Journal of Sound and Vibration (2000) **237**(4), 683–696
doi:10.1006/jsvi.2000.3069, available online at http://www.idealibrary.com on IDEAL®

DENSITY OF ENERGY OF A PERIODICALLY UNSTUCK ELASTIC BEAM

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(*Received* 22 *March* 1999, *and in final form* 12 *April* 2000)

In the present paper, a method to model a periodically stuck constrained elastic layer applied to a beam is proposed. The periodic distribution of the interface conditions enables one to use an homogenization technique. A simplified homogenization method is presented, which relies on kinematic assumptions on the shape of the displacement field in the beam. The potential and kinetic energies are then calculated to carry out a finite element formulation of the problem. Using the homogenized beam element, the first five natural frequencies are computed. These are compared to a conventional finite element modelling. The two models are in good agreement. However, a tremendous reduction of the number of degrees of freedom (d.o.f.) needed is achieved using the homogenized finite element beam (63 d.o.f.), compared to the classical finite element model (1500 d.o.f.).

1. INTRODUCTION

The damping of vibrating structures is usually accomplished by applying damping layers on them. However, it was recently observed experimentally that non-continuous adhesive conditions could be performed better than continuous ones. The purpose of the present paper is to propose a first model and understanding of this type of coating. A simple structure was therefore selected for the present study. In this particular case, a constrained viscoelastic material is set down on a plate in a particular way: the damping layer is only glued to the base plate in specific areas. This is practically achieved by inter-spacing an ultra-thin plastic film, with periodically perforated square holes (see Figure 1). The aim of this work is to predict the dynamic behavior of such a coated plate. To simplify the analysis, only the elastic behavior of the system is considered. This will already give a rough idea of how energy can be dissipated by observing the density of strain energies. Modelling analytically the dynamic behavior of this kind of system is a difficult task, considering the type and the number of interface conditions one has to take into account. On the other

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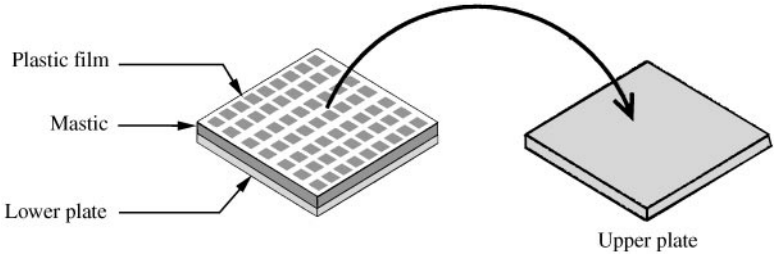

Figure 1. Non-uniformly stuck sandwich plate.

hand, using a finite element calculation will give rise to an enormous number of degrees of freedom. This is because the complicated interface conditions will require an adapted mesh. In the areas where a change of interface conditions is occurring, namely stuck or unstuck, the displacement field varies quickly. To be accurately represented, singularities in mechanics require an extremely refined finite element mesh. This leads to a tremendous number of degrees of freedom, considering that, an equivalent structure with continuous interface condition (classical constrained layer damping) will be one dimensional and geometrically simple.

Taking advantage of the periodicity of the structure, the modelling is carried out using a particular homogenization technique. Homogenization is a well-established technique in composite material science [1, 2]. In order to obtain equivalent mechanical properties, one usually employs the rule of mixture. However, other fields of material or mechanical engineering have benefited from it, like for example in fluid–structure interaction problems or thermal problems. The principle of homogenization is to replace the real structure, which is discontinuous, by an equivalent continuous structure. The discontinuity can be of any type, namely geometrical like here, or due to locally variable mechanical properties like in composite structures or fluid–structure interaction for example. This will induce discontinuous displacement fields as well. At the end of the process, the real system is replaced by an equivalent continuous system, in terms of its geometry or material properties and of its displacement fields, which is representative of the global behavior of the initial system. Discretization of the homogeneous system using a finite element method can then be accomplished at a reasonable computational cost.

Further simplifications have been introduced so as to study the effect of the periodical interface conditions on the vibrational behavior of the beam without introducing too much complexity at this stage. The studied structure is assumed to be of beam type. The discontinuities only exist in the x direction so the problem becomes completely one dimensional.

2. SIMPLIFIED HOMOGENIZATION

Even though some general homogenization procedure exists [1–3], using a simplified approach when possible is advisable. The classical procedure lies in homogenizing the governing and interface equations. The homogenized solution can then be obtained by solving the homogenized equations. An approximate solution can be provided by discretizing the previous equations, using the finite element method. Sometimes, when possible, this goal can be achieved in a more convenient way by calculating directly the homogenized expressions of the kinetic and potential energies of an elementary cell of the structure [4–6]. Performing a finite element formulation of the problem is then

straightforward. Considering an elementary cell, the different steps required by the proposed method are then: (1) assuming *a priori* displacement fields in an elementary cell; (2) calculating the potential and kinetic densities of energy in an elementary cell; (3) choosing the variables to homogenize; (4) calculating the homogenized potential and kinetic energy densities; and (5) performing the formulation of the homogenized beam element.

Obviously, the first stage is the most important and delicate one. It is clear that apart from geometrically simple cases, it will be difficult to complete it. This is the main limitation of the method.

3. HYPOTHESIS AND MODELLING

Assume that the main conditions for using the homogenization technique are fulfilled, namely (1) the system is showing some kind of periodicity, (2) the characteristic dimension of a period, which is one of the elementary cell, is small with respect to the dimension of the system.

Here, the periodicity is given by the stacking sequence of the viscoelastic material. The characteristic dimension is represented by the length ε of the related cell, which is small compared to the total length L of the beam.

An elementary cell is a composite beam made of three layers: the lower beam (1), the mastic (2) and the upper beam (3) (Figure 2). Layer 2 is itself divided into two zones: the "sliding" zone (I) and the "stuck" zone (II) (Figure 2).

The whole system is a composite beam, so that the usual beam assumptions are employed (one-dimensional state of stresses). The upper and lower beams are assumed to be made of aluminium and thin enough to undergo only tension strains (the shearing strains are neglected). On the contrary, shearing effects as well as tension effects have to be considered in layer 2, as the material is less stiffer than the aluminium. The Timoshenko assumption is employed in that case, that is the transverse shear stress is constant through the thickness of layer 2. More precisely, the modelling assumptions are the following:

1. Perfectly sliding interface condition between layers 1 and 2 in zone I.
2. Perfect sticking interface condition between layers 1 and 2 in zone II.
3. Beams 1 and 3 follow the Euler–Bernoulli assumptions.

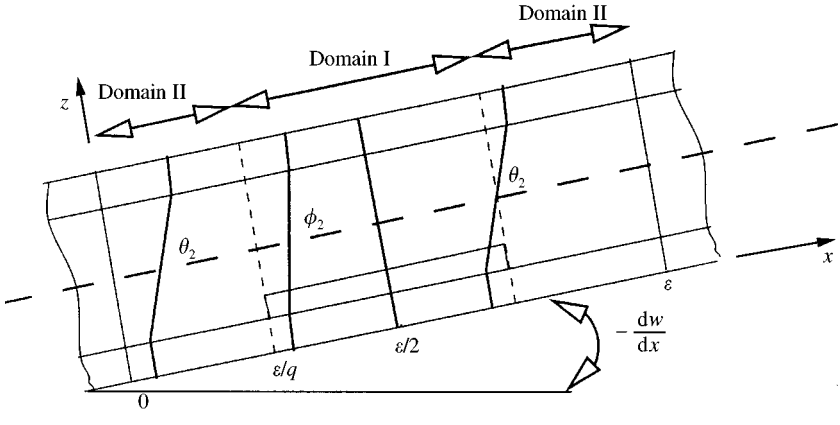

Figure 2. Elementary cell description.

4. In zone II, the system is similar to a classical constrained layer beam (see [7–9] for references on constrained layer beam theory), and the discontinuous gluing condition has no influence on the general behaviour of the system in zone I.
5. In zone I, the system is considered to behave like an association of a single beam (beam 1) and a two-layered beam: actually the selected modeling will make the true in the middle of domain I.
6. Straight sections remain straight after deformation in all layers.

The displacement field components are (1) the longitudinal displacement $v_i(x, z)$: $i = 1$ for layer 1, $i = 2\text{I}$ for layer 2 in domain I, $i = 2\text{II}$ for layer 2 in domain II and $i = 3$ for layer 3, (2) the transverse displacement $w(x)$.

Remark. Assumptions 4 and 5 are actually proposing a model for the section rotation. Hypothesis 4 assumes that the discontinuity effect is negligible in domain II. Obviously, there is a transition area where this is not realistic. In this area, the section rotation will vary from the one of a classical Euler–Bernouilli beam to the one of a three-layered beam. Depending on the length of the unstuck area, the section rotation can behave as a discontinuous or continuous function of x, when varying from one model to another. It is here proposed to approximate the section rotation by a quadratic function of x and to assess *a posteriori* the validity of the assumption.

Layers 1 *and* 3: Both longitudinal displacements $v_1(x, z)$ and $v_3(x, z)$ are of the same kind:

$$\begin{aligned} v_1(x, z) &= v_1^0(x) - z\frac{\mathrm{d}w(x)}{\mathrm{d}x}, \\ v_3(x, z) &= v_3^0(x) - \left(z - \frac{h_1}{2} - \frac{h_3}{2} - h_2\right)\frac{\mathrm{d}w(x)}{\mathrm{d}x}, \end{aligned} \tag{1}$$

the exponent "0" is referring to the value of the variable at mid layer.

Layer 2, *domain* II: As stated before, shearing deformations are dominant in the viscoelastic layer. According to Timoshenko's hypothesis, an additional variable $\theta_2(x)$ is needed:

$$v_{2\text{II}}(x, z) = v_{2\text{II}}^0(x) + \left(z - \frac{h_1}{2} - \frac{h_2}{2}\right)\theta_2(x). \tag{2}$$

Since the sticking conditions between the layers 1/2 and 2/3 are supposed to be perfect. v_2 must fulfill the following continuity equations:

$$\begin{aligned} v_{2\text{II}}\left(x, \frac{h_1}{2}\right) &= v_1\left(x, \frac{h_1}{2}\right), \\ v_{2\text{II}}\left(x, \frac{h_1}{2} + h_2\right) &= v_3\left(x, \frac{h_1}{2} + h_2\right). \end{aligned} \tag{3}$$

Those relationships allow one to express θ_2 and v_2^0 as a function of v_1^0, v_3^0 and $\mathrm{d}w/\mathrm{d}x$:

$$\begin{aligned} v_{2\text{II}}^0(x) &= \frac{v_1^0(x) + v_3^0(x)}{2} + \frac{1}{4}(h_3 - h_1)\frac{\mathrm{d}w}{\mathrm{d}x}(x), \\ \theta_2(x) &= \frac{v_3^0(x) - v_1^0(x)}{h_2} + \frac{h_3 + h_1}{2h_2}\frac{\mathrm{d}w}{\mathrm{d}x}(x). \end{aligned} \tag{4}$$

Layer 2, *domain* I: A variable $\phi_2(x)$ related to the angle of section rotation is introduced in order to take into account the shearing effects:

$$v_{2\mathrm{I}}(x, z) = v_{2\mathrm{I}}^0(x) + \left(z - \frac{h_1}{2} - \frac{h_2}{2}\right)\phi_2(x). \tag{5}$$

Assumptions 4 and 5 and continuity conditions between domains I and II are expressed by the following relationships:

$$\phi_2\left(\frac{\varepsilon}{q}\right) = \theta_2\left(\frac{\varepsilon}{q}\right), \qquad \phi_2\left(\frac{\varepsilon}{2}\right) = -\frac{\mathrm{d}w}{\mathrm{d}x}\left(\frac{\varepsilon}{2}\right), \qquad \phi_2\left(\varepsilon - \frac{\varepsilon}{q}\right) = \theta_2\left(\varepsilon - \frac{\varepsilon}{q}\right), \tag{6}$$

where "$1/q$" stands for the ratio between the length of the sliding area to the one of the elementary cell.

Further assumptions have now to be made in order to determine the function $\phi_2(x)$ for x between ε/q and $\varepsilon - \varepsilon/q$. A quadratic polynomial interpolation approximation is chosen for $\phi_2(x)$:

$$\phi_2(x) = ax^2 + bx + c, \tag{7}$$

where a, b and c are determined using the relationships (6), leading to the following expression:

$$\begin{aligned}\varphi_2(x) = {} & 2\left[\frac{q}{ee(q-2)}\right]^2\left[\theta_2\left(ee - \frac{ee}{q}\right) + \theta_2\left(\frac{ee}{q}\right) + 2\frac{\mathrm{d}w}{\mathrm{d}x}\left(\frac{ee}{2}\right)\right]x^2 \\ & + \frac{q}{ee(q-2)^2}\left[(2-3q)\theta_2\left(\frac{ee}{q}\right) - q(q+2)\theta_2\left(ee - \frac{ee}{q}\right) - 4\mathrm{q}^2\frac{\mathrm{d}w}{\mathrm{d}x}\left(\frac{ee}{2}\right)\right]x \\ & + \frac{1}{(q-2)^2}\left[q(q-1)\theta_2\left(\frac{ee}{q}\right) + 4(q-1)\frac{\mathrm{d}w}{\mathrm{d}x}\left(\frac{ee}{2}\right) + q\theta_2\left(ee - \frac{ee}{q}\right)\right].\end{aligned} \tag{8}$$

Moreover, the displacement must be continuous at the interface between the layers 2 and 3:

$$v_{2\mathrm{I}}\left(x, \frac{h_1}{2} + h_2\right) = v_3\left(x, \frac{h_1}{2} + h_2\right), \tag{9}$$

so that

$$v_{2\mathrm{I}}^0(x) = v_3^0(x) - \frac{h_2}{2}\phi_2(x) + \frac{h_3}{2}\frac{\mathrm{d}w}{\mathrm{d}x}(x). \tag{10}$$

Finally, $v_{2\mathrm{I}}(x, z)$ is given by

$$v_{2\mathrm{I}}(x, z) = v_3^0(x) + \frac{h_3}{2}\frac{\mathrm{d}w}{\mathrm{d}x}(x) + \left(z - \frac{h_1}{2} - h_2\right)\phi_2(x). \tag{11}$$

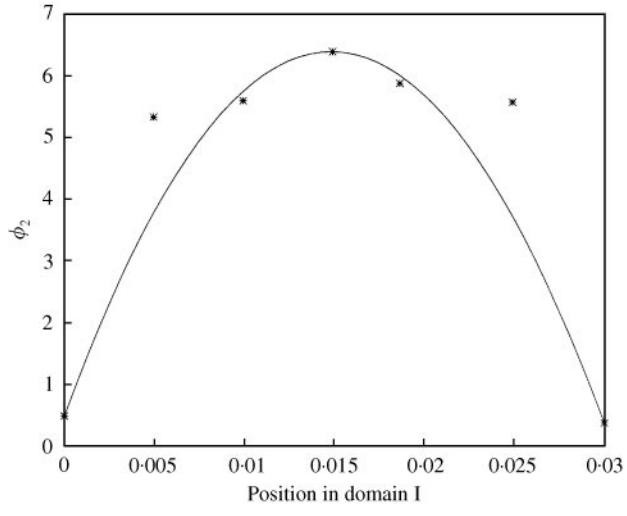

Figure 3. Variation of ϕ_2 in zone I, between ε/q (0) and $\varepsilon - \varepsilon/q$ (0·03): —— quadratic approximation; ⋆ finite element calculation.

Notice that as the following relationships are satisfied:

$$v_{2\mathrm{I}}\left(\frac{\varepsilon}{q}, z\right) = v_{2\mathrm{II}}\left(\frac{\varepsilon}{q}, z\right), \qquad v_{2\mathrm{I}}\left(\varepsilon - \frac{\varepsilon}{q}, z\right) = v_{2\mathrm{II}}\left(\varepsilon - \frac{\varepsilon}{q}, z\right), \tag{12, 13}$$

and the displacement field v_2 is continuous all through layer 2.

Remark. Choosing a quadratic interpolation to model the section rotation in domain I is completely arbitrary and intuitive at this stage. This can only be justified by *a posteriori* results. Alternatively, a finite element calculation can be performed, which shows that the two models follow the same tendencies (Figure 3).

4. ENERGY IN AN ELEMENTARY CELL

4.1. POTENTIAL ENERGY

Once the displacement field is known in the whole elementary cell, the strains and stresses fields can be obtained, enabling to calculate the elastic energy in a cell. It is depending on the functions v_i^0, w, v_2^0, θ_2 and ϕ_2, as well as on the characteristics of each layer i: h_i (thickness), ρ_i (density), E_i (Young's modulus), G_2 (mastic shear modulus) and b (beam width). Actually, among the displacement functions, the only independent displacement functions are v_1^0, v_0^3 and w (refer to equations (4) and (6)). Nevertheless, these notations are used to keep the formulae simple.

Layer 1 *or* 3: According to the Euler–Bernouilli assumptions the only non-zero quantity is the longitudinal strains ε_{xx}:

$$\varepsilon_{xx} = \frac{\mathrm{d}v_i}{\mathrm{d}x} = \frac{1}{E_i}\,\sigma_{xx}. \tag{14}$$

The potential energy is expressed by

$$E_{pt_i} = \frac{E_i \times b}{2} \int_0^{\varepsilon} \left[h_i \left(\frac{\mathrm{d}v_i^0}{\mathrm{d}x}\right)^2 + \frac{h_i^3}{12}\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \right] \mathrm{d}x \quad \text{for } i = 1 \text{ or } 3. \tag{15}$$

As stated before, the displacement field in layer 2 is different from that in the other layers (see equation (11)).

Layer 2: There are two different kinds of strains and stresses:

- tensile: $\sigma_{xx} = E_2 \varepsilon_{xx}$,
- shear: $\sigma_{xx} = 2G_2 \varepsilon_{xz}$.

The expression of the displacements fields is changing depending on the sticking condition encountered, so that

In Domain I: The strains are the following:

$$\varepsilon_{xx} = \frac{\mathrm{d}v_3^0}{\mathrm{d}x} + \frac{h_3}{2}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \left(z - \frac{h_1}{2} - h_2\right)\frac{\mathrm{d}\phi_2}{\mathrm{d}x},$$
$$\varepsilon_{xz} = \frac{1}{2}\left[\phi_2 + \frac{\mathrm{d}w}{\mathrm{d}x}\right], \tag{16}$$

and the tensile strain energy can be expressed by

$$E_{pt_{2\mathrm{I}}} = \frac{b}{2}\int_{h1/2}^{h_1/2+h_2}\int_{\varepsilon/q}^{\varepsilon-\varepsilon/q} \sigma_{xx}\varepsilon_{xx}\,\mathrm{d}x\,\mathrm{d}z$$
$$= \frac{E_2 \times b}{2}\int_{\varepsilon/q}^{\varepsilon-\varepsilon/q}\left[h_2\left(\frac{\mathrm{d}v_3^0}{\mathrm{d}x}\right)^2 + \frac{h_2 h_3^2}{4}\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 + \frac{h_2^3}{12}\left(\frac{\mathrm{d}\phi_2}{\mathrm{d}x}\right)^2\right. \tag{17}$$
$$\left. + h_3 h_2 \frac{\mathrm{d}v_3^0}{\mathrm{d}x}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - h_2^2 \frac{\mathrm{d}v_3^0}{\mathrm{d}x}\frac{\mathrm{d}\phi_2}{\mathrm{d}x} - \frac{h_2^2}{2} h_3 \frac{\mathrm{d}\phi_2}{\mathrm{d}x}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right]\mathrm{d}x$$

the shear strain energy being

$$E_{pt_{2\mathrm{I}}} = b\int_{h1/2}^{h_1/2+h_2}\int_{\varepsilon/q}^{\varepsilon-\varepsilon/q} \sigma_{xz}\varepsilon_{xz}\,\mathrm{d}x\,\mathrm{d}z$$
$$= \frac{G_2 \times b \times h_2}{2}\int_{\varepsilon/q}^{\varepsilon-\varepsilon/q}\left[\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 + \phi_2^2 + 2\phi_2\frac{\mathrm{d}w}{\mathrm{d}x}\right]\mathrm{d}x, \tag{18}$$

where ϕ_2 is given by equation (6).

In Domain II: The tensile strain energy is

$$E_{pt_{2\mathrm{II}}} = \frac{E_2 \times b}{2}\int_{[0,\varepsilon/q]\cup[\varepsilon-\varepsilon/q,\varepsilon]}\left[h_2\left(\frac{\mathrm{d}v_2^0}{\mathrm{d}x}\right)^2 + \frac{h_0^3}{12}\left(\frac{\mathrm{d}\theta_2}{\mathrm{d}x}\right)^2\right]\mathrm{d}x \tag{19}$$

and the shear energy is

$$E_{ps_{2\mathrm{II}}} = \frac{G_2 \times b \times h_2}{2}\int_{[0,\varepsilon/q]\cup[\varepsilon-\varepsilon/q,\varepsilon]}\left[\theta_2^2 + \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 + 2\theta_2\frac{\mathrm{d}w}{\mathrm{d}x}\right]\mathrm{d}x. \tag{20}$$

4.2. KINETIC ENERGY

The kinetic energy can be easily obtained from the displacement fields, and is given by

$$E_{c_i} = \frac{\rho_i \times b \times h_i}{2} \int_0^{\varepsilon} (\dot{v}_i^2 + \dot{w}^2)\,\mathrm{d}x \quad \text{for } i = 1, 2 \text{ or } 3 \tag{21}$$

5. HOMOGENIZATION PRINCIPLE

The main objective to be accomplished is homogenizing the energies calculated before. As seen before, the energies are function of $v_1(x)$, $v_{2\mathrm{I}}(x)$, $v_{2\mathrm{II}}(x)$, $v_3(x)$, $w(x)$, $\theta_2(x)$ and $\varphi_2(x)$. On the other hand, from equations (1), (4), (8) and (11), one can see that these can be expressed as a function of $v_1(x)$, $v_3(x)$ and $w(x)$. Consequently, the energies in the composite beam will only depend on $v_1(x)$, $v_3(x)$ and $w(x)$.

One has to bear in mind that the quantity x that has been used throughout the previous sections is linked to an elementary cell. It is actually describing the behavior of the system at the cell scale. If now a particular cell i is picked up, a macroscopic scale is introduced, which is that of the structure. Let us note the characteristic quantity of this scale X (see Figure 4). Homogenizing the composite structure means determining the displacements $V_1(X)$, $V_3(X)$ and $W(X)$, which are related to $v_1(x)$, $v_3(x)$ and $w(x)$ in a cell i in the following way:

$$v_1^{0i}(x) = V_1^0(x + X) = V_1^0(X) + x\frac{\mathrm{d}V_1^0}{\mathrm{d}X}(X) + \frac{x^2}{2}\frac{\mathrm{d}^2V_1^0}{\mathrm{d}X^2}(X) + \cdots, \tag{22}$$

$$v_3^{0i}(x) = V_3^0(x + X) = V_3^0(X) + x\frac{\mathrm{d}V_3^0}{\mathrm{d}X}(X) + \frac{x^2}{2}\frac{\mathrm{d}^2V_3^0}{\mathrm{d}X^2}(X) + \cdots, \tag{23}$$

$$w^i(x) = W(x + X) = W(X) + x\frac{\mathrm{d}W}{\mathrm{d}X}(X) + \frac{x^2}{2}\frac{\mathrm{d}^2W}{\mathrm{d}X^2}(X) + \cdots. \tag{24}$$

The homogenization process is then equivalent to obtaining expressions for the energies as a function of X, $V_1(X)$, $V_3(X)$ and $W(X)$, and to get rid of x. This can be done, using equations (23) and (24), by integrating the energy densities over the x-domain.

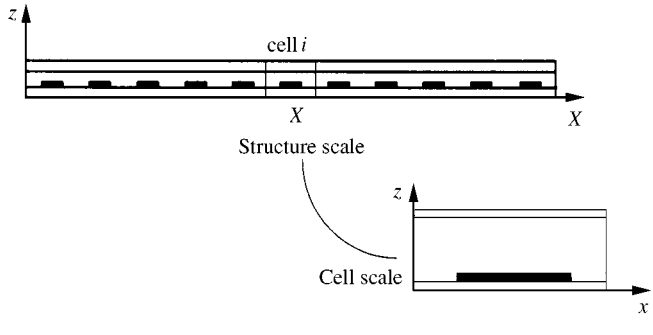

Figure 4. The double scale.

The zeroth order approximation of the densities of potential and kinetic energies are then given by

$$e_{pt_0} = \frac{1}{h_1 + h_2 + h_3}\left(\underbrace{\frac{t_{-2}}{\varepsilon^2} + \frac{t_1}{\varepsilon} + t_0}_{\substack{\text{extension/compressure} \\ \text{in layer 2}}} + \underbrace{c_0}_{\substack{\text{shearing} \\ \text{in layer 2}}} + \underbrace{p_0}_{\substack{\text{extension/compressure} \\ \text{in layer 1 and 3}}} \right) \tag{25}$$

and

$$\varepsilon_{k_0} = \frac{1}{h_1 + h_2 + h_3}(e_{k_{v0}} + e_{c_{kc0}} + e_{c_{kb0}}), \tag{26}$$

where $e_{k_{v0}} + e_{c_{kc0}} + e_{c_{kb0}}$ are the densities of kinetic energy in layers 2, 1 and 3, respectively, and are given together with t_{-2}, t_1, t_0, c_0 and p_0 in Appendix A. Using Equations (23) and (21) derivatives of $V_1(X)$, $V_3(X)$ and $W(X)$ with respect to X will appear. One has to then carefully select the Taylor expansion order (23) and (24) so that all energy quantities are properly represented: (1) insufficient expansion order will incorrectly describe the behavior of the system at the cell scale (the characteristic sticking quantity q must be present), (2) higher expansion order will lead to unnecessary complicated expressions.

Referring to reference [10] one can show that to provide a zeroth order approximation of the energies a second order expansion for $V_1(X)$ and $V_3(X)$ and a third order expansion for $W(X)$ are sufficient.

6. HOMOGENIZED FINITE ELEMENT BEAM

Once the homogenized potential and kinetic energies are determined, formulating an associated finite element is quite straight forward. Appropriate interpolation functions have to be selected now, according to the following criteria:

— the displacement fields are continuous,
— all derivatives appearing in the energies expression are correctly represented using the chosen interpolation functions,
— if n is the highest order of derivatives appearing in the energies expression, all the derivatives up to the $n-1$ order must be continuous.

Energy expressions show derivatives up to

— the second order for the longitudinal displacements $V_1(X)$ and $V_3(X)$,
— the third order for the transverse displacement $W(X)$.

The chosen interpolation functions must then ensure the continuity of $V_i(X)$, $\mathrm{d}V_i(X)/\mathrm{d}X$, $W(X)$, $\mathrm{d}W(X)/\mathrm{d}X$ and $\mathrm{d}^2W(X)/\mathrm{d}X^2$. To meet these conditions, Hermite polynomials of order 1 for the longitudinal displacements and order 2 for the transverse displacement are employed.

The resulting beam element has two nodes and seven-d.o.f. per node. $V_1(X)$, $V_3(X)$ and $W(X)$ are approximated using the following formula:

$$\begin{aligned} V_i^0(X) = &\left(-3\frac{X^2}{L^2} + 1 + 2\frac{X^3}{L^3}\right)V_{i1}^0 + \left(X - 2\frac{X^2}{L} + \frac{X^3}{L^2}\right)\frac{\mathrm{d}V_{i1}^0}{\mathrm{d}X} \\ &+ \left(3\frac{X^2}{L^2} + \frac{X^3}{L^3}\right)V_{i2}^0 + \left(-\frac{X^2}{L} + \frac{X^3}{L^2}\right)\frac{\mathrm{d}V_{i2}^0}{\mathrm{d}X} \quad \text{for } i = 1 \text{ or } 3, \end{aligned} \tag{27}$$

$$W(X) = \left(1 - 10\frac{X^3}{L^3} + 15\frac{X^4}{L^4} - 6\frac{X^4}{L^4}\right) W_1 \left(-6\frac{X^3}{L^2} + X + 8\frac{X^4}{L^3} - 3\frac{X^5}{L^4}\right)\frac{dW_1}{dX}$$
$$+ \left(\frac{X^2}{2} - \frac{3X^3}{2L} + \frac{3X^4}{2L^2} - \frac{X^5}{2L^3}\right)\frac{d^2W_1}{dX^2} + \left(10\frac{X^3}{L^3} - 15\frac{X^4}{L^4} + 6\frac{X^5}{L^5}\right) W_2 \qquad (28)$$
$$+ \left(-4\frac{X^3}{L^2} + 7\frac{X^4}{L^3} - 3\frac{X^5}{L^4}\right)\frac{dW_2}{dX} + \left(\frac{X^3}{2L} - \frac{X^4}{L^2} + \frac{X^5}{2L^3}\right)\frac{d^2W_2}{dX^2},$$

where L is the length of an element.

7. VALIDATION

The purpose of the new beam element is to provide an accurate though cost-effective approximation of the global elastic behavior of the studied composite structure. Validation of the element can be achieved using a reference model, that is believed to faithfully represent the structure. The commercial finite element code ANSYS supplies the reference model, which is constructed using the following elements: (1) BEAM3 to model the upper and lower layer, (2) PLANE42 to model the viscoelastic core.

In section 4, an *a priori* polynomial interpolation was proposed to approximate the section rotation in layer 2, domain 2 (see equation (8)). This is not the only possibility, and Table 1 presents the results obtained using the following models:

— ANSYS model the reference solution,
— linear interpolation for the section rotation in layer 2, between the values at the end of domain I and the value at the middle of it (model 1),
— quadratic interpolation: the chosen one (model 2),
— in layer 2 the continuity condition at the interface between domain I/domain II is relaxed, so that inside domain I, the section rotation is expressed by (model 3)

$$\phi_2(x) = -\frac{dw}{dx}(x). \qquad (29)$$

TABLE 1

Comparison of first five eigenfrequencies obtained through different model

	ANSYS	Model 1	Model 2	Model 3
Eigenfrequency 1	9·82 Hz —	10·41 Hz (6%)	10·30 Hz (4·89%)	8·54 Hz (13·0%)
Eigenfrequency 2	15·26 Hz —	16·16 Hz (5·93%)	15·99 Hz (4·78%)	13·34 Hz (12·58%)
Eigenfrequency 3	22·96 Hz —	24·21 Hz (5·44%)	23·98 Hz (4·44%)	20·48 Hz (10·8%)
Eigenfrequency 4	30·73 Hz —	32·2 Hz (4·78%)	31·93 Hz (3·9%)	27·89 Hz (9·24%)
Eigenfrequency 5	40·71 Hz —	42·44 Hz (4·25%)	42·13 Hz (3·49%)	37·58 Hz (7·69%)

Calculations are carried out using the following characteristics:

— cells:

$$\text{number of cells} = 20,$$
$$\text{cell length} = \varepsilon = 0{\cdot}06\ \text{m},$$

— sticking parameter:

$$q = 4,$$

— mechanical characteristics:

$$E_{1\text{ or }3} = 7{\cdot}037 \times 10^{10}\ \text{Pa},$$
$$\rho_{1\text{ or }3} = 2700\ \text{kg/m}^3,$$
$$h_{1\text{ or }3} = 0{\cdot}001\ \text{m},$$
$$E_2 = 1{\cdot}10^6\ \text{Pa},$$
$$v_2 = 0{\cdot}48,$$
$$\rho_2 = 1500\ \text{kg/m}^3,$$
$$h_2 = 0{\cdot}008\ \text{m},$$

— Number of finite elements:

$$n_{ele} = 8,$$

— boundary conditions: the beams is free–free.

The results in Table 1 show that models 1 and 2 give good results with respect to the reference solution. On the other hand, model 3 appears to be too soft: as it is less constrained than the other two models, the calculated frequencies are lower than the one for models 1 and 2. One can then draw the following conclusions:

— The influence of domain II over domain I cannot be neglected: shearing energy is an important component of the total energy stored in domain I.
— From the comparison between models 1 and 2, one can see that model 2 is the best of the two, enabling one to conclude that the influence of domain II over domain I is decreasing quickly.
— The proposed model (model 2) provides a good approximation of the global elastic behavior of the real structure.

The evolution of the discrepancy between the reference model and the proposed model are at first sight a bit surprising for anyone familiar with finite element calculations, as the percentage of error between the two models is decreasing with the frequency. This is because the homogenized element is incorporating another layer of approximation on top of the usual finite element discretization process, which is due to the homogenization process. Assuming that the convergence is achieved with respect to the finite element discretization, one should really read the results in Table 1 as being a comparison of how good the homogenized model is with respect to a full model. Indeed, further refinement of the finite element model showed the same behavior.

One should notice that 63 d.o.f. (9 nodes) were sufficient to obtain, using the homogenization techniques, results comparable with the reference solution. In return, the ANSYS model needed 1500 d.o.f. achieve the same results.

The proposed method involves quite cumbersome analytical calculations, which can be seen as the main drawback. However, the systematic use of a symbolic calculus software (Maple), enables one to handle it, making the proposed method easy and attractive to use.

8. CONCLUSION

A method to model a three-layered composite beam, with periodically stuck interface conditions, was presented. It is based on a simplified homogenization technique. The main benefit of the proposed method is the tremendous decrease in the number of d.o.f. needed to accurately represent the elastic behavior of the structure. The simplified homogenization technique, which is an energy approach, directly provides expressions for the homogenized energies in the structure, making it very convenient to use for finite element formulations.

However, one has to bear in mind that the proposed model is extremely efficient in the low-frequency range, but will fail in the high-frequency range. This is intrinsically due to the use of the homogenization technique. Modelling the behavior of the structure in the high-frequency range will require a refined meshing which is not compatible with the homogenization principle. When using homogenization, the size of the meshing should be greater or at least equal to the periodicity of the structure (length of an elementary cell).

To overcome the main difficulty involved with the homogenization technique, which is complex analytical calculus, a systematic use of a symbolic computation software was made. In this way, the proposed method happens to be not only accurate but also convenient and easy to use.

This paper was mainly focused on the modelling of the elastic aspect of the dynamic behavior of the composite beam. As the middle layer is made of viscoelastic material, a study of the damping properties of this kind of coating would be a natural sequel to this work. The technical extension of the proposed element to take into account the damping possibility of composite beam involves the use of a viscoelastic rheological law for the damping material. This aspect of the problem will be the subject of further research.

REFERENCES

1. E. SANCHEZ-PALENCIA 1980 *Non-homogeneous Media and Vibration Theory*. Berlin: Springer.
2. E. SANCHEZ-HUBERT and E. SANCHEZ-PALENCIA 1992 *Introduction aux méthodes asymptotiques et à l'homogénéisation*. Paris: Masson.
3. A. BENSOUSSAN, J. L. LIONS and G. PAPANICOLAOU 1978 *Asymptotic Analysis for Periodic Structues*. Amsterdam: North-Holland.
4. E. JACQUELIN 1994 *Ph.d. Thesis*, *Ecole Centrale de Lyon*. Comportement sous solicitations dynamiques de cœurs de réacteurs à eau préssurisée.
5. E. JACQUELIN, J. P. LAINÉ, C. TROLLAT and L. JÉZÉQUEL 1998 *Computer Methods in Applied Mechanics* **155**, 1–13. Modelling the behaviour of a PWR core by an homogenization technique.
6. E. JACQUELIN, C. REMILLAT, J. P. LAINÉ and L. JÉZÉQUEL 1998 *Comptes de l académie des sciences de Paris* Série IIb, vol. **326**, 887–892. Modélisation d'un revêtement précontraint à collage par points.
7. E. M. KERWIN JR. 1958 *Journal of the Acoustical Society of America* **31**, 952–962. Damping of flexural waves by constrained viscoelastic layer.
8. R. A. DITARANTO 1965 *Journal of Applied Mechanics* **32**. (Transactions of the ASME Series E 87,881) 881–886. Theory of vibratory bending for elastic and viscoelastic layered finite-length beams.

9. D. J. MEAD 1969 *Journal of Sound and Vibration* **10**, 163–175. The forced vibration of a three-layer, damped sandwich beam with arbitrary boundary conditions.
10. C. REMILLAT 1997 *Ph.D. Thesis, Ecole Centrale de Lyon*. Modélisation, identification des propriétés de cœur et optimisation de structures sandwiches a âme viscoélastique.

APPENDIX A

$$t_{-2} = \frac{2}{9}\frac{E_2^* h_2 q}{q-2}\Bigg[4(V_1^0)^2 - 8V_3^0 V_1^0 + V_1^0\frac{\mathrm{d}W}{\mathrm{d}X}(-4h_1 - 4h_3 - 8h_2) + 4(V_3^0)^2 + V_3^0\frac{\mathrm{d}W}{\mathrm{d}X}(4h_1 + 4h_3 + 8h_2) + \left(\frac{\mathrm{d}W}{\mathrm{d}X}\right)^2(4h_2h_3 + h_1^2 + 2h_1h_3 + 4h_1h_2 + h_3^2 + 4h_2^2)\Bigg], \tag{A.1}$$

$$t_{-1} = \frac{2}{9}\frac{E_2^* h_2 q}{q-2}\Bigg[4V_1^0\frac{\mathrm{d}V_3^0}{\mathrm{d}X} - 4V_1^0\frac{\mathrm{d}V_3^0}{\mathrm{d}X} + V_1^0\frac{\mathrm{d}W}{\mathrm{d}X^2}(-4h_2 - 2h_1 - 2h_3) - 4\frac{\mathrm{d}V_1^0}{\mathrm{d}X}V_3^0 + \frac{\mathrm{d}V_1^0}{\mathrm{d}X}\frac{\mathrm{d}W}{\mathrm{d}X}(-4h_2 - 2h_1 - 2h_3) + 4V_3^0\frac{\mathrm{d}V_3^0}{\mathrm{d}X} + V_3^0\frac{\mathrm{d}^2W}{\mathrm{d}X^2}(2h_1 + 4h_2 + 2h_3) + \frac{\mathrm{d}V_3^0}{\mathrm{d}X}\frac{\mathrm{d}W}{\mathrm{d}X}(2h_1 + 4h_2 + 2h_3) + \frac{\mathrm{d}W}{\mathrm{d}X}\frac{\mathrm{d}^2W}{\mathrm{d}X^2}(4h_2h_3 + h_1^2 + 2h_3h_1 + 4h_2h_1 + h_3^2 + 4h_2^2)\Bigg], \tag{A.2}$$

$$t_0 = \frac{E_2^* h_2}{72q(q-2)}\Bigg[V_1^0\frac{\mathrm{d}^2V_1^0}{\mathrm{d}X^2}(64 - 64q + 32q^2) + V_1^0\frac{\mathrm{d}^2V_3^0}{\mathrm{d}X^2}(32 - 8q^2 - 32q) + V_1^0\frac{\mathrm{d}^3W}{\mathrm{d}X^3}(32qh_1 - 4q^2h_3 - 16q^2h_1 + 16h_3 - 16h_2q^2 - 32h_1 - 16qh_3) + 4q(7q-6)\left(\frac{\mathrm{d}V_1^0}{\mathrm{d}X}\right)^2 - 4q(5q+6)\frac{\mathrm{d}V_1^0}{\mathrm{d}X}\frac{\mathrm{d}V_3^0}{\mathrm{d}X} - 2q\frac{\mathrm{d}V_1^0}{\mathrm{d}X}\frac{\mathrm{d}^2W}{\mathrm{d}X^2}(14qh_1 + 5qh_3 + 6h_3 + 16h_2q - 12h_1) + \frac{\mathrm{d}^2V_1^0}{\mathrm{d}X^2}V_3^0(64q - 64 - 32q^2) + \frac{\mathrm{d}^2V_1^0}{\mathrm{d}X^2}\frac{\mathrm{d}W}{\mathrm{d}X}(-32h_2q^2 - 32h_1 - 32h_3 + 32qh_1 - 64h_2 - 16q^2h_3 + 32qh_3 - 16q^2h_4 + 64h_2q) + V_3^0\frac{\mathrm{d}^2V_3^0}{\mathrm{d}X^2}(8q^2 - 32 + 32q) + V_3^0\frac{\mathrm{d}^3W}{\mathrm{d}X^3}(4q^2h_3 - 32qh_1 + 16q^2h_1 - 16h_3 + 16h_2q^2 + 32h_1 + 16qh_3)\Bigg] \tag{A.3}$$

$$
\begin{aligned}
&+4q(7q-6)\left(\frac{\mathrm{d}V_3^0}{\mathrm{d}X}\right)^2+2q\frac{\mathrm{d}V_3^0}{\mathrm{d}X}\frac{\mathrm{d}^2W}{\mathrm{d}X^2}(5qh_1+6h_1+16h_2q+12h_3+14qh_3)\\
&+\frac{\mathrm{d}^2V_3^0}{\mathrm{d}X^2}\frac{\mathrm{d}W}{\mathrm{d}X}(4q^2h_3+4q^2h_1+32h_2q-16h_3-32h_2-16h_1+8h_2q^2+16qh_1+16qh_3)\\
&+\frac{\mathrm{d}W}{\mathrm{d}X}\frac{\mathrm{d}^3W}{\mathrm{d}X^3}(12h_2q^2h_3+32h_1h_2-32h_2qh_1-8h_3^2-16qh_1^2+16h_1^2\\
&+2q^2h_3^2+8h_1h_3+16h_2qh_3-16h_2h_3+16h_2^2q^2+8qh_3^2+8q^2h_1^2+10q^2h_1h_3\\
&+24h_2q^2h_1-8qh_1h_3)\\
&+q\left(\frac{\mathrm{d}^2W}{\mathrm{d}X^2}\right)^2(6h_1h_3+7qh_3^2-6h_3^2+16h_2qh_1+7qh_1^2+16h_2qh_3-6h_1^2+5qh_1h_3+16h_2^2q)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
c_0=\frac{G_2^*}{40qh_2}\Bigg[&(V_1^0)^2(4q+32)+V_1^0V_3^0(-8q-64)\\
&+V_1^0\frac{\mathrm{d}W}{\mathrm{d}X}(-32h_3-64h_2-4qh_3-4qh_1-8h_2q-32h_1)\\
&+(V_3^0)^2(4q+32)+V_3^0(32h_3+64h_2+4qh_3+4qh_1+8h_2q+32h_1)\\
&+\left(\frac{\mathrm{d}W}{\mathrm{d}X}\right)^2(8h_1^2+4h_2^2q+16h_1h_3+qh_3^2+4h_2qh_3+32h_2h_3+qh_1^2\\
&+8h_3^2+32h_2^2+4h_2qh_1+32h_1h_2+2qh_1h_3)\Bigg]
\end{aligned}
\tag{A.4}
$$

$$
\rho_0=\frac{E_1}{24}\left[12\left(\frac{\mathrm{d}V_1^0}{\mathrm{d}X}\right)^2+h_1^3\left(\frac{\mathrm{d}^2W}{\mathrm{d}X^2}\right)^2\right]+\frac{E_3}{24}\left[12\left(\frac{\mathrm{d}V_3^0}{\mathrm{d}X}\right)^2+h_3^3\left(\frac{\mathrm{d}^2W}{\mathrm{d}X^2}\right)^2\right]
\tag{A.5}
$$

$$
\begin{aligned}
e_{c_{v0}}=\frac{\rho_2h_2}{360q}\Bigg[&(V_1^0)^2[12q+96)+2V_1^0V_3^0(18q+24)\\
&+2V_1^0\frac{\mathrm{d}W}{\mathrm{d}X}(9qh_3+8h_2q-16h_2-6qh_1-48h_1+12h_3)+(V_3^0)^2(132q-144)\\
&+2V_3^0\frac{\mathrm{d}W}{\mathrm{d}X}(-9qh_1-72h_3+52h_2q-104h_2+66qh_3-12h_1)+180W^2q\\
&+\left(\frac{\mathrm{d}W}{\mathrm{d}X}\right)^2(32h_2^2q+3qh_1^2-36h_3^2+52h_2qh_3+24h_1^2-8h_2qh_1\\
&+33qh_3^2-9qh_1h_3-104h_2h_3-64h_2^2-12h_1h_3+16h_1h_2)],
\end{aligned}
\tag{A.6}
$$

$$
e_{c_{pc0}}=\frac{\rho_3h_3}{24}\left[12(V_3^0)^2+12W^2+\left(\frac{\mathrm{d}W}{\mathrm{d}X}\right)^2h_3^2\right],
\tag{A.7}
$$

$$
\varepsilon_{c_{pb0}}=\frac{\rho_1h_1}{24}\left[12(V_1^0)^2+12W^2+h_1^2\left(\frac{\mathrm{d}W}{\mathrm{d}X}\right)^2\right].
\tag{A.8}
$$