cos\alpha\frac{\partial U_\theta}{\partial\theta} + \frac{P_{36}}{x^2\tan\alpha}U_\theta \\
&\quad + P_{33}\frac{\partial^2 W}{\partial x^2} + \frac{(P_{36}+P_{63})}{x\sin\alpha}\frac{\partial^2 W}{\partial x\,\partial\theta} + \frac{P_{66}}{x^2\sin^2\alpha}\frac{\partial^2 W}{\partial\theta^2} - \frac{P_{36}}{x^2\sin\alpha}\frac{\partial W}{\partial\theta} - \frac{P_{44}}{x^2\tan^2\alpha}W - I_1\frac{\partial^2 W}{\partial t^2} \\
&\quad + \left(P_{33} - \frac{P_{47}}{x\tan\alpha}\right)\frac{\partial\beta_x}{\partial x} + \left(\frac{P_{63}}{x\sin\alpha} - \frac{P_{4,10}\cos\alpha}{x^2\sin^2\alpha}\right)\frac{\partial\beta_x}{\partial\theta} \\
&\quad + \left(P_{36} - \frac{P_{48}}{x\tan\alpha}\right)\frac{\partial\beta_\theta}{\partial x} + \frac{(P_{66} - P_{49}\,\mathrm{cotg}\,\alpha)}{x\sin\alpha}\frac{\partial\beta_\theta}{\partial\theta}
\end{aligned}
$$

$L_4(U_x, U_\theta, W, \beta_x, \beta_\theta, \overline{P_{ij}})$

$$
\begin{aligned}
&= P_{71}\frac{\partial^2 U_x}{\partial x^2} + \frac{(P_{75}+P_{10,1})}{x\sin\alpha}\frac{\partial^2 U_x}{\partial x\,\partial\theta} + \frac{P_{10,5}}{x^2\sin^2\alpha}\frac{\partial^2 U_x}{\partial\theta^2} - \frac{P_{75}}{x^2\sin\alpha}\frac{\partial U_x}{\partial\theta} - I_2\frac{\partial^2 U_x}{\partial t^2} \\
&\quad + P_{72}\frac{\partial^2 U_\theta}{\partial x^2} + \frac{(P_{74}+P_{10,2})}{x\sin\alpha}\frac{\partial^2 U_\theta}{\partial x\,\partial\theta} + \frac{P_{10,4}}{x^2\sin^2\alpha}\frac{\partial^2 U_\theta}{\partial\theta^2} - \frac{P_{74}}{x^2\sin\alpha}\frac{\partial U_\theta}{\partial\theta} + \frac{P_{36}}{x\tan\alpha}U_\theta \\
&\quad + \left(\frac{P_{74}}{x\tan\alpha} - P_{33}\right)\frac{\partial W}{\partial x} + \frac{((P_{10,4}/x)\,\mathrm{cotg}\,\alpha - P_{36})}{x\sin\alpha}\frac{\partial W}{\partial\theta} - \frac{P_{74}}{x^2\tan\alpha}W
\end{aligned}
$$

$$+ P_{77}\frac{\partial^2\beta_x}{\partial x^2} + \frac{(P_{10,7}+P_{7,10})}{x\sin\alpha}\frac{\partial^2\beta_x}{\partial x\,\partial\theta} + \frac{P_{10,10}}{x^2\sin^2\alpha}\frac{\partial^2\beta_x}{\partial\theta^2}$$

$$-\frac{P_{7,10}}{x^2\sin\alpha}\frac{\partial\beta_x}{\partial\theta} - P_{33}\,\beta_x - I_3\frac{\partial^2\beta_x}{\partial t^2} + P_{78}\frac{\partial^2\beta_\theta}{\partial x^2}$$

$$+\frac{(P_{10,8}+P_{79})}{x\sin\alpha}\frac{\partial^2\beta_\theta}{\partial x\,\partial\theta} + \frac{P_{10,9}}{x^2\sin^2\alpha}\frac{\partial^2\beta_\theta}{\partial\theta^2} - \frac{P_{79}}{x^2\sin\alpha}\frac{\partial\beta_\theta}{\partial\theta} - P_{36}\,\beta_\theta, \tag{D.3, 4}$$

$$L_5(U_x, U_\theta, W, \beta_x, \beta_\theta, \overline{P_{ij}})$$

$$= P_{81}\frac{\partial^2U_x}{\partial x^2} + \frac{(P_{85}+P_{91})}{x\sin\alpha}\frac{\partial^2U_x}{\partial x\,\partial\theta} + \frac{P_{95}}{x^2\sin^2\alpha}\frac{\partial^2U_x}{\partial\theta^2} - \frac{P_{85}}{x^2\sin\alpha}\frac{\partial U_x}{\partial\theta}$$

$$+ P_{82}\frac{\partial^2U_\theta}{\partial x^2} + \frac{(P_{84}+P_{92})}{x\sin\alpha}\frac{\partial^2U_\theta}{\partial x\,\partial\theta} + \frac{P_{94}}{x^2\sin^2\alpha}\frac{\partial^2U_\theta}{\partial\theta^2} - \frac{P_{84}}{x^2\sin\alpha}\frac{\partial U_\theta}{\partial\theta} + \frac{P_{66}}{x\tan\alpha}U_\theta - I_2\frac{\partial^2U_\theta}{\partial t^2}$$

$$+\left(\frac{P_{84}}{x\tan\alpha} - P_{63}\right)\frac{\partial W}{\partial x} + \frac{((P_{94}/x)\operatorname{cotg}\alpha - P_{66})}{x\sin\alpha}\frac{\partial W}{\partial\theta} - \frac{P_{84}}{x^2\tan\alpha}\,W$$

$$+ P_{87}\frac{\partial^2\beta_x}{\partial x^2} + \frac{(P_{97}+P_{8,10})}{x\sin\alpha}\frac{\partial^2\beta_x}{\partial x\,\partial\theta} + \frac{P_{9,10}}{x^2\sin^2\alpha}\frac{\partial^2\beta_x}{\partial\theta^2} - \frac{P_{8,10}}{x^2\sin\alpha}\frac{\partial\beta_x}{\partial\theta} - P_{63}\,\beta_x$$

$$+ P_{88}\frac{\partial^2\beta_\theta}{\partial x^2} + \frac{(P_{98}+P_{89})}{x\sin\alpha}\frac{\partial^2\beta_\theta}{\partial x\,\partial\theta}$$

$$+\frac{P_{99}}{x^2\sin^2\alpha}\frac{\partial^2\beta_\theta}{\partial\theta^2} - \frac{P_{89}}{x^2\sin\alpha}\frac{\partial\beta_\theta}{\partial\theta} - P_{66}\beta_\theta - I_3\frac{\partial^2\beta_\theta}{\partial t^2} \tag{D.5}$$

APPENDIX E: CIRCULAR PLATES

The five differential equations of motion are defined as follows:

$$L_1(U_r, U_\theta, W, \beta_r, \beta_\theta, \overline{P_{ij}})$$

$$= P_{11}\frac{\partial^2U_r}{\partial r^2} + \frac{1}{r}(P_{51}+P_{15})\frac{\partial^2U_r}{\partial r\,\partial\theta} + \frac{P_{55}}{r^2}\frac{\partial^2U_r}{\partial\theta^2} - \frac{P_{15}}{r^2}\frac{\partial U_r}{\partial\theta} - I_1\frac{\partial^2U_r}{\partial t^2} + P_{12}\frac{\partial^2U_\theta}{\partial r^2}$$

$$+\frac{1}{r}(P_{52}+P_{14})\frac{\partial^2U_\theta}{\partial r\,\partial\theta} + \frac{P_{54}}{r^2}\frac{\partial^2U_\theta}{\partial\theta^2} - \frac{P_{14}}{r^2}\frac{\partial U_\theta}{\partial\theta}$$

$$+ P_{17}\frac{\partial^2\beta_r}{\partial r^2} + \frac{1}{r}(P_{1,10}+P_{57})\frac{\partial^2\beta_r}{\partial r\,\partial\theta} + \frac{P_{5,10}}{r^2}\frac{\partial^2\beta_r}{\partial\theta^2} - \frac{P_{1,10}}{r^2}\frac{\partial\beta_r}{\partial\theta} - I_2\frac{\partial^2\beta_r}{\partial t^2}$$

$$+ P_{18}\frac{\partial^2\beta_\theta}{\partial r^2} + \frac{1}{r}(P_{19}+P_{58})\frac{\partial^2\beta_\theta}{\partial r\,\partial\theta} + \frac{P_{59}}{r^2}\frac{\partial^2\beta_\theta}{\partial\theta^2} - \frac{P_{19}}{r^2}\frac{\partial\beta_\theta}{\partial\theta}, \tag{E.1}$$

$$L_2(U_r, U_\theta, W, \beta_r, \beta_\theta, \overline{P_{ij}})$$

$$= P_{21}\frac{\partial^2 U_r}{\partial r^2} + \frac{1}{r}(P_{41} + P_{25})\frac{\partial^2 U_r}{\partial r\,\partial\theta} + \frac{P_{45}}{r^2}\frac{\partial^2 U_r}{\partial\theta^2} - \frac{P_{25}}{r^2}\frac{\partial U_r}{\partial\theta} + P_{22}\frac{\partial^2 U_\theta}{\partial r^2}$$

$$+ \frac{1}{r}(P_{42} + P_{24})\frac{\partial^2 U_\theta}{\partial r\,\partial\theta} + \frac{P_{44}}{r^2}\frac{\partial^2 U_\theta}{\partial\theta^2} - \frac{P_{24}}{r^2}\frac{\partial U_\theta}{\partial\theta} - I_1\frac{\partial^2 U_\theta}{\partial t^2}$$

$$+ P_{27}\frac{\partial^2\beta_r}{\partial r^2} + \frac{1}{r}(P_{2,10} + P_{47})\frac{\partial^2\beta_r}{\partial r\,\partial\theta} + \frac{P_{4,10}}{r^2}\frac{\partial^2\beta_r}{\partial\theta^2} - \frac{P_{2,10}}{r^2}\frac{\partial\beta_r}{\partial\theta}$$

$$+ P_{28}\frac{\partial^2\beta_\theta}{\partial r^2} + \frac{1}{r}(P_{29} + P_{48})\frac{\partial^2\beta_\theta}{\partial r\,\partial\theta} + \frac{P_{49}}{r^2}\frac{\partial^2\beta_\theta}{\partial\theta^2} - \frac{P_{29}}{r^2}\frac{\partial\beta_\theta}{\partial\theta} - I_2\frac{\partial^2\beta_\theta}{\partial t^2}, \tag{E.2}$$

$$L_3(U_r, U_\theta, W, \beta_r, \beta_\theta, \overline{P_{ij}})$$

$$= P_{33}\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}(P_{36} + P_{63})\frac{\partial^2 W}{\partial r\,\partial\theta} + \frac{P_{66}}{r^2}\frac{\partial^2 W}{\partial\theta^2} - \frac{P_{36}}{r^2}\frac{\partial W}{\partial\theta} - I_1\frac{\partial^2 W}{\partial t^2}$$

$$+ P_{33}\frac{\partial\beta_r}{\partial r} + \frac{P_{63}}{r}\frac{\partial\beta_r}{\partial\theta} + P_{36}\frac{\partial\beta_\theta}{\partial r} + \frac{P_{66}}{r}\frac{\partial\beta_\theta}{\partial\theta}, \tag{E.3}$$

$$L_4(U_r, U_\theta, W, \beta_r, \beta_\theta, \overline{P_{ij}})$$

$$= P_{71}\frac{\partial^2 U_r}{\partial r^2} + \frac{1}{r}(P_{10,1} + P_{75})\frac{\partial^2 U_r}{\partial r\,\partial\theta} + \frac{P_{10,5}}{r^2}\frac{\partial^2 U_r}{\partial\theta^2} - \frac{P_{75}}{r^2}\frac{\partial U_r}{\partial\theta} - I_2\frac{\partial^2 U_r}{\partial t^2} + P_{72}\frac{\partial^2 U_\theta}{\partial r^2}$$

$$+ \frac{1}{r}(P_{74} + P_{10,2})\frac{\partial^2 U_\theta}{\partial r\,\partial\theta} + \frac{P_{10,4}}{r^2}\frac{\partial^2 U_\theta}{\partial\theta^2} - \frac{P_{74}}{r^2}\frac{\partial U_\theta}{\partial\theta}$$

$$- P_{33}\frac{\partial W}{\partial r} - \frac{P_{36}}{r}\frac{\partial W}{\partial\theta}$$

$$+ P_{77}\frac{\partial^2\beta_r}{\partial r^2} + \frac{1}{r}(P_{7,10} + P_{10,7})\frac{\partial^2\beta_r}{\partial r\,\partial\theta} + \frac{P_{10,10}}{r^2}\frac{\partial^2\beta_r}{\partial\theta^2} - \frac{P_{7,10}}{r^2}\frac{\partial\beta_r}{\partial\theta} - P_{33}\beta_r$$

$$- I_3\frac{\partial^2\beta_r}{\partial t^2} + P_{78}\frac{\partial^2\beta_\theta}{\partial r^2} + \frac{1}{r}(P_{79} + P_{10,8})\frac{\partial^2\beta_\theta}{\partial r\,\partial\theta} + \frac{P_{10,9}}{r^2}\frac{\partial^2\beta_\theta}{\partial\theta^2} - \frac{P_{79}}{r^2}\frac{\partial\beta_\theta}{\partial\theta} - P_{36}\beta_\theta, \tag{E.4}$$

$$L_5(U_r, U_\theta, W, \beta_r, \beta_\theta, P_{ij})$$

$$= P_{81}\frac{\partial^2 U_r}{\partial r^2} + \frac{1}{r}(P_{91} + P_{85})\frac{\partial^2 U_r}{\partial r\,\partial\theta} + \frac{P_{95}}{r^2}\frac{\partial^2 U_r}{\partial\theta^2} - \frac{P_{85}}{r^2}\frac{\partial U_r}{\partial\theta} + P_{82}\frac{\partial^2 U_\theta}{\partial r^2}$$

$$+ \frac{1}{r}(P_{84} + P_{92})\frac{\partial^2 U_\theta}{\partial r\,\partial\theta} + \frac{P_{94}}{r^2}\frac{\partial^2 U_\theta}{\partial\theta^2} - \frac{P_{84}}{r^2}\frac{\partial U_\theta}{\partial\theta} - I_2\frac{\partial^2 U_\theta}{\partial t^2}$$

$$- P_{63}\frac{\partial W}{\partial r} - \frac{P_{66}}{r}\frac{\partial W}{\partial\theta} + P_{87}\frac{\partial^2\beta_r}{\partial r^2}$$

$$+ \frac{1}{r}(P_{8,10} + P_{97})\frac{\partial^2\beta_r}{\partial r\,\partial\theta} + \frac{P_{9,10}}{r^2}\frac{\partial^2\beta_r}{\partial\theta^2} - \frac{P_{8,10}}{r^2}\frac{\partial\beta_r}{\partial\theta} - P_{63}\beta_r$$

$$+ P_{88}\frac{\partial^2\beta_\theta}{\partial r^2} + \frac{1}{r}(P_{89} + P_{98})\frac{\partial^2\beta_\theta}{\partial r\,\partial\theta} + \frac{P_{99}}{r^2}\frac{\partial^2\beta_\theta}{\partial\theta^2} - \frac{P_{89}}{r^2}\frac{\partial\beta_\theta}{\partial\theta} - P_{66}\beta_\theta - I_3\frac{\partial^2\beta_\theta}{\partial t^2}, \tag{E.5}$$

APPENDIX F: NOMENCLATURE

A_1, A_2	Lamé's parameters
A_{ij}	extensional stiffness, equation (21)
a_i, b_i $(i = 1, 2, 3)$	defined by equation (20)
B_{ij}	bending-extensional coupling stiffness, equation (21)
D_{ij}	bending stiffness, equation (21)
$E_{\alpha\beta}$	Young's moduli of elasticity, equation (10)
f_i $(i = 0, 2, 4, \dots, 10)$	coefficients of the characteristic equation (48)
$G_{\alpha\beta}$	rigidity moduli of elasticity, equation (10)
g_i $(i = 1, 2, 3)$	geometrical scale factor quantities, equation (2)
I_i	inertia moment
L_i	motion equations (34)
$M_i (i = 1, 2)$	the moment resultants applied in α_i's direction
$M_{ij} (i, j = 1, 2;\ i \neq j)$	the moment resultants applied on the middle surface in α_j's direction ($\alpha_i = cte$)
$\bar{m}$	defined by equation (47)
$N_i (i = 1, 2)$	the in-plane force resultants applied in α_i's direction
$N_{ij} (i, j = 1, 2;\ i \neq j)$	the in-plane force resultants applied on the middle surface in α_j's direction ($\alpha_i = cte$)
P_{ij}	terms of elasticity matrix ($i = 1, \dots, 10;\ j = 1, \dots, 10$)
$Q_{ij} (i, j = 1, 2, 3)$	the elastic stiffness in the material co-ordinates, equation (10)
$\bar{Q}_{ij}$ $(i, j = 1, 2, 3)$	the elastic stiffness in the global co-ordinates, equation (14)
Q_i $(i = 1, 2)$	the transverse force resultants
q_1, q_2, q_n	the external force vector
R_i $(i = 1, 2)$	curvature radius
h	thickness of the shell
h_k	thickness of the lamina, equation (21)
$h_i (i = 0, 2, 4, \dots, 8)$	coefficients of the characteristic equation (49)
u_1, u_2, w	the displacement vector components
$\ddot{u}_i$ $(i = 1, 2)$ and $\ddot{w}$	defined by equation (15)
$\mathbf{T}_{ij} (i, j = 1, 2, 3)$	transformation matrix elements, equation (13)
α_1 and α_2	curvilinear co-ordinates of the surface
β_1 and β_2	the rotations of tangents to the reference surface
$\ddot{\beta}_i$ $(i = 1, 2)$	defined by equation (15)
ε_i	deformation vector components
σ_i	normal stress vector components, equation (9)
τ_{ij}	shear stress vector components, equation (9)
ρ	density of the shell material
ζ	distance of the point from the corresponding point on the reference surface along the normal direction
η	roots of characteristic equation, equations (48, 49)
ε_1^0, ε_2^0	normal strains of the reference surface
$\gamma_{in} (i = 1, 2)$, γ_{12}	shearing strain components, equation (3)
γ_1^0, γ_2^0	in-plane shearing strains of the reference surface
κ_1, κ_2	change in the curvature of the reference surface
τ_1, τ_2	torsion of the reference surface
μ_1^0, μ_2^0	shearing strains
v_{ij}	the Poisson ratios, equation (10)