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FORCED RESPONSE OF STRUCTURAL DYNAMIC SYSTEMS WITH LOCAL TIME-DEPENDENT STIFFNESSES

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This paper deals with a method which is meant to directly approximate the steady state response of linear differential equations with periodic coefficients under external excitations. The interest lies in the use of particular systems with time-independent characteristics (mass, damping) and with periodically time-varying stiffness. A description of the principle of the method is provided. This method has been successfully tested on a single-degree-of-freedom (s.d.o.f) example and compared to the standard Runge–Kutta method. Moreover, the parameters are assumed to be a modification of initial non-parametric systems and allow us the use of the forced reanalysis methods to improve the direct spectral method (DSM). The description of the reanalysis method is made with its implementation within the direct spectral method. Then, a practical application concerning a clamped/free beam with parametric mounts is presented to demonstrate the ability of the proposed method in the analysis of systems which have many d.o.f.s and localized parameters.

1. INTRODUCTION

The parametric vibrations and the many problems they create have been studied by both engineers and mathematicians. This kind of behaviour occurs in structural systems subjected to turbulent flow, cracked rotors or elastic linkage systems (such as slider–crank mechanism). These are described in references [1–6]. After the study of the stability of such mechanisms, the steady state is often sought and it is not worth studying the transient response. When traditional integration techniques (Runge–Kutta, Newmark) [4, 5] are employed, the transient response is computed until a steady state is obtained. Indeed, a large number of cycles of integration are needed to get a steady state response and thus are computationally expensive. Nevertheless, these conventional methods are well known for their robustness and could be used to compute the reference solution.

Another commonly used procedure to find the forced response consists of using the methods based on Fourier series expansion [6]. Indeed, there is no need to choose an integration time step and these methods directly approximate the steady state forced response. Besides, they are not perfectly adapted to the treatment of systems having large number of d.o.f.s.

The last method computes the steady state in the frequency domain by transforming the set of differential equations into a set of algebraic equations using the Fourier

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transformation. Thus, the spectra of the forced response is given by solving a classical constant linear system using either an iterative (iterative spectral method) [2, 3] or a direct method (the direct spectral method).

This paper deals with the latter method to directly calculate the steady state response of linear differential equations with periodic coefficients. The interest in it lies in the use of particular systems with time-independent characteristics (mass, damping) and with periodically time-varying stiffness. After the description of the principle of the method, some numerical results concerning a s.d.o.f. system are displayed in order to validate this method. Moreover, the parameters are assumed to be a perturbation of initial non-parametric systems and allows one the use of the forced reanalysis methods [7–9] to improve the direct spectral method. After the description of the reanalysis method implementation within the direct spectral method, a practical application is taken into consideration in order to show the ability of the proposed method to deal with large multi-d.o.f. systems with localized parameters.

2. A DIRECT ANALYSIS: THE DIRECT SPECTRAL METHOD (THE DSM)

2.1. THEORY

These parametric systems with n degrees of freedom (d.o.f) can be described by a second order differential equation with time-varying coefficients, which is

$$[M]\cdot\ddot{x}(t) + [C]\cdot\ddot{x}(t) + [K(t)]\cdot x(t) = f(t) \quad (n\times 1), \tag{1}$$

where matrices are denoted by brackets and a superscript dot represents differentiation with respect to the time t.

The n d.o.f. square matrices $[M]$, $[C]$ and $[K(t)]$, respectively, represent the mass, damping and time-varying stiffness matrices. The n d.o.f. vector $x(t)$ and $f(t)$, respectively, represent the generalized displacements and generalized forces.

When calculating the frequency response, the forcing function is given by

$$f(t) = F\cdot \mathrm{e}^{-\mathrm{i}\omega t} \quad (n\times 1), \tag{2}$$

where F is a n d.o.f.-vector of constants, ω the forcing frequency, t the time and the complex value $\mathrm{i}^2 = -1$.

The time-dependent stiffness can be divided into two parts:

- time-independent or initial stiffness $[K_0]$ $(n\times n)$;
- time-varying stiffness $g(t)$. $[k]$ $(n\times n)$

where the scalar $g(t)$ is the periodic modulation and the n d.o.f. square matrix $[k]$ the amplitude of the parameters.

Therefore, the equation of motion (1) can be rewritten as

$$[M]\cdot\ddot{x}(t) + [C]\cdot\dot{x}(t) + [K_0]\cdot x(t) + g(t)\cdot[k]\cdot x(t) = f(t) \quad (n\times 1). \tag{3}$$

The goal of these studies, is to find the steady response. In this case, the Fourier transformation of equation (3) can be obtained by retaining only the stationary terms and this yields the following continuous equation:

$$(-\omega^2[M] + \mathrm{i}\omega\cdot[C] + [K_0])\cdot X(\omega) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} g(t)\cdot[k]\cdot x(t)\,\mathrm{d}t = F(\omega) \quad (n \text{ d.o.f.} \times 1), \tag{4}$$

where $X(\omega)$ and $F(\omega)$ are the Fourier transforms of the n d.o.f. vector of generalized displacements and loadings respectively. Thanks to the convolution theorem (i.e., the Fourier transformation of the product of two time functions is the convolution of their Fourier transformation) the integral which appears in equation (4) can be expressed as

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} g(t)\cdot[k]\cdot x(t)\,\mathrm{d}t = [k]\cdot(G\otimes X)(\omega) \quad (n\times 1), \tag{5}$$

where $\otimes$ shows the convolution.

Introducing equation (5) into equation (4) leads to

$$(-\omega^2[M] + \mathrm{i}\omega\cdot[C] + [K_0])\cdot X(\omega) + [k]\cdot(G\otimes X)(\omega) = F(\omega) \quad (n\times 1). \tag{6}$$

Three parts are emphasized in equation (6): (1) the initial n d.o.f. square complex impedance matrix $(-\omega^2\cdot[M] + \mathrm{i}\omega\cdot[C] + [K_0])$; (2) the effects of parameters contained in the convolution; (3) the loadings $F(\omega)$.

Continuous equation (6) is discretized into $N+1$ discrete frequencies (from 0 to $N\omega_s$) to computerize. Furthermore, the convolution product can be rewritten into a matrix product,

$$[k](G\otimes X)(\omega) = [K]\cdot X_2 \quad (2(N+1)\cdot n)\times 1 \tag{7}$$

with

$$X_2 = \left| \begin{matrix} \Re(X(0)) \\ \Im(X(0)) \\ \vdots \\ \Re(X(N\omega_s)) \\ \Im(X(N\omega_s)) \end{matrix} \right. \quad (2(N+1)\cdot n)\times 1$$

the $(2(N+1)\ n\times 1)$ vector where a distinction is made between imaginary and real parts.

Equation (7) is explained in Appendix A. Convolution becomes, in this case, a frequencies–frequencies coupling and leads to

$$\begin{bmatrix} R(0) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & R(N\omega_s) \end{bmatrix}\cdot X_2 + [\Delta K]X_2 = F_2 \quad (2(N+1)n)\times 1 \tag{8}$$

with

$$R(n\cdot\omega_s) = \begin{bmatrix} -(n\cdot\omega_s)^2\cdot[M] + [K_0] & -(n\cdot\omega_s)\cdot[C] \\ (n\cdot\omega_s)\cdot[C] & -(n\cdot\omega_s)^2\cdot[M] + [K_0] \end{bmatrix}$$

a $(2n)$ square matrix or in a simplified way

$$([R] + [\Delta K])X_2 = F_2 \quad (2(N+1)\,n)\times 1 \tag{9}$$

where $[R]$ is the real initial impedance matrix and $[\Delta K]$ the perturbation coming from the parameters.

Thus, the forced response spectra is given by solving the linear system

$$X_2 = ([R] + [\Delta K])^{-1}\cdot F_2\ (2(N+1)\,n)\times 1. \tag{10}$$

But there is a limit to this method: the loading. Indeed, problems with non-periodic excitations cannot be treated by this method. The latter can only be applied to harmonic or periodic excitations such as the Fourier series decomposed.

2.2. NUMERICAL IMPLEMENTATION

In order to proceed to a good numerical implementation, it is necessary to use the following procedure: (1) initialization; (2) Fourier transformations of the loadings $f(t)$ and the parameters' modulation $g(t)$; (3) calculation and assembly of the real initial impedance $[R]$; (4) convolution matrix construction $[\Delta K]$; (5) the parametric forced response obtained by inverting the modified impedance matrix.

2.3. APPLICATION AND RESULTS

In order to validate the DSM, several periodic damped systems have been studied and numerical results have been systematically compared with those obtained by using a classical time integration schemes (Runge–Kutta second and third or fourth and fifth order). In this part, two examples with a damped/spring/mass system illustrate a comparative study between the proposed method and RK to test the efficiency of the DSM. The characteristics of these cases are displayed in Table 1 and Figure 1.

The results of these two tests (spectra and time responses) are presented in Figures 2–5.

For Figures 3 and 5, the continuous and the dotted lines represent the displacement obtained by the DSM and the Runge–Kutta method respectively.

Typical time histories and spectra of the steady state forced response in the case of different frequencies show the influence of the parameters on the nature of the responses.

TABLE 1

Properties of the parametric system

	Case 1	Case 2
m	10 kg	10 kg
c	0·5 N s/m	2 N s/m
$k(t)$	$10 + 2\cos(\omega_P t)$	$10 + 2\cos(\omega_P t)$
$f(t)$	$\cos(\omega_E t)$	$\cos(\omega_E t)$
ω_E	0·4 rad/s	0·1 rad/s
ω_P	0·3 rad/s	2 rad/s

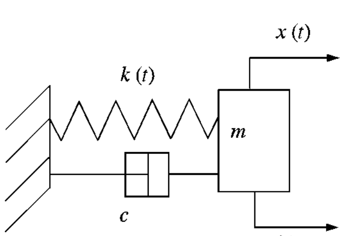

Figure 1. s.d.o.f. parametric system.

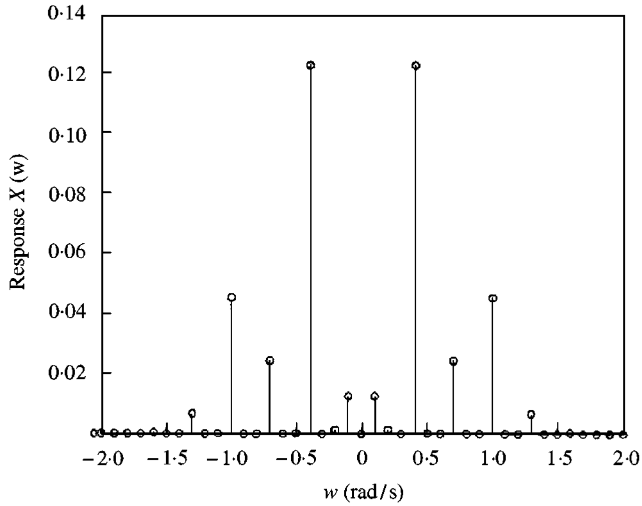

Figure 2. Spectrum of case 1.

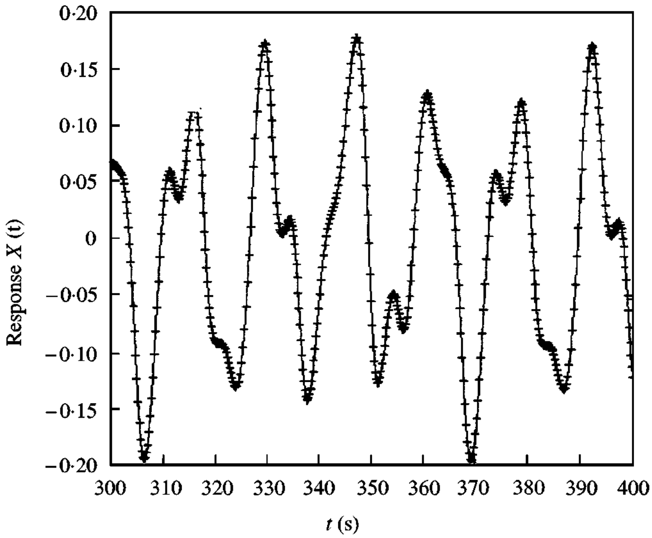

Figure 3. Displacement of case 1.

A comparison between the results of the DSM and those of the Runge–Kutta illustrates the accuracy of the developed method. The maximal relative errors obtained in the chosen examples of the DSM method and the classical method of Runge–Kutta are less than 1% in the first case and around 1% in the second one. This explains why the curbs corresponding to the two methods are superposed on the graphs of Figures 2–5. Moreover, the calculation time has been reduced by 10 times with the DSM.

3. MECHANICAL SYSTEMS LOCALLY PARAMETRIC: DSM AND FORCED REANALYSIS

It is rare to find 100% of the parametric systems. The parameters are often concentrated in a few d.o.f. Furthermore, in DSM, the forced response spectrum is obtained by inverting the modified impedance matrix. Thus, the forced reanalysis could be used to improve the

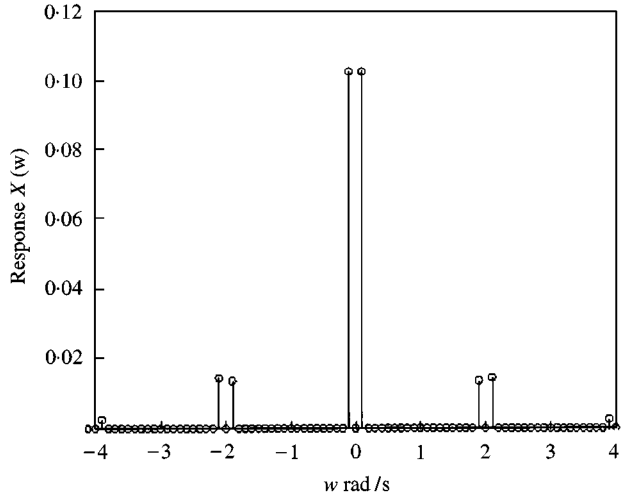

Figure 4. Spectrum of case 2.

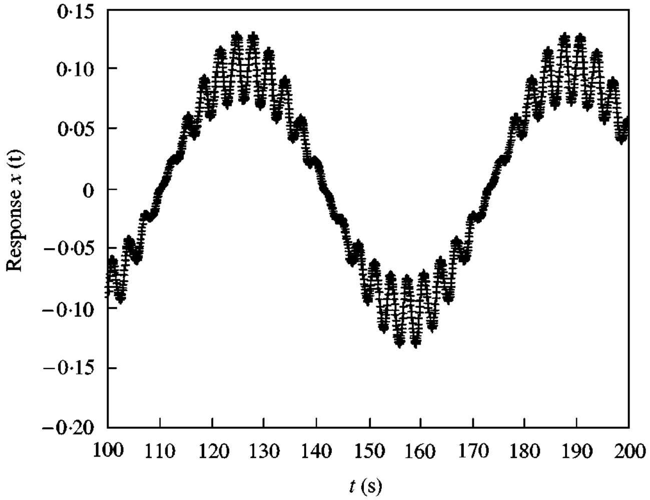

Figure 5. Displacement of case 2.

DSM without loss of accuracy and with time savings [7–9]. A peculiar method has been programmed: the Palazzolo method [9].

After the description of the reanalysis method implementation within the DSM, a practical application is taken into consideration in order to show the ability of the proposed method to deal with a large quantity of d.o.f. systems with localized parameters.

3.1. THEORY AND IMPLEMENTATION

The parametric problem with modified degrees of freedom (m.d.o.f.) in equation (9) could also be partitioned as follows:

$$X_2 = \begin{vmatrix} X_2^1 \\ X_2^2 \end{vmatrix}, \quad F_2 = \begin{vmatrix} F_2^1 \\ F_2^2 \end{vmatrix}, \quad R = \begin{bmatrix} R^{11} & R^{12} \\ R^{21} & R^{22} \end{bmatrix}, \quad R^{-1} = \begin{bmatrix} S^{11} & S^{12} \\ S^{21} & S^{22} \end{bmatrix}, \quad [\Delta K] = \begin{bmatrix} 0 & 0 \\ 0 & \Delta K^{22} \end{bmatrix}, \tag{11}$$

where X_2^1 and F_2^1 are the spectra of non-modified d.o.f., X_2^2 and F_2^2 those of m.d.o.f.s, respectively, and the inverse impedance matrix R^{-1} obtained by Gaussian elimination or by its spectral formulation.

By this condensation, equation (10) is written as

$$\begin{bmatrix} R^{11} & R^{12} \\ R^{21} & R^{22} \end{bmatrix} \begin{vmatrix} X_2^1 \\ X_2^2 \end{vmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \Delta K^{22} \end{bmatrix} \begin{vmatrix} X_2^1 \\ X_2^2 \end{vmatrix} = \begin{vmatrix} F_2^1 \\ F_2^2 \end{vmatrix} \quad (2(N+1)n) \times 1. \tag{12}$$

The impedance matrix could be factorized and inverted. This yields

$$\begin{vmatrix} X_2^1 \\ X_2^2 \end{vmatrix} + \begin{bmatrix} S^{11} & S^{12} \\ S^{21} & S^{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \Delta K^{22} \end{bmatrix} \begin{vmatrix} X_2^1 \\ X_2^2 \end{vmatrix} = \begin{bmatrix} S^{11} & S^{12} \\ S^{21} & S^{22} \end{bmatrix} \begin{vmatrix} F_2^1 \\ F_2^2 \end{vmatrix} = \begin{vmatrix} X_{02}^1 \\ X_{02}^2 \end{vmatrix} \quad (2(N+1)n) \times 1, \tag{13}$$

where the right-hand side (RHS) part is the initial non-parametric forced response spectra (X_{02}^1 and X_{02}^2).

Equation (13) could also be rewritten into two subsets:

$$\begin{aligned} X_2^1 + S^{12}\Delta K \cdot X_2^2 &= X_{02}^1, \quad (2(N+1)(n-m)) * 1, \\ (I + S^{22}\Delta K) \cdot X_2^2 &= X_{02}^2, \quad (2(N+1)m) * 1. \end{aligned} \tag{14}$$

Thus, the new parametric forced response is obtained by

$$\begin{aligned} X_2^2 &= (I + S^{22}\Delta K)^{-1} \cdot X_{02}^2, \\ X_2^1 &= -S^{12}\Delta K \cdot X_2^2 + X_{02}^1. \end{aligned} \tag{15}$$

From this condensed formulation, only a reduced part of the initial impedance is required. This results in a noticeable computation time savings. It also leads to a reduction of memory size needed in its numerical implementation. This last property cannot be neglected since the size of the parametric problems we have to cope with in the study of mechanical systems, increases more and more.

Moreover, this method is more efficient if the forced response spectra are needed for a range of loadings at different frequencies. The main limit comes from the number of d.o.f.s affected by the parameters. If this number is too high, the reanalysis computation can be an inadequate substitute for the DSM. Therefore, a comparative study in terms of operation costs was carried out and, for a given configuration showed a limit in terms of numbers of d.o.f. affected by parameters, a limit beyond which a reanalysis is of no use.

In order to proceed to a good numerical implementation of the forced reanalysis applied to the resolution of parametric systems, the procedure described in Figure 6 is recommended.

3.2. INITIAL DATA

In order to evaluate the influence of the implementation of reanalysis methods in DSM, it is interesting to study a clamped beam with parametric springs. A comparison on CPU time is made between these two methods. Figure 7 presents the modelled structure built with eight finite elements, having 16 d.o.f.s and four parametric springs. The mechanical characteristics can be seen in Table 2.

Two kinds of tests have been carried out:

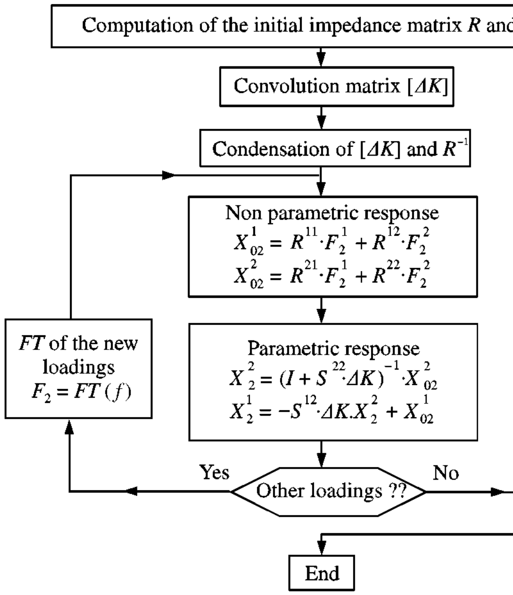

Figure 6. Forced reanalysis applied to the resolution of parametric systems.

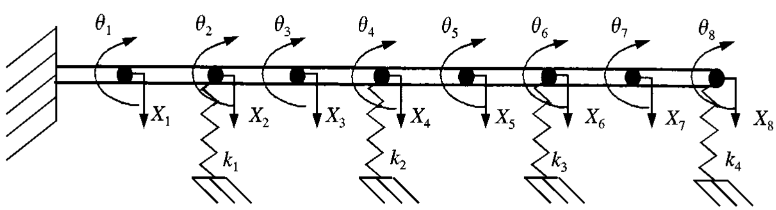

Figure 7. Free/clamped beam with parametric mounts.

TABLE 2

Mechanical characteristics of the parametric clamped beam

Second moment of inertia of the beam cross-section (m^4)	5×10^{-9}
Young's modulus of steel (N/m^2)	$2{\cdot}1 \times 10^{11}$
Mass density (kg/m^3)	$7{\cdot}8 \times 10^{3}$
Number of degree of freedom n d.o.f. (1 translation and 1 rotation per node)	16
External force $f(N)$ On node 8 vertically	$1 \sin(\omega_E t)$
ω_E(rad/s)	20
C(N s/m) or (N s/rad)	1

Case 1: The structure displayed with one parametric spring (k_4). This one has 6·25% of its d.o.f.s modified by the parameters.

Case 2: The structure displayed with two parametric springs (k_3 and k_4). This one has 12·5% of its d.o.f.s modified by the parameters.

TABLE 3

Parameters characteristics

Case	k_1 (N/m)	k_2 (N/m)	k_3 (N/m)	k_4 (N/m)
1	100	100	100	$100(1 + 0{\cdot}5\sin(5t))$
2	100	100	$100(1 + 0{\cdot}3\sin(7t))$	$100(1 + 0{\cdot}5\sin(5t))$

TABLE 4

CPU times for each stages of the calculation of the response

		Case 1			Case 2		
	Calculation stages	DSM rea (s)	DSM (s)	Runge–Kutta	DSM rea (s)	DSM (s)	Runge–Kutta
1	Assembly	0·11	0·11		1·44	1·44	
2	Convolution	0·99	0·99		5·88	5·88	
3	Condensation	0·01			0·05		
4	Invert R	0·09			0·43		
5	Non-parametric response	0·01			0·05		
6	Parametric response by Palazzolo	0·01			0·26		
7	Parametric response		0·11	2000		0·62	2500
	Total	1·22	1·21	2000	8·11	7·94	2500

The characteristics of the parameters are described in Table 3.

3.3. RESULTS

The CPU times for the different methods used are presented in Table 4. The computer used has a pentium 166 MHz processor and 16 Mbytes of ram.

As can be seen, the DSM reanalyzed (DSM rea) or not (DSM), is more efficient than the traditional method. In these particular cases, the forced reanalysis seems to be of no interest. But its efficiency increases when the forced response is needed for a range of frequencies. Thus, it is useless to compute all processes (only 5 6). Only the computation of the non-parametric and parametric response by Palazzolo are necessary. The graphics of Figures 8 and 9 show the CPU time evolution for an increasing number of frequencies. This proves the efficiency of reanalysis methods in the treatment of parametric systems

4. CONCLUSIONS

The direct spectral method represents a powerful technique for the analysis of parametric systems.

The major advantage of this method is the ability to quickly calculate the parametric system forced response in the field of frequency. In addition, the use of forced reanalysis improves this new method and makes the treatment of large parametric systems possible.

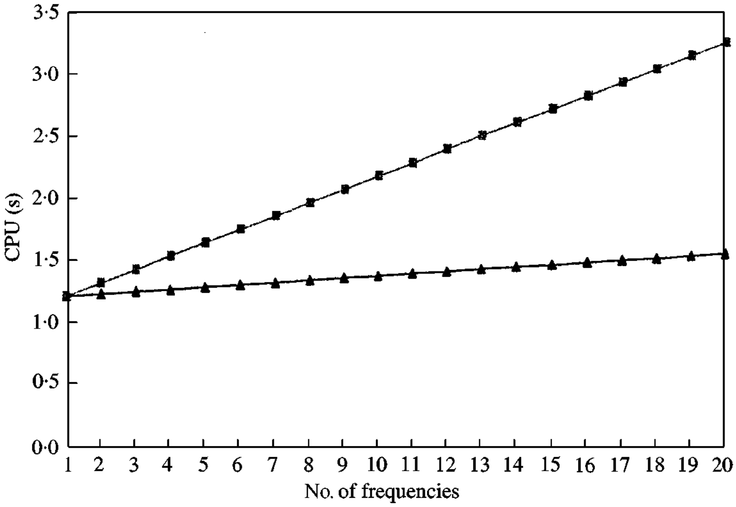

Figure 8. CPU time for a range of frequencies and for case 1.

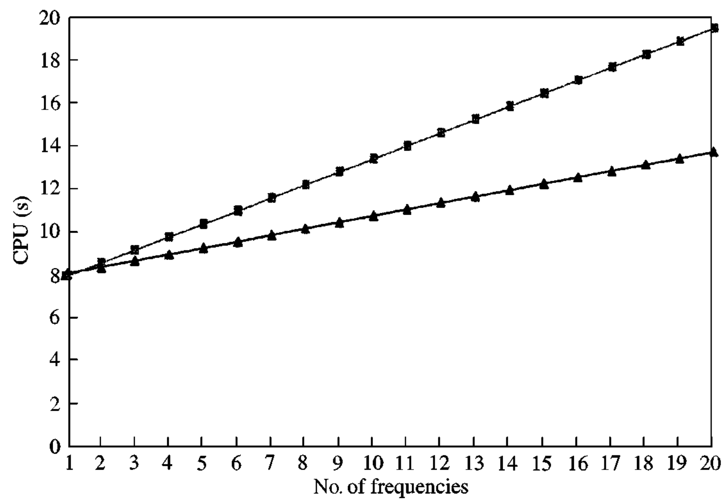

Figure 9. CPU time for a range of frequencies and for case 2.

Thanks to localized parameters, all matrices are sparse and they also reduce the data storage.

The substantial progress shown in this study is the implementation of a method which, by combining the reanalysis techniques, enables considerable time savings together with a very good quality in terms of accuracy of the results obtained. This method reveals its strength during the conception process when many successive analyses are necessary in order to find the optimal structure.

The method displayed here represents, as a consequence, a very attractive substitute for the methods traditionally used in computational codes.

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APPENDIX A: CONVOLUTION

The Fourier transformation of the product of two time functions is the convolution of the Fourier transformations of the time functions (A1). This kind of convolution is called circular convolution due to its properties of symmetry and periodicity:

$$[k]\cdot g(t)\cdot x(t) \xrightarrow{\text{Fourier transform}} [k]\,\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\omega\omega}\, g(t)\cdot x(t)\,\mathrm{d}t = [k]\cdot(G\otimes X)(\omega). \quad \text{(A1)}$$

The integral which appears in equation (A1) can be expressed in a discrete form (A2) and is truncated by the lower $(-N)$ and the upper $(+N)$ boundaries.

$$[k]\cdot(G\otimes X)(n\cdot\omega_s) = \sum_{m=-N}^{+N} [k]\cdot G((n-m)\cdot\omega_s)\cdot X(m\cdot\omega_s) \quad (n\times 1) \qquad \text{(A2)}$$

with

$$X(m\cdot\omega_s) = \left| \begin{matrix} X_1(m\cdot\omega_s) \\ \vdots \\ X_{n\ \mathrm{d.o.f.}}(m\cdot\omega_s) \end{matrix} \right. \quad (n\times 1).$$

Additionally, if $X(-m\cdot\omega_s) = \bar{X}(m\cdot\omega_s)$, then equation (A2) could be rewritten as

$$[k]\cdot(G\otimes X)(n\cdot\omega_s) = \sum_{m=0}^{+N} a\cdot[k]\cdot(G((n-m)\cdot\omega_s)\cdot X(m\cdot\omega_s) + G((n+m)\cdot\omega_s)\cdot\bar{X}(m\cdot\omega_s)) \qquad \text{(A3)}$$

with $a = 1$ for $m > 0$ and $a = 0{\cdot}5$ for $m = 0$.

This complex formulation could be rewritten into a real formulation considering that

$$\left| \begin{matrix} \Re(\bar{X}) \\ \Im(\bar{X}) \end{matrix} \right. = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \left| \begin{matrix} \Re(X) \\ \Im(X) \end{matrix} \right. \quad (2n\times 1), \qquad \text{(A4)}$$

where I is a square $(2n)$ identity matrix.

Substituting equation (A4) into equation (A3), one obtains

$$[k]\cdot(G\otimes X)(n\cdot\omega_s)$$

$$= \sum_{m=0}^{+N} a\begin{pmatrix} \Re(G((n-m)\cdot\omega_s)\cdot[k]) & -\Im(G((n-m)\cdot\omega_s)\cdot[k]) \\ \Im(G((n-m)\cdot\omega_s)\cdot[k]) & \Re(G((n-m)\cdot\omega_s)\cdot[k]) \end{pmatrix}\left|\begin{matrix}\Re(X(m\cdot\omega_s)) \\ \Im(X(m\cdot\omega_s))\end{matrix}\right.$$

$$+ a\begin{pmatrix} \Re(G((n+m)\cdot\omega_s)\cdot[k]) & -\Im(G((n+m)\cdot\omega_s)\cdot[k]) \\ \Im(G((n+m)\cdot\omega_s)\cdot[k]) & \Re(G((n+m)\cdot\omega_s)\cdot[k]) \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix}\left|\begin{matrix}\Re(X(m\cdot\omega_s)) \\ \Im(X(m\cdot\omega_s))\end{matrix}\right.$$

$$= \sum_{m=0}^{+N} a\begin{pmatrix} \Re((G((n-m)\cdot\omega_s)+G((n+m)\cdot\omega_s))\cdot[k]) & \Im((-G((n-m)\cdot\omega_s)+G((n+m)\cdot\omega_s))\cdot[k]) \\ \Im((G((n-m)\cdot\omega_s)+G((n+m)\cdot\omega_s))\cdot[k]) & \Re((G((n-m)\cdot\omega_s)-G((n+m)\cdot\omega_s))\cdot[k]) \end{pmatrix}\left|\begin{matrix}\Re(X(m\cdot\omega_s)) \\ \Im(X(m\cdot\omega_s)).\end{matrix}\right. \tag{A5}$$

Thus, one obtains a generalized formulation

$$[k]\cdot(G\otimes X)\,(\omega) = [\Delta K]\cdot X_2 \;((2(N+1)\,n)\times 1), \tag{A6}$$

where

$$X_2 = \left|\begin{matrix}\Re(X(0)) \\ \Im(X(0)) \\ \vdots \\ \Re(X(N)) \\ \Im(X(N))\end{matrix}\right. .$$

APPENDIX B: NOMENCLATURE

n	number of degrees of freedom
RK	Runge–Kutta method
DSM	direct spectral method
DSM rea	direct spectral method reanalyzed

Time domain

t	time

Time invariant characteristics

$[M]$	mass matrix
$[K_0]$	initial stiffness matrix
$[C]$	damping matrix

Parametric characteristics

$[k]$	parametric stiffness's amplitude
$g(t)$	parametric stiffness's modulation
$x(t)$	generalized displacement
$\dot{x}(t)$	generalized velocity
$\ddot{x}(t)$	generalized acceleration
$f(t)$	generalized forces

Frequency domain

$\Re$	real part
$\Im$	imaginary part
i	complex value $i^2 = -1$
ω	forcing frequency
ω_s	sampling frequency
FT	Fourier transformation
N	no. of discretized frequencies
$\otimes$	circular convolution
$G(\omega)$	amplitude of parameters
$R(\omega)$	real initial impedance matrix at ω
$F(\omega)$	complex or $F_2(\omega)$ real amplitude force
$X(\omega)$	complex or $X_2(\omega)$ real amplitude displacement
$\overline{(\ )}$	complex conjugate

Forced reanalysis

$\Delta(\ )$	changes