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NUMERICAL SOLUTION FOR DIFFERENTIAL EQUATIONS OF DUFFING-TYPE NON-LINEARITY USING THE GENERALIZED DIFFERENTIAL QUADRATURE RULE

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The generalized differential quadrature rule (GDQR) proposed recently by the authors is applied for the first time to second and fourth order initial-value differential equations with Duffing-type non-linearity. Procedures are given in detail to convert these non-linear differential equations into a set of linear algebraic equations in an iterative loop using the Frechet derivative. The effectiveness of the GDQR for obtaining the periodic solution of the Duffing equation has been demonstrated through a number of examples. It is also shown that the use of the Frechet derivative makes it easier for the GDQR to handle non-linearity. The wide applicability of the GDQR is manifested further through this work.

1. INTRODUCTION

Today, system-governing equations of mechanics and other scientific disciplines are often solved using numerical methods. Topics concerned with computational methods have emerged as key areas of research and education throughout the world. In structural dynamics, the numerical solution of the governing equations in time domain is often obtained through the finite-difference (FD) techniques. The essence of the FD method is to replace the derivatives at a discrete point with difference quotients using only a few adjacent points. The FD method often employs low order polynomial test functions and obtains results of medium accuracy.

As a general method to solve boundary/initial-value differential equations, a generalized differential quadrature rule (GDQR) was proposed recently by the authors [1–4]. The GDQR employs high order test functions at all the discrete grid points to obtain a solution of high accuracy. The GDQR is a generalization of differential quadrature method (DQM) [1] and has been applied to initial-value linear differential equations of second through fourth orders [2]. The newly proposed DQ [3] is precisely the GDQR though the term GDQR is not used. The GDQR has also been applied to two-point boundary-value differential equations of fourth, sixth and eighth orders in structural mechanics [4–10]. Static and free vibrational analyses of two-dimensional rectangular plates have been completed using the GDQR [10]. Third order non-linear Blasius differential equations and sixth order Onsager equations in fluid mechanics have been successfully solved using the GDQR, although their domains are unbounded [11]. Coupled differential equations have also been effectively dealt with [4, 11].

The focus in this paper is on a novel application of the GDQR to the non-linear initial value differential equations, such as Duffing's equations encountered in structural dynamics. A method using trigonometric series has been proposed and applied to second and fourth order Duffing's equations by Groves [12]. However, two computational difficulties were encountered—the lengthy and complex algebraic manipulations and the difficulty of accurately solving the resultant non-linear algebraic equations. The GDQR is applied here to overcome these difficulties through simple programmable procedures. The non-linearity is dealt with using the Frechet derivative [11, 13] to convert the non-linear differential equation into linear differential equation in an iteration process. A comparative study is conducted using the present GDQR and the method of trigonometric series. Very good results have been obtained. It is also shown that the use of the Frechet derivative makes easier the manipulation of the non-linear differential equations in the GDQR.

This work applies for the first time the GDQR to second and fourth order initial-value differential equations with Duffing-type non-linearity. Procedures are given in detail to convert these non-linear differential equations into a set of linear algebraic equations in an iterative loop using the Frechet derivative. The wide applicability of the GDQR is manifested further through some examples in this work.

2. DUFFING'S EQUATIONS

Duffing's equation is a group of classical non-linear initial-value differential equations

$$y^{(2)} + y + Ry^3 = F \sin \omega t, \tag{1}$$

where $y^{(r)} = \mathrm{d}^r y(t)/\mathrm{d}t^r$, y is an unknown function, t is the axis, ω is a given parameter (frequency), and F and R are given constants.

The trigonometric series method has been applied to approximate its periodic solutions [12]

$$y = a_1 \sin \omega t + a_3 \sin 3\omega t + a_5 \sin 5\omega t + \cdots . \tag{2}$$

Substitution of the series, after truncation, into equation (1) yields a set of non-linear algebraic equations for the coefficients a_n after equating coefficients of the same terms. Periodic solution of the forced Duffing equation for the case of $F = 2$, $\omega = 1$, and $R = -1/6$ was obtained as [12]

$$y = -2{\cdot}5425 \sin t - 0{\cdot}07139 \sin 3t - 0{\cdot}00219 \sin 5t. \tag{3}$$

Its initial conditions at $t = 0$ should thus be

$$y = 0, \qquad y^{(1)} = -2{\cdot}7676. \tag{4}$$

Physically, the homogeneous Duffing equation ($F = 0$) represents the free vibration of a pendulum. The frequency of the oscillations depends on the initial conditions of the pendulum. Periodic solution for the case of $\omega = 0{\cdot}7$ and $R = -1/6$ was found as [12]

$$y = 2{\cdot}058 \sin 0{\cdot}7t + 0{\cdot}0816 \sin 2{\cdot}1t + 0{\cdot}00337 \sin 3{\cdot}5t. \tag{5}$$

One should have the initial conditions at $t = 0$ as

$$y = 0, \qquad y^{(1)} = 1{\cdot}62376. \tag{6}$$

The series solution method of equation (2) is also applicable to higher order equations, when it is applied to the following Duffing equation:

$$y^{(4)} + 5y^{(2)} + 4y - \tfrac{1}{6}y^3 = 0. \tag{7}$$

The solution was obtained for $\omega = 0{\cdot}9$ as follows:

$$y = 2{\cdot}1906 \sin 0{\cdot}9t - 0{\cdot}02247 \sin 2{\cdot}7t + 0{\cdot}000045 \sin 4{\cdot}5t. \tag{8}$$

The appropriate initial conditions are

$$y = 0, \qquad y^{(1)} = 1{\cdot}91103, \qquad y^{(2)} = 0, \qquad y^{(3)} = -1{\cdot}15874. \tag{9}$$

3. THE GDQR

The GDQR is briefly reviewed here for completeness of the present work. When the field function $\psi(x)$, governed by a differential equation, is constrained by one or more than one condition at any individual point, we first divide the solution domain with the points x_i $(i = 1, 2, \ldots, N)$ that include all the points with the given conditions. Note that only governing equation is to be satisfied for some points. If n_i conditions (equations) are to be satisfied at point x_i, the GDQR is expressed as follows [1–4]:

$$\frac{\mathrm{d}^r\psi(x_i)}{\mathrm{d}x^r} = \sum_{j=1}^{N} \sum_{l=0}^{n_j-1} E_{ijl}^{(r)} \psi_j^{(l)} = \sum_{m=1}^{M} E_{im}^{(r)} P_m \quad (i = 1, 2, \ldots, N), \tag{10}$$

where $E_{im}^{(r)}$ (which is a convenient expression of $E_{ijl}^{(r)}$) is the weighting coefficient corresponding to the rth order derivative at point x_i, and $M = \sum_{i=1}^{N} n_i$ is the number of the total independent variable P_m, which is expressed in series as

$$\{P\}^{\mathrm{T}} = \{P_1, P_2, \ldots, P_m \ldots, P_M\} = \{\psi_1^{(0)}, \psi_1^{(1)}, \ldots, \psi_1^{(n_1-1)}, \ldots, \psi_N^{(0)}, \psi_N^{(1)}, \ldots, \psi_N^{(n_N-1)}\},$$

where $\psi_i = \psi_i^{(0)} = \psi(x_i)$ is the function value, and $\psi_i^{(l)} = \psi^{(l)}(x_i)$ $(l = 1, 2, \ldots, n_i - 1)$ are its derivatives.

It is clearly shown from equation (10) that the GDQR forces the same number of independent variables $\psi^{(l)}(x_i)$ $(l = 0, 1, 2, \ldots, n_i - 1)$ as that of the equations at a point, and that its independent variables are chosen as function value and its derivatives of possible lowest order wherever necessary. The DQM chooses only function values as independent variables. The reason why the term—generalized—is employed in the GDQR is that the GDQR is fully generalized for the differential equations of any finite order. The conventional DQM is usually confined within second order boundary-value differential equations or first order initial-value differential equations [14]. In a most recent review work, Bellomo [14] mentioned the generalization of the DQM but gave no suggestions.

4. APPLICATION OF THE GDQR

For a kth order initial-value differential equation, there are k initial conditions at the initial point, and all the other discrete points just need to satisfy the governing equation. Therefore, the GDQR expression can be inferred from equation (10) as follows:

$$y^{(r)}(t_i) = \sum_{j=1}^{N+k-1} E_{ij}^{(r)} G_j \quad (i = 1, 2, \ldots, N), \tag{11}$$

where $\{G_1, G_2, \ldots, G_N, G_{N+1}, \ldots, G_{N+k-1}\} = \{y_1, y_2, \ldots, y_N, y_N^{(1)}, \ldots, y_N^{(k-1)}\}$. For the second and fourth order differential equations which will be solved in this work, we have $k = 2$ and 4 respectively.

Special attention should be paid to the numbering of sampling points. When the solution domain is chosen as $[0, T]$ and the number of sampling points is N, the initial point is denoted as t_N and the initial conditions are expressed as $y_N, y_N^{(1)}, \ldots, y_N^{(k-1)}$, since we use an inverse node numbering convention in initial-value problems [2]. The numbering for the discrete points can be arbitrary on the principle of the interpolation theory. The inverse numbering is used just for the convenience of notation and programming. Equation (11) for the cases of $k = 2$ and 4 has been derived and applied to linear differential equations [2]. The corresponding weighting coefficients have been obtained there by these authors and will be used directly in this work later.

Non-linear differential equations (1) and (7) can be converted into linear differential equations using the Frechet derivative and then into a set of linear algebraic equations using equation (11). Consider equation (1) as an example. Newton's approach is first adopted wherein one begins with assumed function values consistent with the initial condition—equation (4). Then successively refined solutions are obtained through the following iteration scheme:

$$y^{[m+1]} = y^{[m]} + \theta^{[m]}, \tag{12}$$

where $y^{[m]}$ and $\theta^{[m]}$ are the function value and its refinement and m is the iteration count.

The function value refinement can be obtained through solving the following equation written in an operator form as [11, 13]

$$L^{(1)}(\theta) + L(y) = 0, \tag{13}$$

where the operator $L(y)$ is obtained according to equation (1), i.e.,

$$L(y) = \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + y + Ry^3 - F\sin\omega t \tag{14}$$

and $L^{(1)}(\theta)$ is the Frechet derivative defined as

$$L^{(1)}(\theta) = \frac{\partial}{\partial\varepsilon} L(y + \varepsilon\theta)|_{\varepsilon=0}. \tag{15}$$

In order to evaluate the Frechet derivative, y is replaced by $y + \varepsilon\theta$ in equation (14) to obtain

$$L(y + \varepsilon\theta) = \frac{\mathrm{d}^2(y + \varepsilon\theta)}{\mathrm{d}t^2} + (y + \varepsilon\theta) + R(y + \varepsilon\theta)^3 - F\sin\omega t.$$

The above equation is differentiated partially with respect to ε, and then ε is set equal to zero in the resultant derivative. Substituting the Frechet derivative so determined into equation (13), one obtains the following equation:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + (1 + 3Ry^2)\theta = -\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - y - 3Ry^3 + F\sin\omega t. \tag{16}$$

Since θ is correction function, the initial condition equations (4), (6) and (9) can be combined and obtained as follows, where $k = 2$ for equations (4) and (6) and $k = 4$ for equation (9), respectively,

$$\theta(t_N) = \theta^{(1)}(t_N) = \cdots = \theta^{(k-1)}(t_N) = 0. \tag{17}$$

The cosine-type grid points are often used in the differential quadrature since it usually produces accurate results. The grid points are then expressed as

$$t_i = \frac{1}{2}\left[1 - \cos\frac{N-i}{N-1}\pi\right]T, \tag{18}$$

where it is clearly shown that the initial point is t_N. Equation (16) is a linear second order differential equation for the function refinement θ, along with two initial conditions at point t_N given by equation (17). For the other points, only the differential equation (16) is to be satisfied. Based on the GDQR definition about independent variables, we have k ($n_N = k$) independent variables $\theta_N, \theta_N^{(1)}, \ldots, \theta_N^{(k-1)}$ at t_N, and one ($n_i = 1$) independent variable θ_i at all the other points t_i ($i = 1, 2, 3, \ldots, N-1$). Therefore, the corresponding GDQR expression for the function θ can be written as

$$\theta^{(r)}(t_i) = \sum_{j=1}^{N+k-1} E_{ij}^{(r)} U_j \quad (i = 1, 2, \ldots, N), \tag{19}$$

which is similar to equation (11) where $\{U_1, U_2, \ldots, U_N, U_{N+1}, \ldots, U_{N+k-1}\} = \{\theta_1, \theta_2, \ldots, \theta_N, \theta_N^{(1)}, \ldots, \theta_N^{(k-1)}\}$, and $k = 2$ for second order differential equation and $k = 4$ for fourth order differential equation. Since equation (17) must be satisfied, the superscript $N + k - 1$ in equation (19) can be changed as $N - 1$.

Then successively refined independent variables are obtained through the following iteration scheme:

$$\{G\}^{[m+1]} = \{G\}^{[m]} + \{U\}^{[m]}, \tag{20}$$

where $\{G\}^{[m]}$ and $\{U\}^{[m]}$ are their respective values and m is the iteration count.

The differential equation (16) is then expressed using equations (11) and (19) as follows, where the count number m is omitted:

$$\sum_{j=1}^{N-1} E_{ij}^{(2)}\theta_j + (1 + 3Ry_i^2)\theta_i = b_i \quad (i = 1, 2, 3, \ldots, N-1), \tag{21}$$

where the coefficient

$$b_i = -\sum_{j=1}^{N+1} E_{ij}^{(2)} G_j - y_i - 3Ry_i^3 + F\sin\omega t_i. \tag{22}$$

The iteration procedures are as follows. First, the initial values of independent variables $\{G\}^{\mathrm{T}} = \{y_1, y_2, \ldots, y_N, y_N^{(1)}, \ldots, y_N^{(k-1)}\}$ are assumed according to the initial conditions. Second, coefficient b_i in equation (21) is calculated using equation (22). The $N - 1$ variables $\{\theta_1, \theta_2, \ldots, \theta_{N-1}\}$ can be calculated from the $N - 1$ linear algebraic equations in equation

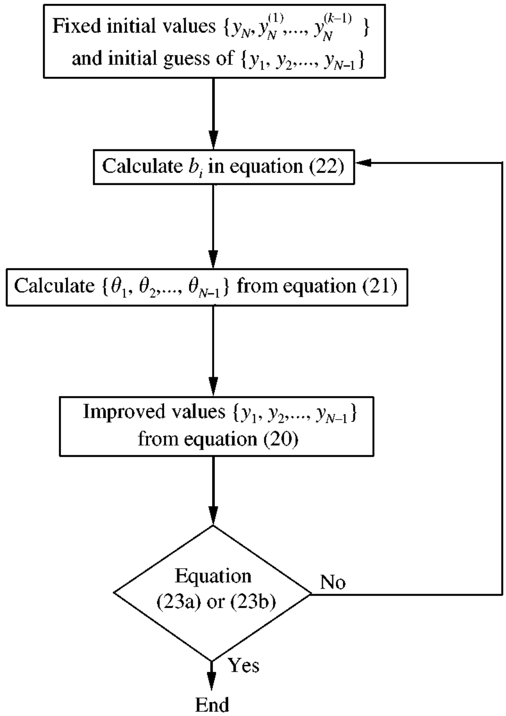

Figure 1. Diagram for the iterative solution procedure of second order Duffing equation.

(21). The refinement of $\{G\}$ is obtained from equation (20). Third, apply the refined $\{G\}$ to the next round of iteration starting with the above-mentioned second step and thus obtain a new $\{U\}$. The following convergence criterion of equation (23a) or (23b) was used for controlling the iteration scheme:

$$\left(\sum_{j=1}^{N+k-1} (U_j)^2\right)^{[m+1]} \Bigg/ \left(\sum_{j=1}^{N+k-1} (U_j)^2\right)^{[m]} \leqslant 10^{-6}, \tag{23a}$$

$$|U_j| \leqslant 10^{-6} \quad (j = 1, 2, \ldots, N-1). \tag{23b}$$

If the convergence criterion is satisfied, we obtain the required independent variables $\{G\}$. If not, continue the iteration starting with the second step. In order to elaborate the iterative procedure, a block diagram in Figure 1 shows all the major steps in a logical sequence. The solution procedures of unforced second order Duffing equations are the same if one imposes $F = 0$ in equation (1).

For the fourth order initial value differential equation (7) together with the initial condition equation (9), similar solution procedures are applied. If we still employ the refinement function θ, the GDQR expression for function θ can be written as equation (19) for $k = 4$.

The solution procedures for fourth order equations are similar to those for second order equations. Therefore, only the GDQR expansion for the refinement function is written accordingly as

$$\sum_{j=1}^{N-1} E_{ij}^{(4)}\theta_j + 5\sum_{j=1}^{N-1} E_{ij}^{(2)}\theta_j + \left(4 - \frac{y_i^2}{2}\right)\theta_i = b_i \quad (i = 1, 2, 3, \ldots, N-1), \tag{24}$$

where the coefficient

$$b_i = -\sum_{j=1}^{N+3} E_{ij}^{(4)} G_j - 5\sum_{j=1}^{N+3} E_{ij}^{(2)} G_j - 4y_i + \frac{y_i^3}{6}. \tag{25}$$

5. DISCUSSIONS AND CONCLUSIONS

The generalized differential quadrature rule is used here to deal with the second and fourth order initial value differential equations with Duffing-type non-linearity. Based on the definition of the GDQR, the second order equation has two independent variables at the initial point and one independent variable at any other discrete point. The non-linear Blasius equation encountered in fluid mechanics [11] has also been solved using the GDQR. Third order boundary value Blasius equation has two boundary conditions at one point and one boundary condition at the other point. Both second order initial-value differential equation and third order boundary-value Blasius equation have two independent variables at one point and one independent variable at all the other discrete points. Therefore, their respective GDQR expressions are one and the same. In a word, the GDQR expressions neither distinguish between the initial-value equation and the boundary-value one nor make a distinction between the different orders.

For both the second and fourth order Duffing equations, the GDQR expression and their corresponding weighting coefficients are directly adopted from reference [2]. Non-linear Duffing equations are converted into linear differential equations using the Frechet derivative and then into a set of linear algebraic equations using the GDQR. The use of Frechet derivative makes it easier to overcome the non-linearity through simple programmable procedures. There is no restriction on the choice of the independent variables at the first iteration step, which are usually taken as zero. However, a non-linear algebraic equation has to be solved using pattern search or Newton's method in employing the trigonometric series method [12]. Newton's method sometimes yields only the trivial solution of $a_1 = a_3 = a_5 = a_7 = 0$, even when it starts very close to the desired solutions. Another disadvantage of trigonometric series method is that the frequency must be assumed beforehand and the initial condition is obtained later. This is contrary to the normal case where the initial value is known at first.

The cosine type of sampling-point distribution is used in all cases of this work. The number of sampling points is employed as follows: $N = 9$, 17, 15 for Tables 1, 2, 3, respectively, $N = 15$, 25 for Figures 2 and 3, respectively, and $N = 35$ for Figure 4. The calculated results are interpolated using the Lagrange interpolation. In initial-value problems, a decision has to be made to properly choose the domain length T. The accuracy decreases with the increase in domain length. For the forced Duffing equation, one will not obtain convergent results if the domain length is increased beyond more than one period. Therefore, only half-period solution is obtained and shown in Figure 2. For second and fourth order unforced equations, one can obtain the convergent solutions for an interval of more than one period. Figures 3 and 4 display one-period solutions. Tables 1–3 list convergent solutions for even longer domain length. Very good results are observed. Some

Table 1

Comparison of results for the forced Duffing equation of second order

	Displacement			Velocity			Acceleration		
t	GDQR	Equation (3)	Error (%)	GDQR	Equation (3)	Error (%)	GDQR	Equation (3)	Error (%)
4	1·9682	1·9605	– 0·392	1·4781	1·4767	– 0·098	– 2·2111	– 2·2189	0·353
3·8	1·6285	1·6210	– 0·466	1·9160	1·9159	– 0·002	– 2·1324	– 2·1381	0·268
3·6	1·2044	1·1968	– 0·636	2·3139	2.3144	0·022	– 1·7980	– 1·7965	– 0·084
3·4	0·7094	0·7018	– 1·089	2·6148	2·6141	– 0·028	– 1·1618	– 1·1520	– 0·845
3·2	0·1688	0·1615	– 4·500	2·7622	2·7595	– 0·097	– 0·2844	– 0·2762	– 2·967
3	– 0·3830	– 0·3896	1·692	2·7242	2·7205	– 0·137	0·6566	0·6592	0·395
2·8	– 0·9090	– 0·9149	0·639	2·5094	2·5053	– 0·162	1·4535	1·4550	0·106
2·6	– 1·3779	– 1·3829	0·362	2·1617	2·1571	– 0·210	1·9724	1·9752	0·144
2·4	– 1·7688	– 1·7728	0·226	1·7402	1·7353	– 0·284	2·1975	2·1979	0·020
2·2	– 2·0726	– 2·0757	0·146	1·2972	1·2927	– 0·347	2·2059	2·2010	– 0·221
2	– 2·2885	– 2·2907	0·100	0·8648	0·8616	– 0·372	2·1096	2·1026	– 0·333
1·8	– 2·4199	– 2·4217	0·074	0·4538	0·4517	– 0·456	2·0059	2·0021	– 0·191
1·6	– 2·4711	– 2·4725	0·057	0·0588	0·0571	– 3·008	1·9556	1·9555	– 0·001
1·4	– 2·4437	– 2·4447	0·042	– 0·3337	– 0·3354	0·520	1·9820	1·9815	– 0·027
1·2	– 2·3368	– 2·3375	0·031	– 0·7384	– 0·7397	0·177	2·0739	2·0701	– 0·182
1	– 2·1468	– 2·1474	0·027	– 1·1643	– 1·1648	0·040	2·1814	2·1776	– 0·173
0·8	– 1·8699	– 1·8704	0·028	– 1·6061	– 1·6063	0·010	2·2153	2·2164	0·052
0·6	– 1·5050	– 1·5054	0·028	– 2·0381	– 2·0389	0·039	2·0650	2·0690	0·194
0·4	– 1·0584	– 1·0586	0·022	– 2·4137	– 2·4148	0·048	1·6402	1·6387	– 0·090
0·2	– 0·5472	– 0·5473	0·010	– 2·6741	– 2·6745	0·016	0·9173	0·9140	– 0·364
0	0	0	0	– 2·7676	– 2·7676	0·001	0·0121	0	0

TABLE 2

Comparison of results for the homogeneous Duffing equation of second order

	Displacement			Velocity			Acceleration		
t	GDQR	Equation (5)	Error (%)	GDQR	Equation (5)	Error (%)	GDQR	Equation (5)	Error (%)
12	1·7679	1·7612	− 0·383	− 0·5740	− 0·5818	1·349	− 0·8570	− 0·8481	0·138
11·4	1·9709	1·9686	− 0·120	− 0·1181	− 0·1248	5·349	− 0·6952	− 0·6993	0·595
10·8	1·9171	1·9182	0·057	0·3021	0·2969	− 1·744	− 0·7428	− 0·7406	− 0·299
10·2	1·5915	1·5964	0·305	0·8015	0·7943	− 0·907	− 0·9198	− 0·9191	− 0·075
9·6	0·9435	0·9523	0·923	1·3482	1·3427	− 0·411	− 0·8030	− 0·8081	0·629
9	0·0281	0·0390	27·998	1·6242	1·6233	− 0·054	− 0·0281	− 0·0386	27·198
8·4	− 0·8965	− 0·8868	− 1·094	1·3759	1·3806	0·337	0·7749	0·7695	− 0·698
7·8	− 1·5634	− 1·5572	− 0·401	0·8332	0·8387	0·655	0·9288	0·9294	0·062
7·2	− 1·9065	− 1·9031	− 0·178	0·3277	0·3327	1·520	0·7506	0·7524	0·236
6·6	− 1·9744	− 1·9738	− 0·032	− 0·0941	− 0·0913	− 3·107	0·6908	0·6950	0·593
6	− 1·7875	− 1·7881	0·034	− 0·5440	− 0·5415	− 0·467	0·8356	0·8323	− 0·398
5·4	− 1·3004	− 1·3030	0·201	− 1·0915	− 1·0872	− 0·389	0·9374	0·9376	0·020
4·8	− 0·4937	− 0·4986	0·993	− 1·5495	− 1·5477	− 0·119	0·4681	0·4744	1·321
4·2	0·4659	0·4613	− 1·002	− 1·5571	− 1·5587	0·102	− 0·4457	− 0·4414	− 0·972
3·6	1·2801	1·2766	− 0·273	− 1·1074	− 1·1097	0·209	− 0·9310	− 0·9330	0·221
3	1·7773	1·7749	− 0·137	− 0·5602	− 0·5616	0·235	− 0·8416	− 0·8402	− 0·165
2·4	1·9725	1·9714	− 0·058	− 0·1055	− 0·1080	2·272	− 0·6945	− 0·6970	0·362
1·8	1·9116	1·9108	− 0·043	0·3138	0·3147	0·310	− 0·7459	− 0·7464	0·055
1·2	1·5775	1·5771	− 0·027	0·8185	0·8165	− 0·253	− 0·9212	− 0·9245	0·353
0·6	0·9201	0·9198	− 0·040	1·3617	1·3618	0·008	− 0·8012	− 0·7894	− 1·493
0	0	0	0	1·6238	1·6238	0·000	0·1185	0	0

TABLE 3

Comparison of results for the homogeneous Duffing equation of fourth order

	y		$y^{(1)}$		$y^{(2)}$		$y^{(3)}$		$y^{(4)}$	
t	GDQR	Error (%)	GDQR	Error (%)	GDQR	Error (%)	GDQR	Error (%)	GDQR	Error (%)
14	0·0706	1·057	1·9102	− 0·015	− 0·0433	0·345	− 1·1590	− 0·089	− 0·070	4·021
13·3	− 1·2093	− 0·071	1·6442	0·004	0·8368	− 0·088	− 1·4140	− 0·002	0·356	0·466
12·6	− 2·0741	− 0·030	0·7180	0·084	1·7539	− 0·047	− 0·9222	− 0·007	− 1·962	− 0·127
11·9	− 2·1166	− 0·002	− 0·6014	− 0·160	1·8099	0·008	0·7871	− 0·350	− 2·163	− 0·016
11·2	− 1·3136	0·037	− 1·5864	− 0·026	0·9299	0·096	1·4333	0·081	0·228	− 1·333
10·5	− 0·0529	1·146	− 1·9106	− 0·001	0·0325	1·374	1·1585	− 0·004	0·052	− 1·434
9·8	1·2243	− 0·044	− 1·6363	0·010	− 0·8500	0·007	1·4173	0·050	− 0·340	− 1·038
9·1	2·0805	− 0·021	− 0·7015	0·028	− 1·7625	− 0·022	0·9037	− 0·104	1·993	− 0·132
8·4	2·1111	− 0·009	0·6183	− 0·079	− 1·8025	− 0·013	− 0·8072	− 0·098	2·136	0·013
7·7	1·2990	0·013	1·5951	− 0·032	− 0·9165	0·016	− 1·4311	− 0·035	− 0·249	0·147
7	0·0352	1·276	1·9108	− 0·014	− 0·0216	2·509	− 1·1581	− 0·041	− 0·035	− 1·854
6·3	− 1·2395	− 0·039	1·6283	0·011	0·8632	− 0·064	− 1·4204	− 1·049	0·323	0·704
5·6	− 2·0868	− 0·014	0·6850	0·047	1·7708	0·005	− 0·8849	0·043	− 2·022	0·132
4·9	− 2·1051	− 0·004	− 0·6351	− 0·040	1·7948	0·007	0·8270	− 0·056	− 2·108	− 0·072
4·2	− 1·2842	0·003	− 1·6036	− 0·006	0·9032	0·013	1·4287	0·092	0·268	− 1·040
3·5	− 0·0177	0·813	− 1·9110	− 0·015	0·0108	− 2·718	1·1578	− 0·110	0·018	22·25
2·8	1·2545	− 0·027	− 1·6202	− 0·011	− 0·8765	− 0·033	1·4233	0·026	− 0·306	0·273
2·1	2·0933	− 0·019	− 0·6685	0·008	− 1·7790	− 0·034	0·8659	− 0·097	2·052	− 0·165
1·4	2·0993	− 0·010	0·6518	− 0·059	− 1·7870	− 0·001	− 0·8466	− 0·219	2·080	0·007
0·7	1·2693	− 0·002	1·6119	− 0·007	− 0·8898	0·047	− 1·4261	0·071	− 0·287	− 0·561
0	0	0	1·9110	0·000	0	0	− 1·1577	0·000	0	0

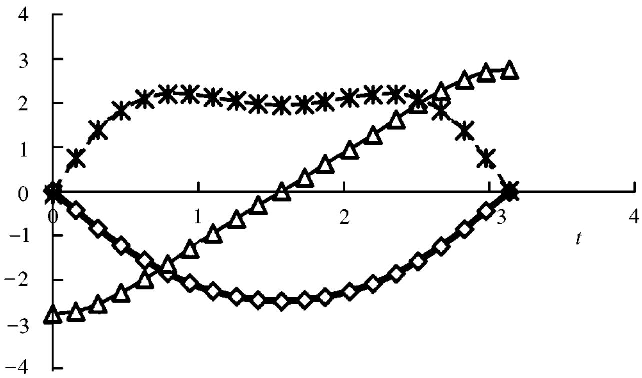

Figure 2. Comparison of results for the forced Duffing equation of second order: ◇, y GDQR; ——, y equation (3); △, y' GDQR; ———, y' equation (3); ✕, y'' GDQR; —‐‐‐—, y'' equation (3).

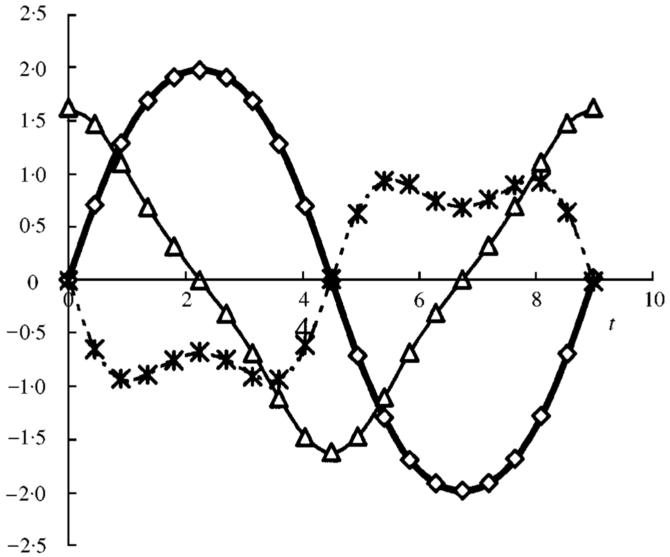

Figure 3. Comparison of results for the homogeneous Duffing equation of second order: ◇, y GDQR; ——, y equation (5); △, y' GDQR; ———, y' equation (5); ✕, y'' GDQR; —‐‐‐—, y'' equation (5).

big relative errors in Tables 1–3 correspond to very small absolute values, while no evident difference can be seen if the function is displayed graphically.

This work applies for the first time the GDQR to second and fourth order initial-value non-linear differential equations. Procedures are given in detail to convert these non-linear differential equations into a set of linear algebraic equations in an iterative loop using the Frechet derivative. The wide applicability of the GDQR is manifested further through examples.

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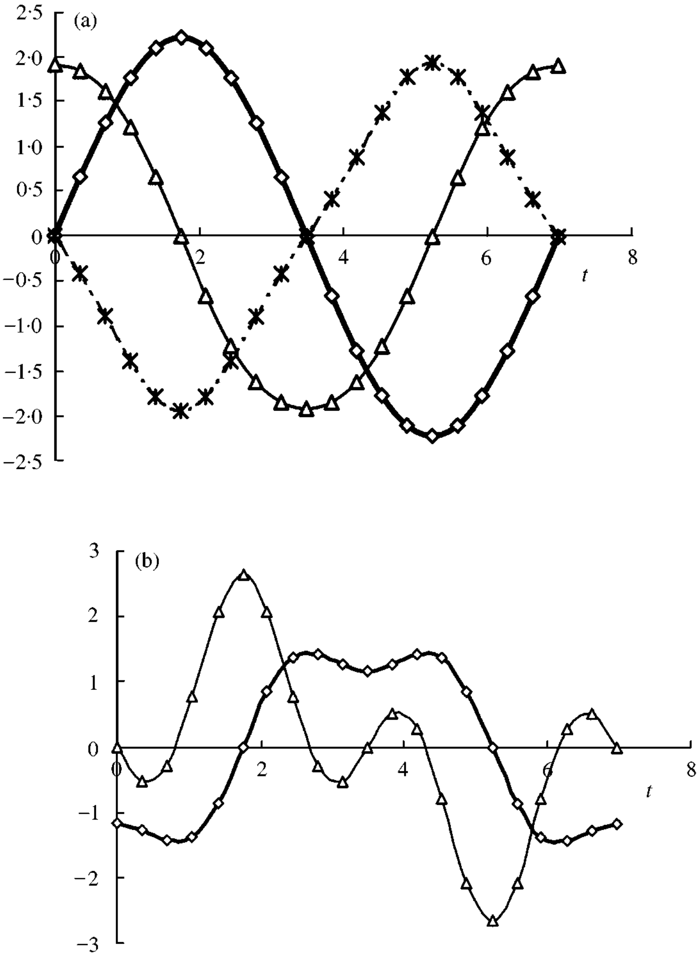

Figure 4. (a) Comparison of solutions for the fourth order Duffing equation: ◇, y GDQR; ——, y equation (8); △, y' GDQR; ———, y' equation (8); ✶, y'' GDQR; —‐‐‐—, y'' equation (8). b. ◇, $y^{(3)}$ GDQR; ——, $y^{(3)}$ equation (8); △, $y^{(4)}$ GDQR; ———, $y^{(4)}$ equation (8).

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