Journal of Sound and Vibration (2000) **237**(4), 733–744
doi:10.1006/jsvi.2000.2990, available online at http://www.idealibrary.com on IDEAL®

FREE VIBRATION OF A SIMPLY SUPPORTED BEAM CARRYING A RIGID MASS AT THE MIDDLE

C. H. CHANG†

Department of Engineering, Science and Mechanics, The University of Alabama, U.S.A.

(*Received* 26 *August* 1999, *and in final form* 27 *January* 2000)

1. INTRODUCTION

The free vibration of a simply supported beam with a constant cross-sectional area carrying a concentrated rigid mass at the mid-point of the beam has been studied in two manners: The first way treated the problem as a single-degree-of-freedom system by neglecting the mass of the beam [1]. The second method included the mass of the beam and treated the configuration as a continuous elastic system [2–4]. In these treatments only the symmetrical vibration of the beam was considered.

Restricting this system to vibrate in symmetrical modes simplified this problem and made unnecessary any consideration of the rotatory inertia of the mounted mass. But if this system is excited by a force applied anywhere on the system other than at the mid-span, or by a couple, antisymmetric vibration modes will be produced.

In this study, this problem is approached from the general case, allowing the concentrated mass to deflect in the z direction and also to rotate about its central axis which is parallel to the y-axis in Figure 1. A general solution including both the rotatory inertia of the beam and of the concentrated mass is obtained by applying elementary beam theory in which the effect of transverse shear deformation is neglected. The latter effect may be significant but was omitted both for simplicity and because it was desired to isolate the effect of rotatory inertia.

The solutions with the rotatory inertia of the beam being neglected are deduced from the general solutions. The antisymmetric mode frequencies for two extreme cases of mounted mass are discussed. Roots of the frequency equations are obtained and plotted for certain ranges of parameters involving the ratio of the concentrated mass to the mass of the beam and its moments of inertia.

2. FORMULATION OF THE PROBLEM

The reference co-ordinates for this system are oriented as shown in Figure 1. The xy-plane coincides with the neutral plane of the undeflected beam. Consider the whole beam as two spans each of length $l/2$.

The positive values of bending moment M, shear force V, deflection W, and rotation θ of the beam are shown in Figure 2. The subscripts 1 and 2 indicate the corresponding spans. When no subscript is given, they apply to both spans.

†Professor emeritus. Address: 5180 Courton street, Alpharetta, GA 30022, U.S.A.

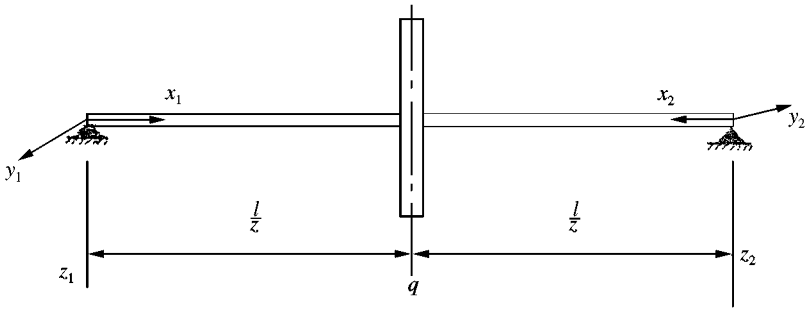

Figure 1. The co-ordinates of the system.

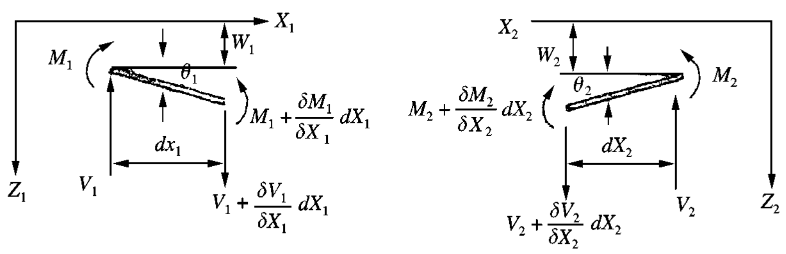

Figure 2. The conventions.

The equation for the lateral vibration of the beam may be given in the following form [5]:

$$a^2 \frac{\partial^4 W}{\partial x^4} + \frac{\partial^2 W}{\partial t^2} - s^2 \frac{\partial^4 W}{\partial x^2 \partial t^2} = 0, \tag{1}$$

where

$$a^2 = EI_b/\rho A \quad \text{and} \quad s^2 = I_b/A, \tag{2}$$

E is the modulus of elasticity, ρ, the mass density, and I_b is the moment of inertia of the beam with cross-sectional area A, hence s is the radius of gyration and t is the time.

The boundary conditions at both ends of the beam are that the deflections and moments vanish:

$$W = 0 \quad \text{and} \quad -EI_b \frac{\partial^2 W}{\partial x^2} = 0 \quad \text{at } x = 0. \tag{3, 4}$$

The conditions of continuity at the middle are that the deflections are equal and slopes equal but having opposite directions:

$$W_1 = W_2 \quad \text{at } x = l/2, \tag{5}$$

and

$$\frac{\partial W_1}{\partial x_1} = -\frac{\partial W_2}{\partial x_2} \quad \text{or} \quad \frac{\partial W_1}{\partial x_1} + \frac{\partial W_2}{\partial x_2} = 0 \quad \text{at } x = l/2. \tag{6}$$

By applying the equation of motion in the z direction to the free body of the concentrated mass, m at the mid-point as shown in Figure 3,

$$\Sigma F_z = ma_z,$$

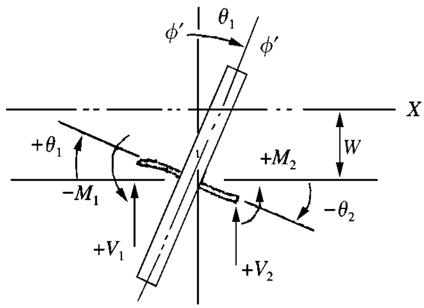

Figure 3. The free body of the concentrated mass.

where F_z and a_z are force and acceleration of the mass in the z direction respectively. One has

$$EI_b\frac{\partial^3 W_1}{\partial x_1^3}+EI_b\frac{\partial^3 W_2}{\partial x_2^3}=m\frac{\partial^2 W_1}{\partial t^2}+\rho I_b\frac{\partial^3 W}{\partial x_1\,\partial t^2}+\rho I_b\frac{\partial^2 W}{\partial x_2\,\partial t^2}\quad \text{at } x=l/2$$

which, by using equation (6), becomes

$$EI_b\frac{\partial^3 W_1}{\partial x_1^3}+EI_b\frac{\partial^3 W_2}{\partial x_2^3}=m\frac{\partial^2 W_1}{\partial t^2}\quad \text{at } x=l/2. \tag{7}$$

The equation for rotatory motion of the concentrated mass about its central axis and parallel to y-axis is

$$\Sigma M_y = I_m\alpha$$

in which α is the angular acceleration, takes the form

$$EI_b\frac{\partial^2 W}{\partial x_1^2}-EI_b\frac{\partial^2 W_2}{\partial x_2^2}=-\,mr^2\,\frac{\partial^3 W_1}{\partial x_1\,\partial t^2}\quad \text{at } x=l/2, \tag{8}$$

where r is the radius of gyration of the mounted mass with respect to its central axis.

3. THE GENERAL SOLUTION

Using the method of separation of variables, let

$$W_n(x,t)=\sum_n X_n(x)T_n(t).$$

Equation (1) results in

$$X_n''''+k_n^4 s^2 X_n''-k_n^4 X_n=0,\qquad T_n+p_n^2\ddot{T}_n=0, \tag{9, 10}$$

where a prime indicates differentiation with respect to x, a dot with respect to time t, and

$$k_n^2=p_n/a, \tag{11}$$

where p_n is the circular frequency.

The solutions of equations (9) and (10) are, respectively [6],

$$X_n = C_n \cosh \alpha_n k_n x + D_n \sinh \alpha_n k_n x + F_n \cos k_n x/\alpha_n + H_n \sin k_n x/\alpha_n, \tag{12}$$

$$T_n = A_n \cos p_n t + B_n \sin p_n t, \tag{13}$$

where

$$\alpha_n = [(k_n^4 s^4/4 + 1)^{1/2} - k_n^2 s^2/2]^{1/2} \tag{14}$$

and A_n, B_n, C_n, D_n, F_n, and H_n are arbitrary constants.

Because of the boundary conditions of equations (3) and (4), $C_n = F_n = 0$, the solution of equation (12) is reduced to

$$X_n = D_n \sinh \alpha_n k_n x + H_n \sin k_n x/\alpha_n.$$

The solutions for each span become

$$X_{1n} = D_{1n} \sinh \alpha_n k_n x_1 + H_{1n} \sin k_n x_1/\alpha_n, \tag{15}$$

$$X_{2n} = D_{2n} \sinh \alpha_n k_n x_2 + H_{2n} \sin k_n x_2/\alpha_n. \tag{16}$$

Let $k_n l/2 = \beta_n$, hence

$$\alpha_n = \{[4(s/l)^4 \beta_n^4 + 1]^{1/2} - 2(s/l)^2 \beta_n^2\}^{1/2} \tag{17}$$

and let the mass ratio

$$R = m/m_b = m/\rho A l. \tag{18}$$

By using solutions (15) and (16), conditions (5) and (6) become

$$(D_{1n} - D_{2n}) \sinh \alpha_n \beta_n + (H_{1n} - H_{2n}) \sin \beta_n/\alpha_n = 0, \tag{19}$$

$$\alpha_n^2 (D_{1n} + D_{2n}) \cosh \alpha_n \beta_n - (H_{1n} + H_{2n}) \cos \beta_n/\alpha_n = 0. \tag{20}$$

The equations of motion (7) and (8) take the forms

$$\begin{aligned} &\alpha_n^6 (D_{1n} + D_{2n}) \cosh \alpha_n \beta_n - (H_{1n} + H_{2n}) \cos \beta_n/\alpha_n \\ &\quad = -2R\beta_n (D_{1n} \sinh \alpha_n \beta_n + H_{1n} \sin \beta_n/\alpha_n)\alpha_n^3 \end{aligned} \tag{21}$$

$$\begin{aligned} &\alpha_n^4 (D_{1n} - D_{2n} \sinh \alpha_n \beta_n - (H_{1n} - H_{2n}) \sin \beta_n/\alpha_n \\ &\quad = 8R(r/l)^2 \beta_n^3 (\alpha_n^2 D_{1n} \cosh \alpha_n \beta_n + \alpha_n H_{1n} \cos \beta_n/\alpha_n). \end{aligned} \tag{22}$$

From the above four equations, the frequency equation is obtained for the non-trivial solution of four constants involved, and has the following form, since $\alpha_n \neq 0$;

$$\begin{aligned} &[R\beta_n \alpha_n (\alpha_n^2 \tanh \alpha_n \beta_n - \tan \beta_n/\alpha_n) - (1 + \alpha_n^4)], \\ &[4R(r/l)^2 \beta_n^3 \alpha_n (\alpha_n^2 \tan \beta_n/\alpha_n - \tanh \alpha_n \beta_n) - (1 + \alpha_n^4) \tan \beta_n/\alpha_n \tanh \alpha_n \beta_n] = 0. \end{aligned} \tag{23}$$

This equation may be separated into two parts, each of which is equated to zero. The first part is

$$R\beta_n \alpha_n (\alpha_n^2 \tanh \alpha_n \beta_n - \tan \alpha_n/\beta_n) - (1 + \alpha_n^4) = 0. \tag{24}$$

When the second part is divided by $\tanh \alpha_n \beta_n \tan \alpha_n/\beta_n$, it becomes

$$4R(r/l)^2 \beta_n^3 \alpha_n (\alpha_n^2 \coth \alpha_n \beta_n - \cot \alpha_n/\beta_n) - (1 + \alpha_n^4) = 0. \tag{25}$$

From the four equations (19)–(22), the following ratio is obtained:

$$D_{1n}/D_{2n} = 1 - [8R(r/l)^2\beta_n^3\alpha_n/(1+\alpha_n^4)]\{\alpha_n^2 \coth \alpha_n\beta_n - (H_{1n}/D_{1n})[\cos(\beta_n/\alpha_n)/\sinh \alpha_n\beta_n)]\}. \quad (26)$$

From equations (19–22) and using equation (24), one has

$$H_{1n}/D_{1n} = H_{2n}/D_{2n} = -\alpha_n^2 \cosh \alpha_n\beta_n/\cos(\beta_n/\alpha_n). \quad (27)$$

Substituting this result into equation (26), one obtains

$$D_{1n}/D_{2n} = 1 \quad \text{or} \quad D_{1n} = D_{2n}. \quad (28)$$

Then, the normal functions (15), (16) take the forms

$$X_{1n}(x_1) = D_{1n}\{\sinh \alpha_n k_n x_1 - [\alpha_n \cosh \alpha_n\beta_n/\cos(\beta_n/\alpha_n)]\sin(k_n x_1/\alpha_n)]\}, \quad (29)$$

$$X_{2n}(x_2) = X_{1n}(x_2). \quad (30)$$

Thus, the normal mode functions resulting from the use of frequency equation (24) are symmetrical with respect to the middle point of the beam; equation (24) is, therefore, termed the frequency equation for symmetric modes.

If instead of equation (24), equation (25) is used in equation (26), the ratios become

$$H_{1n}/D_{1n} = H_{2n}/D_{2n} = -\sinh \alpha_n\beta_n/\sin(\beta_n/\alpha_n), \qquad D_{2n}/D_{1n} = -1. \quad (32, 33)$$

For these ratios, the mode functions take the forms

$$X_{1n}(x_1) = D_{1n}\{\sinh \alpha_n k_n x_1 - [\sinh \alpha_n\beta_n/\sin(\beta_n/\alpha_n)]\sin(k_n x_1/\alpha_n)\}, \quad (34)$$

$$X_{2n}(x_2) = -X_{1n}(x_2). \quad (35)$$

These vibration modes have a nodal point at the middle of the beam and are antisymmetrical with respect to this point. These modes, therefore, are referred to as antisymmetric modes, and equation (25) is called the antisymmetric mode frequency equation.

The determination of the roots of the frequency equations (24) and (25) will be discussed later. However, it should be noted that these roots when arranged in ascending order arise alternately with the lowest, or fundamental root being derived from equation (24). Thus, the symmetric modes are associated with $n = 1, 3, 5, \ldots$, and the antisymmetric modes with $n = 2, 4, 6, \ldots$.

The final solution for lateral vibration of the first half span of the beam is

$$W_1 = \sum_{n=1,3,5,\ldots} (A_n \cos p_n t + B_n \sin p_n t)X_{1n} + \sum_{n=2,4,6,\ldots} (A_n \cos p_n t + B_n \sin p_n t)X_{1n}. \quad (36)$$

The constants A_n and B_n are to be determined by initial conditions. The normal functions, X_{1n} have been determined by equation (29) for odd n and equation (34) for even n. When equations (30) and (35) are used for X_{2n}, the deflection W_2 for the other half span can be obtained.

When the rotatory inertia of the beam is neglected, $s/l = 0$; hence, from equation (14), $\alpha_n = 1$, the frequency equations (24) and (25) are simplified. The equation for symmetric modes is

$$R\beta_n(\tan \beta_n - \tanh \beta_n) - 2 = 0 \quad (37)$$

and that for antisymmetric modes is

$$2R(r/l)\beta^3(\coth \beta_n - \cot \beta_n) - 1 = 0. \quad (38)$$

The normal mode equations for the symmetric case are

$$X_{1n}(x_1) = D_{1n}[\sinh k_n x_1 - (\cosh \beta_n/\cos \beta_n)\sin k_n x_1], \quad (39)$$

$$X_{2n}(x_2) = X_{1n}(x_2) \quad (40)$$

and for the antisymmetric case they are

$$X_{1n}(x_1) = D_{1n}[\sinh k_n x_1 - (\sinh \beta_n/\sin \beta_n)\sin k_n x_1], \quad (41)$$

$$X_{2n}(x_2) = -X_{1n}(x_2). \quad (42)$$

The final deflection solutions are the same as given in equation (36).

Equations (37) and (39), (40) agree with the results presented in reference [3].

4. DISCUSSION OF FREQUENCY EQUATION

It will be shown that the effect of the rotatory inertia of the beam vibration is appreciable for second and higher modes only, the discussion given will be for the basic mode with this effect neglected.

Furthermore, since the discussion of the frequency equation associated with the symmetric mode vibration has been given in reference [2], the antisymmetric frequency equation alone will be discussed herein.

4.1. APPROXIMATE FORMULA FOR THE LOWEST ANTISYMMETRIC MODE FREQUENCY

The frequency equation for antisymmetric modes, equation (25) may be written as

$$K = \beta_n^3(\coth \beta_n - \cot \beta_n), \quad (43)$$

where

$$K = 1/[2R(r/l)^2] = m_b l^2/(2mr^2). \quad (44)$$

For the lowest frequency, β_2, for $\beta_2 < \pi$,

$$\coth \beta_2 = 1/\beta_2 + \beta_2/3 - \beta_2^3/45 + 2\beta_2^5/945 - \beta_2^7/4725 + 2\beta_2^9/93555 - \cdots,$$

$$\cot \beta_2 = 1/\beta_2 - \beta_2/3 - \beta_2^3/45 - 2\beta_2^5/945 - \beta_2^7/4725 - 2\beta_2^9/93555 - \cdots.$$

Substituting these series into equation (43), one has

$$K = \beta_2^3(2\beta_2/3 + 4\beta_2^5/945 + 4\beta_2^9/93555 + \cdots).$$

If only the first term is kept, then

$$\beta_2 = (3K/2)^{1/4}. \quad (45)$$

It will be shown later that the results obtained from this simple formula are quite satisfactory.

From equations (11), (17) and (44), the circular frequency is

$$p_2 = a(2\beta_2/l)^2 = (12EI_b/I_m l)^{1/2}. \quad (46)$$

This result may also be obtained by considering the problem as a single-degree-of-freedom system as shown in Figure 4.

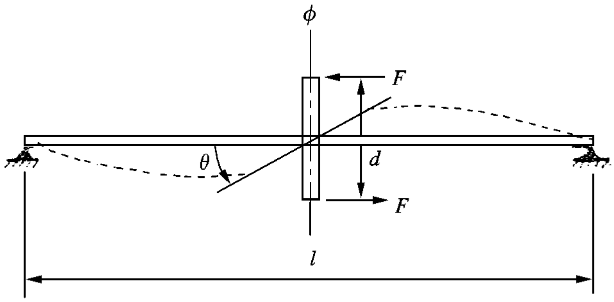

Figure 4. A single-degree-of-freedom system.

If a torque Fd is applied to the mounted mass as shown and then is released, the equation of motion is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + (K_s/I_m)\theta = 0, \tag{47}$$

where K_s is the spring constant of the beam in resistance to the change of the slope at the middle point and it can be shown that

$$K_s = 12EI_b/l.$$

From the solution of equation (47), the circular frequency is

$$p_2 = (K_s/I_m)^{1/2} = (12EI_b/I_m)^{1/2},$$

as given by equation (46).

4.2. ANTISYMMETRIC MODE FREQUENCIES FOR TWO EXTREME CASES OF THE MOUNTED MASS

There are two limiting values of the mounted mass, m: infinite and zero. From equation (44), $K = 0$ and ∞ respectively. These two cases are discussed below.

Case A: m approaches infinity. Physically, this means that the mounted mass is extremely large in comparison with the mass of the beam, so that the midpoint of the beam is essentially fixed. Since $K = 0$, equation (43) becomes,

$$\tanh \beta_n = \tan \beta_n.$$

This is the frequency equation for a beam simply supported at one end and built in at the other end [3]. The roots of this equation are, approximately, given by the expression

$$\beta_n = (n/2 - 3/4)\pi, \quad n = 4, 6, 8, \dots . \tag{48}$$

The root β_n is degenerated to zero as K approaches zero.

Case B: m approaches zero. This means that there is no mass mounted on the beam. Since K approaches infinity, for finite value of β, from equation (43), one obtains

$$\cot \beta_n = - \infty .$$

Therefore,

$$\beta_n = n\pi/2, \quad n = 2, 4, 6, \dots . \tag{49}$$

These are the roots of the frequency equation for the antisymmetric modes of a simply supported beam [1].

5. NUMERICAL RESULTS

For a beam with mass ratio $R = 1$ and $K = 500$ in equation (44), the first six modes based on equations (37), (39), and (40) for odd symmetric modes and equations (38), (41) and (42) for even antisymmetric modes are presented in Figure 5.

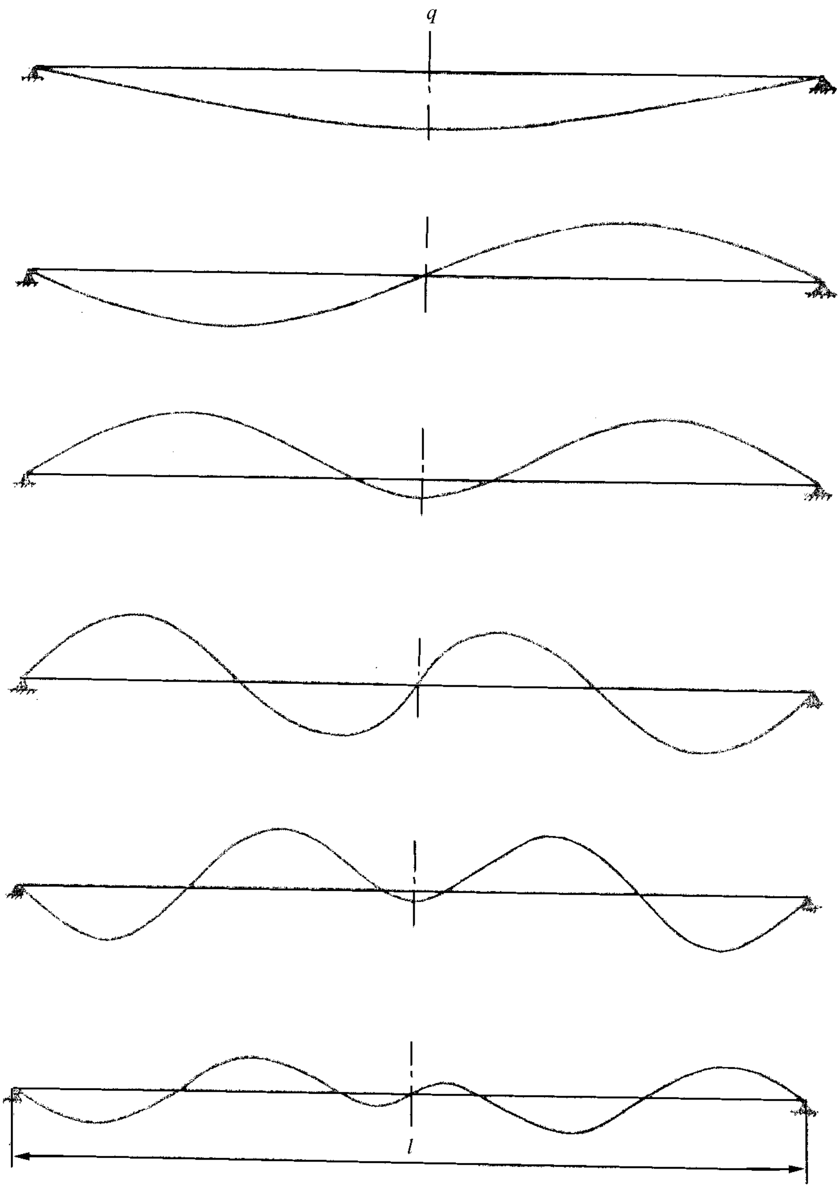

Figure 5. The first six normal modes for $R = 1$, $K = 500$, and $s/l = 0$.

From equation (17), the influence of the rotatory inertia of a beam depends on the slenderness ratio s/l and the frequency parameter β_n. For small s/l ratio and low frequency, the effect may be neglected. In order to see this effect on normal modes, the fifth and sixth modes for beams with $s/l = 0$ and 0·05 are shown in Figure 6 for $R = 1$ and $K = 500$.

The roots, β_n, for $s/l = 0{\cdot}05$, and different R values and for the 1st, 5th, and 9th symmetric modes from equation (24) have been computed and given in Table 1. β_n values for the 2nd, 6th, and 10th antisymmetric modes for different K values obtained from equation (25) are listed in Table 2.

For the same ranges of R and K but for $s/l = 0$, β_n values are obtained from equations (37) and (43), and tabulated in Tables 3 and 4 respectively. These results are also depicted in Figures 7 and 8, in which the results given in Tables 1 and 2 are also denoted by dotted lines.

For the first and second modes, the β_1 and β_2 values for $s/l = 0$ and 0·05 are so close, as seen from the tables, that no difference can be observed, in Figures 7 and 8. Therefore, for the basic modes the rotatory inertia of the beam, practically, may be neglected.

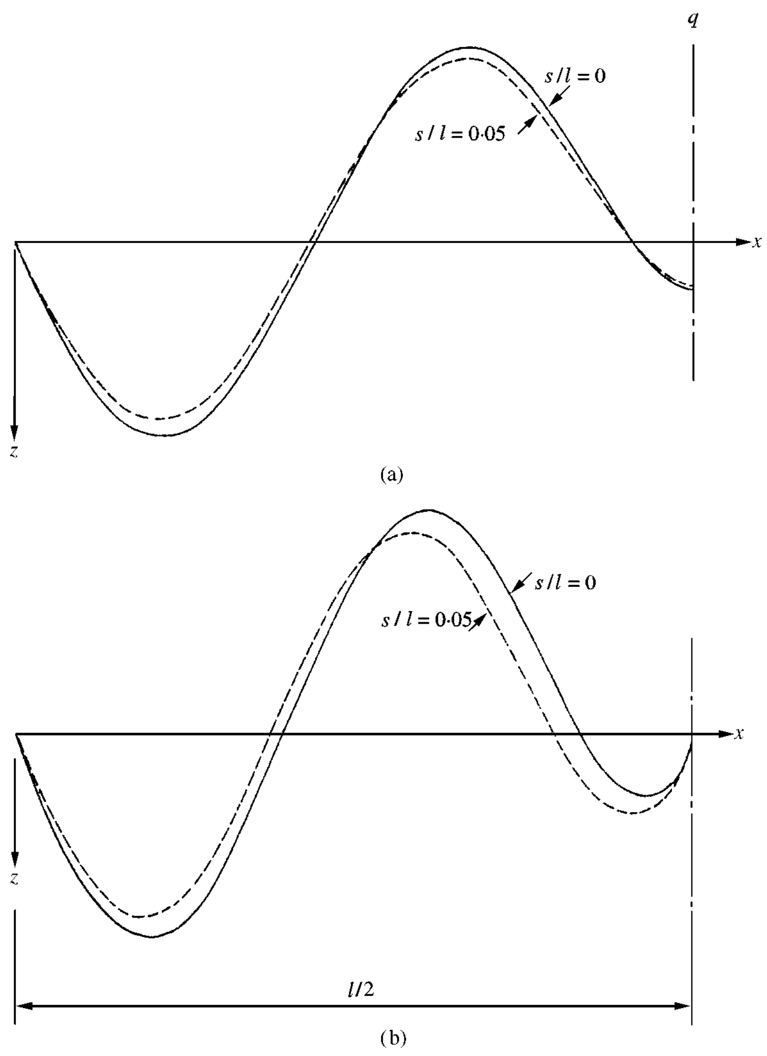

Figure 6. The (a) fifth and (b) sixth normal modes for $R = 1$, $K = 500$, $s/l = 0$ and $s/l = 0{\cdot}05$.

Table 1
β_n values of symmetric modes for $s/l = 0{\cdot}05$

n	$R = 0$	$R = \frac{1}{4}$	$R = 1$	$R = 2$	$R = 3$	$R = 4$	$R = 5$	$R = \infty$
1	1·561	1·413	1·189	1·047	0·962	0·904	0·860	0·785
5	6·965	6·705	6·555	6·515	6·498	6·490	6·486	6·465
9	11·539	10·567	10·503	10·488	10·482	10·478	10·477	10·468

Table 2
β_n values of antisymmetric modes for $s/l = 0{\cdot}05$

n	$K = 0$	$K = 1$	$K = 3$	$K = 10$	$K = 50$	$K = 150$	$K = 500$	$K = 1500$	$K = 5000$	$K = \infty$
2	0	1·104	1·445	1·917	2·597	2·888	3·013	3·050	3·064	3·069
6	6·465	6·467	6·470	6·482	6·547	6·723	7·247	7·753	8·023	8·040
10	10·469	10·470	10·470	10·472	10·484	10·515	10·633	10·950	11·328	11·511

Table 3
β_n values of symmetric modes for $s/l = 0$

n	$R = 0$	$R = \frac{1}{4}$	$R = 1$	$R = 2$	$R = 3$	$R = 4$	$R = 5$	$R = \infty$
1	1·571	1·419	1·191	1·048	0·963	0·904	0·860	0·785
3	4·712	4·363	4·120	4·037	4·003	3·981	3·975	3·927
5	7·854	7·406	7·207	7·134	7·113	7·103	7·096	7·069
7	10·996	10·470	10·297	10·256	10·242	10·234	10·229	10·210
9	14·137	13·575	13·421	13·387	13·376	13·370	13·367	13·352

Table 4
β_n values of antisymmetric modes for $s/l = 0$

n	$K = 0$	$K = 1$	$K = 3$	$K = 10$	$K = 50$	$K = 150$	$K = 500$	$K = 1500$	$K = 5000$	$K = \infty$
2	0	1·104	1·446	1·921	2·624	2·940	3·080	3·121	3·136	π
4	3·927	3·935	3·952	4·011	4·341	4·947	5·739	6·106	6·213	2π
6	7·069	7·070	7·073	7·083	7·142	7·303	7·877	8·747	8·256	3π
8	10·210	10·210	10·211	10·215	10·234	10·283	10·478	11·094	11·936	4π
10	13·352	13·352	13·352	13·354	13·362	13·384	13·465	13·738	14·666	5π

6. CLOSING REMARKS

A comprehensive study on the lateral vibration of a simply supported beam carrying a concentrated mass at the center of beam has been made. It has been found that it is not necessary to limit the vibration to either symmetric or antisymmetric vibration, beforehand. They will emerge from the general approach.

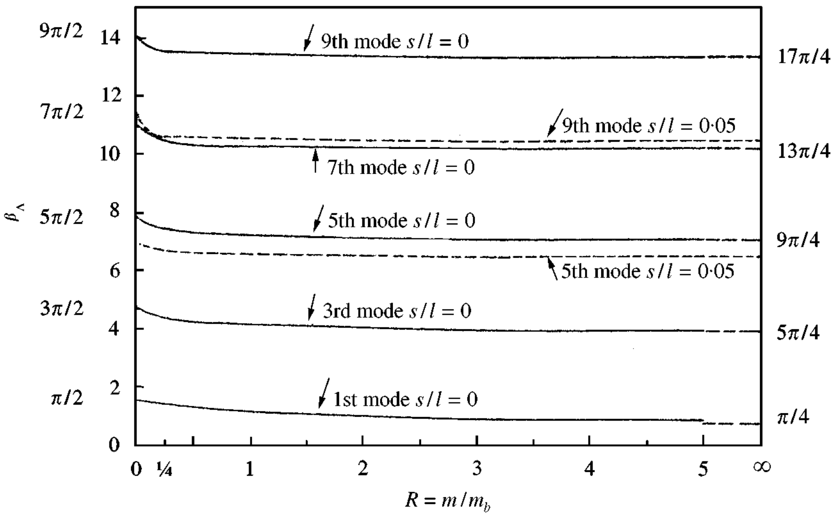

Figure 7. β–R curves for symmetric modes.

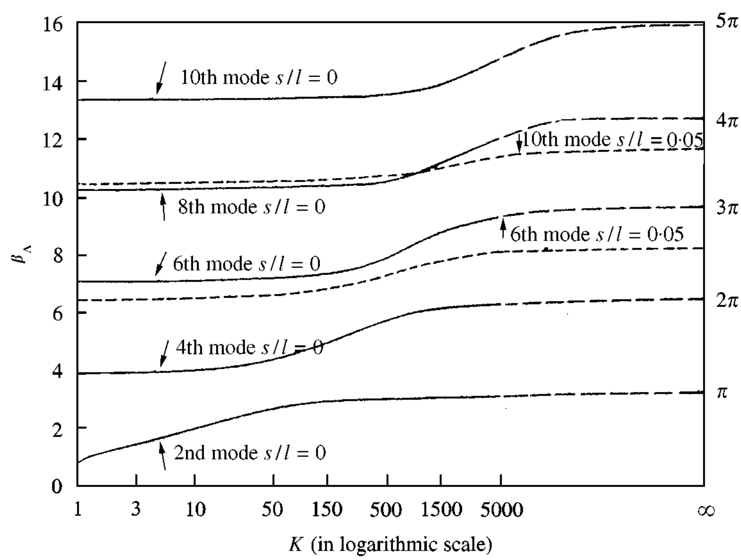

Figure 8. β–K curves for antisymmetric modes.

It has been shown that the frequencies of symmetric modes depend on R, the ratio between the mass carried and the mass of the beam. The antisymmetric frequencies are a function of the value K which, in addition to the value R, is also a function of the moment of inertia of the carried mass about an axis though its center.

When the effect of the moment of inertia of the beam is included, $s/l \neq 0$, the frequencies of both symmetric and antisymmetric for high modes are lower than those when $s/l = 0$, as seen from Figures 7 and 8. The effect is quite significant. It is shown in these figures that the dotted line curves for the 9th and 10th frequencies for $s/l = 0 \cdot 05$ are very close to those of the 7th and 8th modes for $s/l = 0$ respectively. For large K and antisymmetric modes, the

10th frequency for $s/l = 0{\cdot}05$ could be lower than the 8th frequency for $s/l = 0$. It may be concluded that for high modes vibration, both symmetric and antisymmetric, the rotatory inertial effect needs to be taken into consideration.

ACKNOWLEDGMENTS

This study was suggested by Dr Norris Jackson Huffington, Jr for the author's thesis for the degree of Master of Science in Applied Mechanics at Virginia Polytechnic Institute in 1956.

REFERENCES

1. S. TIMOSHENKO 1995 *Vibration Problem in Engineering*. New York: D. Van Nostrand Company, Third edition.
2. KARRMAN and BIOT 1940 *Mathematical Method in Engineering*. New York, London: McGraw-Hill Company. William.
3. H. Hoppmann 1952 *Journal of Applied Mechanics*. Forced lateral vibration of beam carrying a concentrated mass.
4. YU CHEN 1966 *Vibrations*: *Theoretical Methods*. Reading, MA: Addison-Wesley.
5. LORD RAYLEIGH *The theory of Sound*, Vol. 1. New York: Dover Publications, Second edition.
6. C. H. CHANG and Y. C. JUAN 1985 *Journal of Sound and Vibration* **101**, 171–180. Effect of rotatory inertia and shear deformation on vibration of an inclined bar with an end constraint.