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FREE VIBRATIONS OF BEAMS WITH GENERAL BOUNDARY CONDITIONS

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A simple and unified approach is presented for the vibration analysis of a generally supported beam. The flexural displacement of the beam is sought as the linear combination of a Fourier series and an auxiliary polynomial function. The polynomial function is introduced to take all the relevant discontinuities with the original displacement and its derivatives at the boundaries and the Fourier series now simply represents a residual or conditioned displacement that has at least three continuous derivatives. As a result, not only is it always possible to expand the displacement in a Fourier series for beams with any boundary conditions, but also the solution converges at a much faster speed. The reliability and robustness of the proposed technique are demonstrated through numerical examples.

1. INTRODUCTION

A wide spectrum of techniques has been developed for the vibrations of beams with various boundary conditions. Among them, the modal superposition technique is probably the most popular one in which the beam displacement is expressed as a linear combination of eigenfunctions or mode shapes. Although eigenfunctions generally exist in the forms of trigonometric and hyperbolic functions, they also include some integration and frequency constants that have to be determined from boundary conditions. Consequently, each boundary condition essentially calls for a particular set of natural frequencies and mode shapes. However, consider even the simplest end conditions (i.e., pinned, clamped, free and sliding), altogether they can make up 10 different boundary conditions for a beam and 55 different boundary conditions for a (rectangular) plate. Therefore, the use of mode shapes as the basis functions could become a very tedious procedure in reality. Although the frequency equation generally exists for beams with arbitrary boundary conditions [1], most investigations in the literature have been primarily focused on some degenerate cases in which the rotational and/or translational springs are arranged in certain special ways [2–8].

The beam displacement can also be sought in terms of, say, polynomials [9, 10], Fourier series [11] or other functions [12, 13]. Fourier series have been widely used to determine the vibrations of simply supported beams. In such cases, all the required derivatives of the displacement function can be directly calculated from its Fourier series through term-by-term differentiation. For other boundary conditions, however, a Fourier series tends to become slow converged, if it converges at all, and its derivatives may not be so easily obtained. Consequently, the Fourier series technique is basically confined to the simply supported boundary conditions.

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Chung [14] used a Fourier sine series to represent the axial displacement in estimating the natural frequencies of circular cylindrical shells under general boundary conditions which actually only refer to the various combinations of the simplest homogeneous end conditions. This approach was also used by Lin and Wang [11] to study the vibrations of generally supported beams. However, as pointed out later in this study, such a technique is essentially feasible only for the pinned–pinned beams with rotational restraints at ends. Other works based on Fourier expansions can be found in references [15–17] which have been well reviewed by Maurizi and Robledo [18].

The objective of this study is to develop a reliable and unified technique for the vibration analysis of generally supported beams. A brief review is first given of the traditional Fourier series technique and its potential problems in applications. A new technique is then derived in which the beam displacement is sought as the linear superposition of a Fourier cosine series and an auxiliary polynomial. The roles of the polynomial are described with sufficient details. Finally, numerical examples are presented to show the excellent accuracy and remarkable convergence of the current solution.

2. VIBRATION ANALYSIS OF A BEAM USING FOURIER SERIES

2.1. BASIC EQUATIONS

The governing differential equation for the free vibration of a beam is

$$D\mathrm{d}^4 w(x)/\mathrm{d}x^4 - \rho A\omega^2 w(x) = 0 \tag{1}$$

or

$$w''''(x) - \rho_D \omega^2 w(x) = 0, \tag{2}$$

where D, ρ and A are, respectively, the flexural rigidity, the mass density and the cross-sectional area of the beam, ω is frequency in radian, and $\rho_D = \rho A/D$ (a list of symbols is given in Appendix A).

The boundary conditions for a generally supported beam can be expressed as

$$k_0 w = -Dw''', \quad K_0 w' = Dw'' \quad \text{at } x = 0 \tag{3, 4}$$

and

$$k_1 w = -Dw''', \quad K_1 w' = Dw'' \quad \text{at } x = L \tag{5, 6}$$

where k_0 and k_1 are the linear spring constants, and K_0 and K_1 are the rotational spring constants at $x = 0$ and L, respectively.

Many familiar boundary conditions may be considered as the special cases of equations (3–6). For example, the simply supported or pinned–pinned boundary condition can be easily obtained by assuming that at each end the translational and rotational spring constants are extremely large and small, respectively.

The generic solution of equation (1) has been well known as a combination of the trigonometric and hyperbolic functions. However, the integration and frequency parameters in the eigenfunctions need to be determined from the boundary conditions which typically involves solving a transcendental equation, a normally tedious process especially for the elastically restrained beams.

2.2. TRADITIONAL FOURIER SERIES SOLUTIONS

The solution of equation (1) may also be expanded in a Fourier series. If certain continuity conditions are satisfied by the displacement function, some of its derivatives can be simply obtained through term-by-term differentiation. To better understand this, let us start with a couple of mathematical theorems that are related to the differentiation of Fourier series [19].

Theorem 1. *Let $f(x)$ be a continuous function defined on $[0, L]$ with an absolutely integrable derivative, and let $f(x)$ be expanded in Fourier sine series*

$$f(x) = \sum_{m=1}^{\infty} b_m \sin \lambda_m x, \quad 0 < x < L \quad (\lambda_m = m\pi/L) \tag{7}$$

then

$$f'(x) = \frac{f(L) - f(0)}{L} + \sum_{m=1}^{\infty} \left(\frac{2}{L} [(-1)^m f(L) - f(0)] + \lambda_m b_m \right) \cos \lambda_m x. \tag{8}$$

Theorem 2. *Let $f(x)$ be a continuous function defined on $[0, L]$ with an absolutely integrable derivative, and let $f(x)$ be expanded in Fourier cosine series*

$$f(x) = a_0 + \sum_{m=1}^{\infty} a_m \cos \lambda_m x, \quad 0 < x < L, \tag{9}$$

then

$$f'(x) = -\sum_{m=1}^{\infty} \lambda_m a_m \sin \lambda_m x. \tag{10}$$

These two theorems basically tell that while a cosine series can always be differentiated term-by-term, this can be done to a sine series only if $f(0) = f(L) = 0$.

With these in mind, now consider a pinned–pinned beam with rotational end restraints, as shown in Figure 1. This problem was previously studied by Wang and Lin [11]. For the sake of completeness, the related results will be briefly reviewed here. If the displacement function is expanded in a Fourier sine series

$$w = \sum_{m=1}^{\infty} A_m \sin \lambda_m x, \quad 0 \leqslant x \leqslant L, \tag{11}$$

then, according to the above theorems, one will have

$$w' = \sum_{m=1}^{\infty} \lambda_m A_m \cos \lambda_m x, \quad 0 \leqslant x \leqslant L, \tag{12}$$

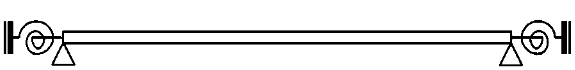

Figure 1. A pinned–pinned beam with rotational springs at both ends.

$$w'' = -\sum_{m=1}^{\infty} \lambda_m^2 A_m \sin \lambda_m x, \quad 0 < x < L, \tag{13}$$

$$w''' = \frac{B_1 - B_0}{L} + \sum_{m=1}^{\infty} \left(\frac{2}{L} (B_1(-1)^m - B_0) - \lambda_m^3 A_m \right) \cos \lambda_m x, \quad 0 \leqslant x \leqslant L, \tag{14}$$

and

$$w'''' = -\sum_{m=1}^{\infty} \left(\frac{2}{L} (B_1(-1)^m - B_0)\lambda_m - \lambda_m^4 A_m \right) \sin \lambda_m x, \quad 0 < x < L, \tag{15}$$

where

$$B_0 = w''(0) \quad \text{and} \quad B_1 = w''(L). \tag{16, 17}$$

It should be noted that the two end points, $x = 0$ and L, have been excluded in equations (13) and (15) because the sine series may not actually converge to the true displacement values there. This can be explained by referring to Figure 2, where a simple linear function $f(x) = cx + d$ $(c, d > 0)$ is defined on $[0, L]$. The extension of $f(x)$ onto $[-L, 0]$ which is even for the cosine series leads to a function that is continuous on $[-L, L]$ and has an identical value at $x = \pm L$. Thus, one obtains a continuous function of period $2L$ whose Fourier series will converge everywhere. However, this is not necessarily the case for a sine series which represents the odd extension of $f(x)$ onto $[-L, 0]$, as shown by the bottom line. Obviously, the new function is piecewise smooth and the corresponding Fourier series only converges to zero at $x = 0$ and $\pm L$ regardless of the actual values of $f(0)$ and $f(L)$. This argument also applies to the beam displacement and its derivatives. Since the displacement of the pinned–pinned beam is identically zero at each end, the sine series, equation (11), will converge over the entire x-axis including $x = 0$ and L.

Combining equations (15) and (2) results in

$$\sum_{m=1}^{\infty} \left\{ -\lambda_m \left(\frac{2}{L} (B_1(-1)^m - B_0) - \lambda_m^3 A_m \right) - \rho_D \omega^2 A_m \right\} \sin \lambda_m x = 0 \tag{18}$$

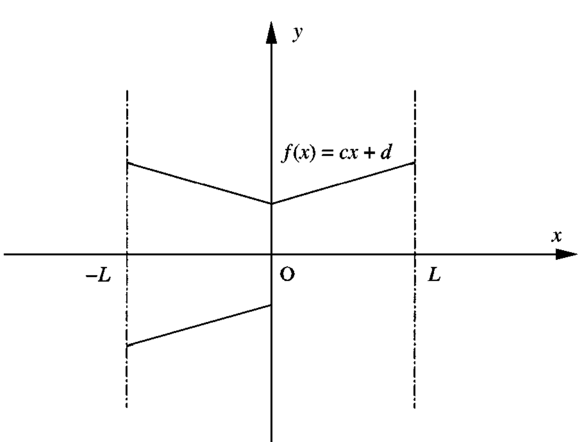

Figure 2. An (odd and even) extension of a function $f(x)$ onto $[-L, 0]$.

or

$$A_m = \frac{\lambda_m}{\lambda_m^4 - \rho_D \omega^2} \left(\frac{2}{L} (B_1(-1)^m - B_0) \right). \tag{19}$$

Substitution of equations (12), (16), (17) and (19) into equations (4) and (6) will lead to two homogeneous equations about B_0 and B_1 in which the coefficients are the functions of frequency. Mathematically, the natural frequencies are simply obtained by requiring the determinant of the coefficient matrix to vanish [11]. Such a procedure involves solving a non-linear equation, which may not always be an easy job numerically.

Alternatively, a much simpler procedure can be used as described below. In light of equation (12), the boundary values of the second derivative, B_0 and B_1, can be determined from equations (4) and (6):

$$B_0 = \hat{K}_0 w'(0) = \hat{K}_0 \sum_{m=1}^{\infty} \lambda_m A_m \quad (\hat{K}_0 = K_0/D), \tag{20}$$

and

$$B_1 = -\hat{K}_1 w'(L) = \hat{K}_1 \sum_{m=1}^{\infty} (-1)^{m+1} \lambda_m A_m \quad (\hat{K}_1 = K_1/D). \tag{21}$$

Substituting equations (20) and (21) into equation (18) gives

$$\sum_{m=1}^{\infty} \left\{ \lambda_m^4 A_m + \frac{2\lambda_m}{L} \left(\hat{K}_1 \sum_{m'=1}^{\infty} (-1)^{m'+m} \lambda_{m'} A_{m'} + \hat{K}_0 \sum_{m'=1}^{\infty} \lambda_{m'} A_{m'} \right) - \rho_D \omega^2 A_m \right\} \sin \lambda_m x = 0. \tag{22}$$

It should be realized that equation (22) is a standard characteristic equation for a matrix from which the eigenvalues and eigenfunctions can be determined without any difficulty. Equation (22) can also be derived from the modal superposition technique if the rotational springs are viewed as structural features added to the simply supported beam.

2.2. A NEW SOLUTION IN TERMS OF THE FOURIER SERIES

Before proceeding to discuss the problems associated with the traditional Fourier series solution, let us introduce another important mathematical theorem [19]:

Theorem 3. *Let $f(x)$ be a continuous function of period $2L$, which has m derivatives, where m-1 derivatives are continuous and the mth derivative is absolutely integrable (the mth derivative may not exist at certain points). Then, the Fourier series of all m derivatives can be obtained by term-by-term differentiation of the Fourier series of $f(x)$, where all the series, except possibly the last, converge to the corresponding derivatives. Moreover, the Fourier coefficients of the function $f(x)$ satisfy the relations*

$$\lim_{n \to \infty} a_n \lambda_n^m = \lim_{n \to \infty} b_n \lambda_n^m = 0. \tag{23}$$

As an example, this theorem will be used to examine the convergence of the previous Fourier sine solution for the beam illustrated in Figure 1. Since the Fourier series, equation (11), represents a continuous function of period $2L$ which has a continuous derivative and piecewise continuous second derivative, one knows from Theorem 3 that the Fourier coefficients satisfies

$$\lim_{n\to\infty} A_n\lambda_n^2 = 0. \tag{24}$$

The condition that the second derivative is absolutely integrable is obviously not a problem here because it has been widely used in Rayleigh–Ritz or other energy methods.

For the cases when the beam is allowed to move at any end due to, say, a translational elastic support, there is no guarantee that the displacement function remains continuous at the end point(s). Hence, the convergence of the Fourier series, if it converges at all, can be expected to be much slower. This problem is further compounded by the fact that the displacement values at the boundaries have to be determined in terms of the third derivative of the Fourier series by making use of the boundary conditions, equations (3) and (5).

In order to overcome this difficulty, an auxiliary polynomial function will be introduced here, that is,

$$w = \bar{w} + p, \tag{25}$$

where the polynomial p is chosen to take all the relevant discontinuities with the original beam displacement and its three derivatives at the end points so that the "residual" displacement $\bar{w}$ is a continuous function and has at least three continuous derivatives. If this new displacement function $\bar{w}$ is expanded, say, in a Fourier cosine series

$$\bar{w} = \sum_{m=0}^{\infty} A_m \cos\lambda_m x, \quad 0 \leqslant x \leqslant L, \tag{26}$$

then all the required differentiations can be simply carried out on term-by-term basis. By setting that

$$p'''(0) = w'''(0) = \alpha_0, \quad p'''(L) = w'''\alpha_1, \tag{27, 28}$$

$$p'(0) = w'(0) = \beta_0, \quad p'(L) = w'(L) = \beta_1, \tag{29, 30}$$

the lowest order polynomial that satisfies equations (27)–(30) can be written as

$$p = p_1 + p_2, \tag{31}$$

where

$$p_1''' = \alpha_0(1 - x/L) + \alpha_1 x/L \tag{32}$$

and

$$p_2' = \beta_0(1 - x/L) + \beta_1 x/L. \tag{33}$$

Integrating equation (32) three times, and choosing the integration constants in such a way that

$$p_1'(0) = 0, \quad p_1'(L) = 0 \tag{34, 35}$$

and

$$\int_0^L p_1 \, \mathrm{d}x = 0, \tag{36}$$

one obtains

$$p_1 = \frac{\alpha_1}{24L}(x^4 - 2L^2x^2) - \frac{\alpha_0}{24L}(4L^2x^2 - 4Lx^3 + x^4) + \frac{L^3}{360}(8\alpha_0 + 7\alpha_1). \tag{37}$$

Similarly, the second part of the polynomial can be expressed as

$$p_2 = \frac{\beta_1}{6L}(3x^2 - L^2) + \frac{\beta_0}{6L}(6Lx - 2L^2 - 3x^2) \tag{38}$$

with

$$\int_0^L p_2 \, \mathrm{d}x = 0. \tag{39}$$

Substitution of equations (25), (26) and (31) into equation (2) leads to

$$\frac{\alpha_1 - \alpha_0}{L} + \sum_{m=1}^{\infty} \lambda_m^4 A_m \cos \lambda_m x - \rho_D \omega^2 \left(\sum_{m=0}^{\infty} A_m \cos \lambda_m x + p_1(x) + p_2(x) \right) = 0. \tag{40}$$

Multiplying equation (40) with $2/L \cos \lambda_m x$ ($m = 0, 1, 2, \ldots$) and integrating it from 0 to L results in

$$\lambda_m^4 A_m - \rho_D \omega^2 (A_m + P^m) = 0, \quad m = 1, 2, \ldots \tag{41}$$

and

$$\frac{\alpha_1 - \alpha_0}{L} - \rho_D \omega^2 A_0 = 0, \tag{42}$$

where

$$P_m = \frac{2}{L} \int_0^L p \cos \lambda_m x \, \mathrm{d}x = \frac{2}{L} \left(- \frac{\alpha_1 (-1)^m - \alpha_0}{\lambda_m^4} + \frac{\beta_1 (-1)^m - \beta_0}{\lambda_m^2} \right). \tag{43}$$

In equation (43) the values of the first and third derivatives at the boundaries need to be determined from equations (3)–(6). In light of equations (25)–(31), (37) and (38), it is not difficult to obtain that

$$\hat{k}_0 \left(\sum_{m=0}^{\infty} A_m + \frac{8L^3\alpha_0}{360} + \frac{7L^3\alpha_1}{360} - \frac{\beta_0 L}{3} - \frac{\beta_1 L}{6} \right) = -\alpha_0, \tag{44}$$

$$\hat{k}_1 \left(\sum_{m=0}^{\infty} (-1)^m A_m - \frac{7L^3\alpha_0}{360} - \frac{8L^3\alpha_1}{360} + \frac{\beta_0 L}{6} + \frac{\beta_1 L}{3} \right) = \alpha_1, \tag{45}$$

$$\hat{K}_0 \beta_0 = \left(- \sum_{m=1}^{\infty} \lambda_m^2 A_m - \frac{\alpha_0 L}{3} - \frac{\alpha_1 L}{6} + \frac{\beta_1}{L} - \frac{\beta_0}{L} \right) \tag{46}$$

and

$$\hat{K}_1\beta_1 = -\left(\sum_{m=1}^{\infty}(-1)^{m+1}\lambda_m^2 A_m + \frac{\alpha_0 L}{6} + \frac{\alpha_1 L}{3} + \frac{\beta_1}{L} - \frac{\beta_0}{L}\right). \tag{47}$$

Equations (44)–(47) can be rewritten in more concise form as

$$\mathbf{H}\bar{\boldsymbol{\alpha}} = \sum_{m=0}^{\infty} \mathbf{Q}_m A_m, \tag{48}$$

where

$$\bar{\boldsymbol{\alpha}} = \{\alpha_0, \alpha_1, \beta_0, \beta_1\}^{\mathrm{T}}, \tag{49}$$

$$\mathbf{H} = \begin{bmatrix} \dfrac{8\hat{k}_0 L^3}{360} + 1 & \dfrac{7\hat{k}_0 L^3}{360} & \dfrac{-\hat{k}_0 L^3}{3} & \dfrac{-\hat{k}_0 L}{6} \\ \dfrac{7\hat{k}_1 L^3}{360} & \dfrac{8\hat{k}_1 L^3}{360} + 1 & \dfrac{-\hat{k}_1 L}{6} & \dfrac{-\hat{k}_1 L}{3} \\ \dfrac{L}{3} & \dfrac{L}{6} & \hat{K}_0 + \dfrac{1}{L} & \dfrac{-1}{L} \\ \dfrac{L}{6} & \dfrac{L}{3} & \dfrac{-1}{L} & \hat{K}_1 + \dfrac{1}{L} \end{bmatrix} \tag{50}$$

and

$$\mathbf{Q}_m = \{-\hat{k}_0 \,(-1)^m \hat{k}_1 \, -\lambda_m^2 \,(-1)^m \lambda_m^2\}^{\mathrm{T}}. \tag{51}$$

Combining equations (41), (42) and (48) gives

$$\lambda_m^4 A_m - \rho_D \omega^2 \left(A_m + \sum_{m'=0}^{\infty} S_{mm'} A_{m'}\right) = 0, \quad m = 1, 2, 3, \ldots \tag{52}$$

and

$$\sum_{m'=0}^{\infty} \mathbf{c}^{\mathrm{T}} \mathbf{H}^{-1} \mathbf{Q}_{m'} A_{m'} - \rho_D \omega^2 A_0 = 0, \tag{53}$$

where

$$\mathbf{c} = \{-1/L \; 1/L \; 0 \; 0\}^{\mathrm{T}}, \quad S_{mm'} = \mathbf{P}_m^{\mathrm{T}} \mathbf{H}^{-1} \mathbf{Q}_{m'} \tag{54, 55}$$

and

$$\mathbf{P}_m = \frac{2}{L}\left\{\frac{1}{\lambda_m^4} \, \frac{(-1)^{m+1}}{\lambda_m^4} \, \frac{-1}{\lambda_m^2} \, \frac{(-1)^m}{\lambda_m^2}\right\}^{\mathrm{T}}. \tag{56}$$

If the Fourier series is truncated to $m = M$, equations (52) and (53) can be rewritten as

$$(\mathbf{K} - \rho_D \omega^2 \mathbf{M})\mathbf{A} = 0, \tag{57}$$

where

$$\mathbf{A} = \{A_0, A_1, \ldots, A_M\}^{\mathrm{T}}, \tag{58}$$

$$\mathbf{K} = \begin{bmatrix} \mathbf{c}^{\mathrm{T}}\mathbf{H}^{-1}\mathbf{Q}_0 & \mathbf{c}^{\mathrm{T}}\mathbf{H}^{-1}\mathbf{Q}_1 & \cdots & \mathbf{c}^{\mathrm{T}}\mathbf{H}^{-1}\mathbf{Q}_{m'} & \cdots & \mathbf{c}^{\mathrm{T}}\mathbf{H}^{-1}\mathbf{Q}_M \\ 0 & \lambda_1^4 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_m^4 \delta_{mm'} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \lambda_M^4 \end{bmatrix} \tag{59}$$

and

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ S_{10} & 1 + S_{11} & \cdots & S_{1m'} & \cdots & S_{1M} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ S_{m0} & S_{m1} & \cdots & \delta_{mm'} + S_{mm'} & \cdots & S_{mM} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ S_{M0} & S_{M1} & \cdots & S_{Mm'} & \cdots & 1 + S_{MM} \end{bmatrix}. \tag{60}$$

Comparing equation (57) with equation (22), one will notice that the current procedure has led to a different explanation of the physical impact of the boundary springs: instead of affecting the stiffness matrix, they change the effective mass of the beam in the current formulation.

The natural frequencies and eigenvectors can now be easily determined by solving a standard matrix eigenproblem. The eigenvectors are actually the expansion coefficients of the Fourier series from which, however, the mode shapes are readily obtained:

$$w = \sum_{m=0}^{M} (\cos \lambda_m x + \mathbf{X}^{\mathrm{T}} \bar{\mathbf{S}}_m) A_m, \tag{61}$$

where

$$\mathbf{X} = \{1 \;\; x \;\; x^2 \;\; x^3 \;\; x^4\}^{\mathrm{T}}, \quad \bar{\mathbf{S}}_m = \mathbf{L}\mathbf{H}^{-1}\mathbf{Q}_m \tag{62, 63}$$

and

$$\mathbf{L} = \begin{bmatrix} 8L^3/360 & 7L^3/360 & -L/3 & -L/6 \\ 0 & 0 & 1 & 0 \\ -L/6 & -L/12 & -1/2L & 1/2L \\ 1/6 & 0 & 0 & 0 \\ -1/24L & 1/24L & 0 & 0 \end{bmatrix}. \tag{64}$$

Finally, making use of the equation

$$\int_0^L w^2 \,\mathrm{d}x = L, \tag{65}$$

the normalized mode shape can be written as

$$\phi = \sum_{m=0}^{M} (\cos \lambda_m x + \mathbf{X}^{\mathrm{T}} \bar{\mathbf{S}}_m) \bar{A}_m, \tag{66}$$

where

$$\bar{A}_m = A_m/\chi, \quad m = 0, 1, 2, \ldots, M, \tag{67}$$

$$\chi = \left(\sum_{m,m'=0} (\delta_{0m}\delta_{0m'} + \tfrac{1}{2}\delta_{mm'} + \bar{\mathbf{S}}_m^{\mathrm{T}} \bar{\mathbf{X}} \bar{\mathbf{S}}_{m'} + 2\mathbf{C}_m^{\mathrm{T}} \bar{\mathbf{S}}_{m'}) A_m A_{m'} \right)^{1/2}, \tag{68}$$

$$\bar{\mathbf{X}} = \frac{1}{L} \int_0^L \mathbf{X}\mathbf{X}^{\mathrm{T}} \,\mathrm{d}x = \{\bar{X}_{ij}\}, \quad (\bar{X}_{ij} = L^{i+j-2}/(i+j-1)) \tag{69}$$

and

$$\mathbf{C}_m = \frac{1}{L} \int_0^L \mathbf{X} \cos \lambda_m x \,\mathrm{d}x$$

$$= \begin{cases} \left\{ 0 \quad -\dfrac{1+(-1)^{m+1}}{\lambda_m^2 L} \quad \dfrac{2(-1)^m}{\lambda_m^2} \quad \dfrac{6[1-(-1)^m] + 3(-1)^m \lambda_m^2 L^2}{\lambda_m^4 L} \quad \dfrac{[-24 + 4\lambda_m^2 L^2](-1)^m}{\lambda_m^4} \right\}^{\mathrm{T}}, & m = 1, 2, \ldots, M. \\ \left\{ 1 \quad \dfrac{L}{2} \quad \dfrac{L^2}{3} \quad \dfrac{L^3}{4} \quad \dfrac{L^4}{5} \right\}^{\mathrm{T}} & m = 0. \end{cases} \tag{70}$$

Because the Fourier series now represents a “conditioned” displacement having at least three continuous derivatives, one can expect it to converge at a faster speed which, according to Theorem 3, should satisfy

$$\lim_{n \to \infty} A_n \lambda_n^4 = 0. \tag{71}$$

Actually, for a generally supported beam, the convergence of the Fourier series solutions can be estimated in a more direct manner. Making use of equations (19) and (41), the expansion coefficients can be expressed as

$$A_m = \frac{2\lambda_m^3}{(\lambda_m^4 - \rho_D \omega^2)L} \left(\frac{B_1(-1)^m - B_0}{\lambda_m^2} - [w(L)(-1)^m - w(0)] \right) \tag{72}$$

and

$$A_m = \frac{\rho_D \omega^2 P_m}{\lambda_m^4 - \rho_D \omega^2} = \frac{2\rho_D \omega^2/\lambda_m^2}{(\lambda_m^4 - \rho_D \omega^2)L} \left(-\frac{\alpha_1(-1)^m - \alpha_0}{\lambda_m^2} + \beta_1(-1)^m - \beta_0 \right). \quad (73)$$

In equation (72), the possible displacement discontinuities at the end points are also taken into account. If all the involved boundary constants are somehow known *a priori*, then equations (72) and (73) can be, respectively, used to estimate the convergence rates of the traditional and current Fourier solutions. In reality, however, the boundary constants have to be determined from equation (48) in terms of the Fourier coefficients. Each time the displacement is differentiated, the convergence speed of the corresponding Fourier series is slowed by a factor of λ_m. Therefore, it can be concluded that the highest order of the derivatives appeared in the boundary conditions actually controls the convergence speed of the mode shape and natural frequency. For instance, if a beam is simply supported having only rotational restraints, the first derivative of the Fourier series has to be used to determine the values of the second derivatives or “bending moments” at the boundaries. Thus, according to equation (72), the natural frequencies and mode shapes will both converge at the speed λ_m^2. It is also clear that, if the beam rests on a translational spring at any end, equation (11) may not actually converge at all. In contrast, the solution in the form of equation (25) can still converge at the speed of λ_m^4.

It must be pointed out that the beam displacement can also be expanded in a Fourier sine series. Although a sine series is perhaps more suited for pinned–pinned beams with rotational restraints, it theoretically converges at a slower speed, for beams with a translational restraint at any end.

3. RESULTS AND DISCUSSIONS

First consider a pinned–pinned beam with rotational constraints at both ends, as shown in Figure 1. Assume the spring constant is very large, say, $\hat{K}_0 L = 10^{10}$, at one end. Table 1 shows the first eight frequency parameters, $\mu_i = a/\pi(\omega_i \sqrt{\rho A/D})^{1/2}$, for the various stiffnesses at the other end. For the two extreme stiffnesses, $\hat{K}_1 L = 0$ and 10^{10}, this problem essentially turns into the classical clamped–pinned and clamped–clamped cases for that the first four frequency parameters are, respectively, as follows [20]: $\mu_i = 1{\cdot}24988$, 2·25, 3·25, 4·25 and $\mu_i = 1{\cdot}50562$, 2·49975, 3·50001, 4·5. The current solution has shown an excellent agreement with them as evidenced in Table 1.

In the above calculations, the Fourier series is truncated to $M = 20$. To examine the convergence of the solution, Table 2 compares the first 10 frequency parameters of the

TABLE 1

Frequency parameters, $\mu_i = L/\pi\,(\omega_i \sqrt{\rho A/D})^{1/2}$, for various stiffnesses of the rotational springs

Mode	$\mu_i = L/\pi(\omega_i \sqrt{\rho A/D})^{1/2}$				
	$\hat{K}_1 L = 0$	$\hat{K}_1 L = 1$	$\hat{K}_1 L = 10$	$\hat{K}_1 L = 100$	$\hat{K}_1 L = 10^{10}$
1	1·24988	1·28656	1·4102	1·49137	1·50562
2	2·25005	2·27081	2·37138	2·47681	2·49975
3	3·25014	3·26491	3·34927	3·46884	3·50001
4	4·25032	4·26175	4·3337	4·46108	4·5

TABLE 2

Frequency parameters, $\mu_i = L/\pi\,(\omega_i\sqrt{\rho A/D})^{1/2}$, for various numbers of terms in Fourier series

	$\mu_i = L/\pi(\omega_i\sqrt{\rho A/D})^{1/2}$			
Mode	$M = 5$	$M = 10$	$M = 15$	$M = 20$
1	1·50563	1·50562	1·50562	1·50562
2	2·49985	2·49976	2·49975	2·49975
3	3·50392	3·50003	3·50001	3·50001
4	4·5073	4·5002	4·50001	4·5
5	—	5·50044	5·50005	5·5
6	—	6·50289	6·5001	6·50002
7	—	7·50421	7·50045	7·50004
8	—	8·52423	8·5007	8·50014
9	—	9·52852	9·50251	9·50022
10	—	—	10·5033	10·5007

TABLE 3

Frequency parameters, $\mu_i = L/\pi\,(\omega_i\sqrt{\rho A/D})^{1/2}$, for various numbers of terms in Fourier series

	$\mu_i = L/\pi(\omega_i\sqrt{\rho A/D})^{1/2}$			
Mode	$M = 50$	$M = 75$	$M = 100$	$M = 150$
1	1·50984	1·49997	1·53034	1·53753
2	2·52008	2·51498	2·50879	2·51103
3	3·52993	3·51930	3·51461	3·50949
4	4·53881	4·52850	4·52151	4·50753
5	5·54829	5·53089	5·52319	5·51528

clamped–clamped beam which are estimated by using different numbers of terms in the Fourier series. It is seen that the Fourier series converges so fast that just a few terms can lead to an excellent prediction. In comparison, as illustrated in Table 3, the traditional solution obtained from equation (22) does not converge nearly as well even though this boundary condition may actually represent an ideal scenario for it: the displacement vanishes at both ends.

As aforementioned, the eigenfunctions or mode shapes can be claculated from equation (66) with an accuracy (or more appropriately, convergence speed) comparable to that of the natural frequencies. In Figures 3–6, the mode shapes are plotted for the first four modes of the clamped–clamped beam. The classical solution for this case is well known as [20]

$$\phi_i = \cosh\frac{\pi\mu_i x}{L} - \cos\frac{\pi\mu_i x}{L} - \sigma_i\left(\sinh\frac{\pi\mu_i x}{L} - \sin\frac{\pi\mu_i x}{L}\right), \tag{74}$$

where

$$\sigma_i = \frac{\cosh\pi\mu_i - \cos\pi\mu_i}{\sinh\pi\mu_i - \sin\pi\mu_i}. \tag{75}$$

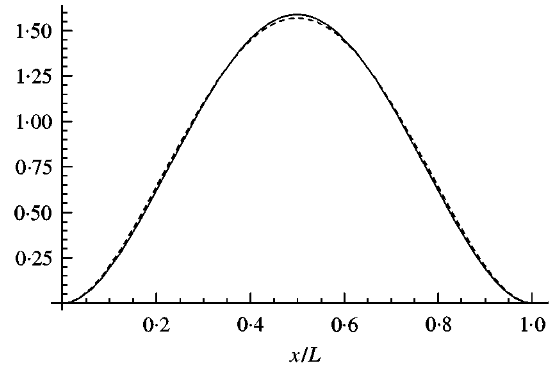

Figure 3. The mode shape of the first mode: ——, equation (74); – – –, equation (66); $M = 1$.

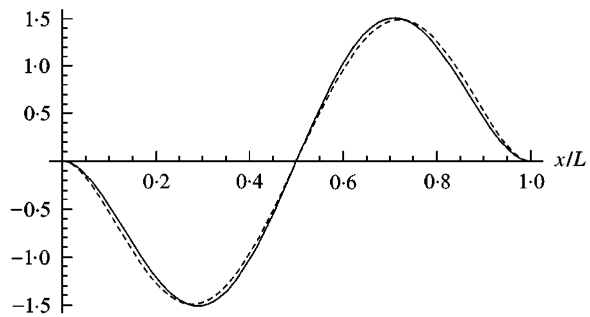

Figure 4. The mode shape for the second mode: ——, equation (74); – – –, equation (66); $M = 2$.

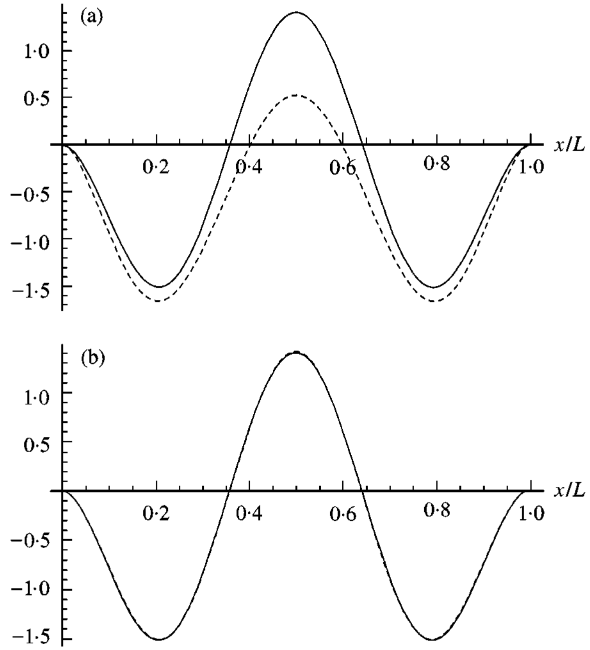

Figure 5. The mode shape for the third natural frequency: ——, equation (74); – – –, equation (66); (a) $M = 3$; (b) $M = 4$.

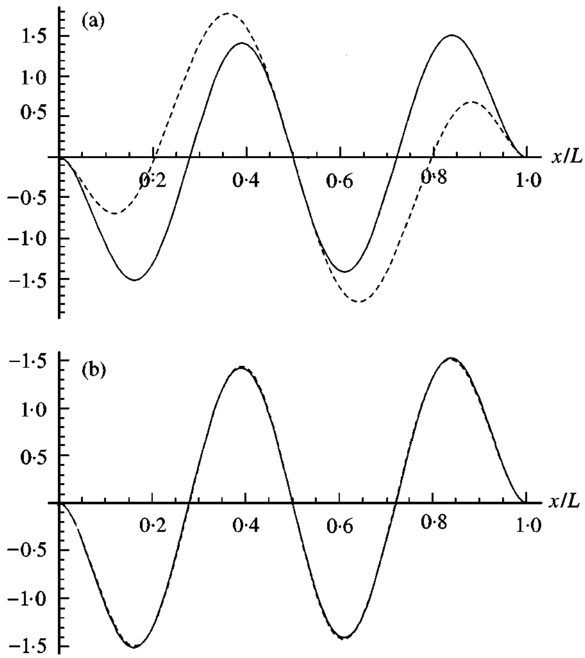

Figure 6. The mode shape for the fourth natural frequency: ——, equation (74); – – –, equation (66); (a) $M = 4$; (b) $M = 5$.

TABLE 4

Fundamental natural frequency, $\mu_1 = (L^2\omega_1\sqrt{\rho A/D})^{1/2}$, of a beam with various combinations of the rotational and translational springs

	$\hat{k}_1L^3$		
$\hat{K}_0L$	0·01	1	100
0·01	0·4948	1·3134	2·9901
1	1·2520	1·5358	3·1085
100	1·8583	1·9940	3·6134

In each of the plots, M denotes the number of cosine terms actually used in the Fourier series, and the curves for any larger M's are not presented simply because they then become virtually identical to the classical solution. All these results have indicated that the mode shapes can also be accurately obtained by taking only a few terms in the Fourier series. However, the remarkable convergence of the current solution is perhaps best manifested in Figures 5 and 6 where only one term has shown a great impact.

Let us now consider two examples that involve both rotational and translational restraints. The first one is concerned with a pinned–free beam with a rotational spring at the pinned end and a translational spring at the free end. Table 4 lists the (dimensionless) fundamental frequencies of the beam for the various combinations of the spring constants. The result are almost exactly the same as those given by Maurizi *et al.* [5]. The second example deals with a beam with both translational and rotational spring supports at each end. Assuming $\hat{k}_0L^3 = \hat{k}_1L^3 = 1$, listed in Table 5 are the five lowest natural

TABLE 5

Frequency parameters, $\mu_i = (L^2\omega_i\sqrt{\rho A/D})^{1/2}$, of a beam with various stiffnesses of the rotational springs

	$\hat{K}_1L = \hat{K}_0L$		
Mode	0·01	1	100
1	1·1843	1·18564	1·1883
2	1·57925	2·23332	3·14418
3	4·75304	5·06326	6·22722
4	7·8607	8·07739	9·33698
5	11·0009	11·1628	12·4499

TABLE 6

Frequency parameters, $\mu_i = (L^2\omega_i\sqrt{\rho A/D})^{1/2}$, calculated using different numbers of terms in the Fourier series, $\hat{K}_1L = \hat{K}_0L = 100$

Mode	$M = 5$	$M = 10$	$M = 20$	$M = 40$	Reference [1]
1	1·188301	1·188301	1·188301	1·188301	1·188301
2	3·14418	3·14418	3·14418	3·14418	3·144179
3	6·22726	6·227224	6·227221	6·22722	6·22722
4	9·33717	9·337013	9·336975	9·336971	9·336969
5	12·4514	12·45001	12·44990	12·44988	12·44988

frequencies, $\mu_i = (a^2\omega_i\sqrt{\rho A/D})^{1/2}$, for the three different rotational stiffnesses, $\hat{K}_1L = \hat{K}_0L = 0{\cdot}01$, 1, 100. This problem was previously studied in reference [1] and an excellent agreement has been observed between the two solutions. The remarkable convergence of the current solution is again demonstrated in Table 6 for this elastically restrained beam.

4. CONCLUSIONS

A new simple and unified approach has been presented for the dynamic analysis of a beam with general boundary conditions. The beam displacement is sought as the superposition of a Fourier series and an auxiliary polynomial that is used to take care of the discontinuities with the original displacement function and its related derivatives. The modal parameters of the beam can be readily and systematically obtained from solving a standard matrix eigenproblem, instead of the non-linear hyperbolic equations as in the traditional techniques.

It has been shown through numerical examples that the natural frequencies and mode shapes can both be accurately calculated for beams with various boundary conditions. The remarkable convergence of the current solution is demonstrated both theoretically and numerically. It should be noted that the proposed technique can be easily extended to certain two-dimensional structures such as plates and shells with general boundary conditions and the fast convergence of the Fourier series makes the extension numerically viable.

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APPENDIX A: NOMENCLATURE

A	cross-sectional area
A_m	Fourier coefficient
B_0, B_1	$= w''(0), w''(L)$
D	flexural rigidity
K_0, K_1	rotational stiffnesses at $x = 0$ and L respectively
$\hat{K}_0, \hat{K}_1$	$= K_0/D, K_1/D$
k_0, k_1	translational stiffnesses at $x = 0$ and L respectively
$\hat{k}_0, \hat{k}_1$	$= k_0/D, k_1/D$
L	beam length
M	number of cosine terms used in the Fourier series
p	polynomial function
p_1, p_2	polynomials
w	beam displacement

$\bar{w}$	conditioned beam displacement
α_0, α_1	$= w'''(0), w'''(L)$
β_0, β_1	$= w'(0), w'(L)$
λ_m	$= \dfrac{m\pi}{L}$
μ_i	$= L/\pi(\omega_i\sqrt{\rho A/D})^{1/2}$ or $(L^2\omega_i\sqrt{\rho A/D})^{1/2}$
ρ	mass density
ω	frequency in radian