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FREE VIBRATION ANALYSIS OF COMPLETELY FREE RECTANGULAR PLATES BY THE SUPERPOSITION-GALERKIN METHOD

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The superposition-Galerkin method for analyzing the free vibration of thin isotropic and orthotropic plates as well as transverse-shear deformable plates was introduced in recent years. It has an advantage over the traditional superposition method in that it gives equally accurate results but requires much less work on the part of the analyst. Unfortunately, it has not been possible up to this time to apply it to plates with free edge conditions. This was due to mixed derivatives appearing in the formulation of free edge boundary conditions. In this paper it is shown how, with the superposition of specially selected sets of forced vibration solutions (building blocks), the above limitations are avoided. While the technique is applied here to the analysis of fully symmetric modes, only, it is demonstrated how the superposition-Galerkin method can be applied to any of the above plate problems regardless of the combination of free edge boundary conditions to be imposed.

1. INTRODUCTION

It has been well demonstrated that the superposition method constitutes a powerful analytical technique for analyzing the free vibration of a wide family of rectangular plates. Initially, the method was applied to thin plates with classical boundary conditions. It subsequently became apparent that the method could be successfully applied to point supported plates, plates resting on elastic edge support, and orthotropic plates. Later the method was applied to thick plates whose behavior was governed by the Mindlin equations. The latter approach was extended to cover the case of transverse shear-deformable composite plates.

During this series of studies one characteristic of the method became apparent. This relates to the fact that as one moved from thin isotropic plates, to orthotropic plates, and subsequently to transverse shear-deformable plates, the amount of work required to conduct the analysis became greater and greater. This, in turn, was largely due to the additional forms of solutions of the governing differential equation(s) which had to be provided for in the analysis as one moved to the more complicated plate problems.

It was for this reason that the author and his colleagues introduced the superposition-Galerkin method for the analysis of these problems. This latter method differed from the traditional superposition method only in the manner whereby solutions were obtained for the response of forced vibration solutions (building blocks) utilized in the analysis. With the newer approach it was not necessary to solve the differential equation(s) directly, but rather, solutions were obtained by the Galerkin method. As explained in

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reference [1], vast savings in computational efforts were achieved, particularly, in the case of transverse shear-deformable plates. It was for this reason that the superposition-Galerkin method was utilized for obtaining solutions to composite plate problems as reported in references [2, 3].

It remained a fact, nevertheless, that this new approach was found not to be applicable to the analysis of plates with combinations of free edges. The reason was very simple. In the classical Galerkin approach to obtaining solutions for building blocks, it is necessary to represent these solutions in series of functions, each term of which satisfies exactly the prescribed boundary conditions. This presented no difficulty for the case of building blocks where only clamped or simply supported edges were involved. In the case of plates with free edges, because of mixed derivatives utilized in formulating the edge conditions mathematically, it became impossible to choose appropriate series of functions. The computational advantages related to the superposition-Galerkin method could therefore not be exploited.

It has now been found that by employing specially selected sets of building blocks the superposition-Galerkin method can, in fact, be utilized to resolve these problems involving plates with combinations of free edge conditions. The objective of this paper is to demonstrate just how this end is accomplished.

2. MATHEMATICAL PROCEDURE

2.1. ANALYSIS OF THE THIN COMPLETELY FREE ORTHOTROPIC PLATE

The orthotropic plate has been selected for demonstration purposes as its analysis is more complicated than that of the isotropic plate, at least by means of the traditional superposition method. In fact, the isotropic plate constitutes a special limiting case of the orthotropic plate. Since the main function of this paper is to demonstrate a method of analysis, only the fully symmetric modes are analyzed. It will be seen that the analysis described is easily extended to handle all families of mode shapes.

2.1.1. *Analysis by the Traditional Superposition Method*

Analysis of this problem by the traditional superposition method will be described briefly for the sake of completeness. This traditional approach was employed, and described in detail, in reference [4], where the same completely free orthotropic plate resting on four symmetrically distributed point supports was examined. It will be appreciated that by deleting the portion of the eigenvalue matrix related to the point supports of this earlier problem, one arrives at the eigenvalue matrix for the problem of immediate interest here.

Fully symmetric modes are analyzed by superimposing the two building blocks represented schematically in Figure 1. Only one-quarter of the plate is considered in the analysis. Edges with two adjacent small circles are free of vertical edge reaction and the slope taken normal to these edges is everywhere zero.

The driven edge of the first building block is also free of vertical edge reaction. It is driven by a harmonic bending moment. The spatial distribution of the amplitude of this bending moment is expressed in series form as

$$\frac{Mb^2}{aD_y} = \sum_{m=0,1,2}^{\infty} \mathrm{E}_m \cos(m\pi\xi). \tag{1}$$

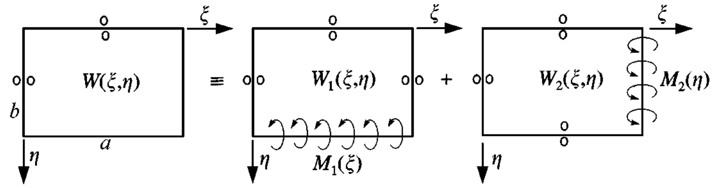

Figure 1. Building blocks utilized in analysis of completely free thin orthotropic plate by the traditional superposition method.

The solution for the building block response is expressed as

$$W(\xi, \eta) = \sum_{m=0,1,2}^{\infty} Y_m(\eta) \cos(m\pi\xi). \tag{2}$$

The governing differential equation is written as

$$\frac{\partial^4 W(\xi, \eta)}{\partial \eta^4} + 2\frac{H}{D_y}\phi^2 \frac{\partial^4 W(\xi, \eta)}{\partial \xi^2 \partial \eta^2} + \frac{D_x}{D_y}\phi^4 \left[\frac{\partial^4 W(\xi, \eta)}{\partial \xi^4} - \lambda^4 W(\xi, \eta)\right] = 0. \tag{3}$$

Upon substituting equation (2) into the governing differential equation, and separating variables, one obtains the ordinary fourth order differential equation governing the functions $Y_m(\eta)$,

$$Y_m^{\mathrm{IV}}(\eta) + \alpha_1 Y_m^{\mathrm{II}}(\eta) + \alpha_2 Y(\eta) = 0, \tag{4}$$

where roman superscripts indicate the order of differentiation with respect to the co-ordinate η, and where $\alpha_1 = -2DHY\phi^2(m\pi)^2$, $\alpha_2 = DXY\phi^4\{(m\pi)^4 - \lambda^4\}$.

The possible forms of solution for equation (4) are well known. The applicable form will depend on the input parameters DHX, DHY, etc. Each will contain four unknowns; however, two are immediately eliminated in view of symmetry required about the ξ-axis. The possible forms of solution are then

$$Y_m(\eta) = A_m \cosh\beta_m\eta + B_m \cos\gamma_m\eta, \quad Y_m(\eta) = A_m \cosh\beta_m\eta + B_m \cosh\gamma_m\eta$$

and

$$Y_m(\eta) = A_m \sin R\eta \sinh S\eta + B_m \cos R\eta \cosh S\eta, \tag{5}$$

where A_m and B_m are constants to be determined. The reader will find the other symbols carefully defined in reference [4].

Enforcing the zero vertical edge reaction, and dynamic moment equilibrium conditions, along the driven edge one obtains the remaining two boundary conditions to be enforced respectively. They are

$$Y_m'''(\eta) + \nu^*\phi^2 Y_m'(\eta) = 0|_{\eta=1} \tag{6}$$

and

$$Y_m''(\eta) + \nu\phi^2 Y_m(\eta) = -E_m|_{\eta=1}. \tag{7}$$

Enforcing the above boundary conditions, the two unknowns of equation (5) are evaluated and the building block response is available in terms of the driving coefficients E_m, regardless of the form the solution takes. Solution for the second building block, with driving coefficients E_n, is obtained from the first through a simple interchange of axis. Upon superimposing the two building blocks, the net contributions to bending moments along each of the boundaries, $\eta = 1$, and $\xi = 1$, are expanded in a cosine series of k terms, where k equals the number of terms utilized in the building block solutions.

Each net coefficient in these new series is set equal to zero, thereby giving rise to a set of $2k$ homogeneous algebraic equations relating the unknowns E_m and E_n. The coefficient matrix of this set of equations forms the required eigenvalue matrix and eigenvalues are obtained by searching for those values of the parameter λ^2 which cause the determinant at the eigenvalue matrix to vanish. Mode shapes are obtained by setting one of the non-zero coefficients, E_m and E_n, equal to unity and then solving the resultant set of non-homogeneous equations in order to evaluate the remaining coefficients.

2.1.2. *Analysis by the superposition-Galerkin method*

We now turn to the problem of obtaining the response of the first building block of Figure 1 by the Galerkin method. We wish to represent the functions $Y_m(\eta)$, in series form, where each term in the series satisfies all of the prescribed boundary conditions. There is no difficulty selecting series of functions for which the condition of symmetry with respect to the ξ-axis is satisfied. The difficulty arises when we try to choose these same functions so that each one satisfies exactly the boundary conditions expressed by equations (6) and (7). This turns out to be an insurmountable task. In fact, this was why earlier attempts to solve completely free plate problems by the superposition-Galerkin method had to be abandoned.

In this paper we choose to analyze the same basic problem as discussed above by means of the alternate set of building blocks of Figure 2. In fact, the first two building blocks of this set replace the first building block of Figure 1.

The first and second building blocks of Figure 2 differ from the first of Figure 1 only in the boundary conditions which are imposed along their driven edges. The first of this pair is

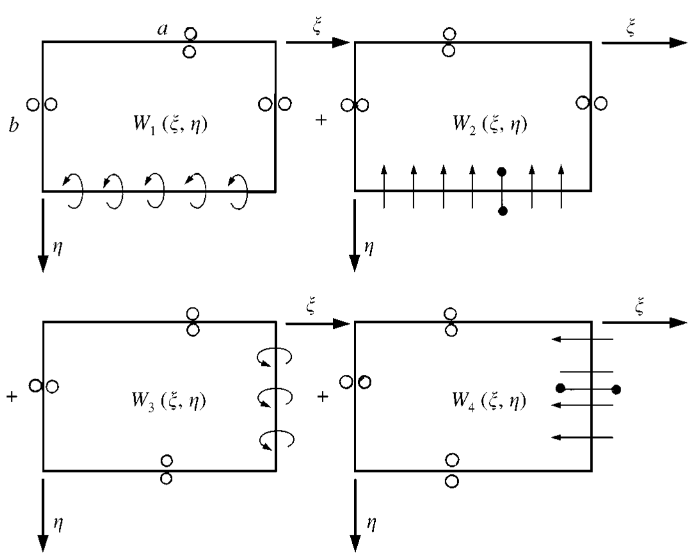

Figure 2. Building blocks utilized in analysis of completely free thin orthotropic plate by the superposition-Galerkin method.

driven by a distributed harmonic bending moment; however, lateral displacement along the driven edge is forbidden. The second building block of this set is driven by a distributed harmonic vertical edge reaction, while slope normal to this edge is forbidden. This latter condition is indicated in the figure by two connected solid dots, one on either side of the driven edge.

It is important to note that no mixed derivatives are involved in the formulation of the boundary conditions of these building blocks. It will thus be found that Galerkin-type solutions can be obtained for their response. It will be noted that any required bending moment can be imposed along the driven edge of the first building block, however, some residual vertical edge reaction is to be expected. Conversely, any required vertical edge reaction can be imposed along the driven edge of the second building block, however, some residual bending moment is to be expected.

When superimposed it will be found that these two building blocks compliment with each other. Driving coefficients can be adjusted so that while one building block enforces the condition of zero net bending moment along the edge $\eta = 1$, of the superimposed set, the other will enforce the condition of zero net vertical edge reaction along the same edge. The ultimate goal of enforcing free edge conditions along the edge is thus accomplished.

Focusing the attention now on the first and second building blocks of Figure 2 we express their solutions in the form of equation (2). Amplitude of the applied bending moment for the first building block is again expressed as in equation (1). Boundary conditions to be enforced along the driven edge are

$$Y_m(\eta)|_{\eta=1} = 0, \quad \mathbf{Y}''_m(\eta)|_{\eta=1} = -E_m. \tag{8, 9}$$

For the second building block the corresponding boundary conditions become

$$Y'_m(\eta)|_{\eta=1} = 0, \quad \mathbf{Y}'''_m(\eta)|_{\eta=1} = -E_m. \tag{10, 11}$$

It will be obvious that exact solutions for the above building blocks can be obtained following steps similar to those described for the first building block of Figure 1. One can obtain solutions for the third and fourth building blocks through a simple interchange of axes and superimpose the set of four. One can then obtain an eigenvalue matrix by enforcing a condition of zero net bending moment and zero net vertical edge reaction along the edges $\eta = 1$ and $\xi = 1$.

In fact, this has been done and it is found that the eigenvalues so obtained are identical to those obtained by utilization of the two building blocks of Figure 1.

We now turn to our main objective, i.e., we wish to demonstrate how solutions for the response of the first two building blocks of Figure 2 can be obtained by the Galerkin method.

Let us choose the following series to represent the functions $Y_m(\eta)$ of the first building block:

$$Y_m(\eta) = \sum_{i=1,2}^{\infty} E_i \frac{\cos(2i-1)\pi\eta}{2} + E_m \left[\frac{1}{2} - \frac{\eta^2}{2}\right]. \tag{12}$$

It will be noted that each term on the right-handside of equation (12) is symmetric with respect to the ξ-axis, as required, and furthermore, because of the addition of the polynomial terms, each term in the summation satisfies exactly the required boundary conditions.

The functions $Y_m(\eta)$ for the second building block may be expressed as

$$Y_m(\eta) = \sum_{i=0,1,2}^{\infty} E_i \cos i\pi\eta + E_m\left[\frac{\eta^2}{12} - \frac{\eta^4}{24}\right]. \tag{13}$$

Again it will be seen that required conditions of symmetry, and driven edge boundary conditions, are satisfied.

From here on standard Galerkin procedures are followed in order to obtain building block response. Let kk equal the number of terms utilized in equations (12) and (13). Focusing on the first building block we wish to obtain the coefficients E_i as a function of the driving coefficient E_m for any value of 'm'. We begin by differentiating the right-hand side of equation (12), term by term, and substituting into the governing differential equation (4). Thus, we arrive at a quantity which we designate as Q_m. Q_m is a function of η and involves the kk unknowns, E_i, and the driving coefficient, E_m. Next we expand the quantity Q_m in a series of kk terms, setting each term in the new series equal to zero. In order to take advantage of orthogonality it is appropriate here to use a series of the type employed in equation (12).

We thus arrive at a set of kk non-homogeneous algebraic equations relating the kk unknowns, E_i, and the quantity E_m. One can easily solve this set of equations utilizing standard computer routines and thereby arrive at a simple linear relationship between the quantities, E_i, and E_m. The response of the first building block to any driving coefficient, E_m is now available. A solution for the response of the second building block is obtained in an identical fashion. The Galerkin procedure as employed here has been described in detail in reference [1].

Response of the third and fourth building blocks is obtained in a manner identical to that described above after appropriate interchange of axes.

Having the response of all building blocks available we turn next to the process of superimposing all four and computation of the eigenvalue matrix. This is achieved in a manner identical to that described above. We again enforce the conditions of zero net bending moment and zero net vertical edge reaction along the edges $\eta = 1$ and $\xi = 1$. Computed eigenvalues based on the traditional superposition method and the superposition-Galerkin method will be presented later. Advantages of the latter method will also be discussed.

2.2. THE COMPLETELY FREE MINDLIN PLATE

2.2.1. *Analysis by the Traditional Superposition Method*

A thorough analysis of this plate by the traditional superposition method was presented in reference [5]. Here it is necessary to give only a brief review of this earlier work. The same basic equations are employed.

Again for illustrative purposes we will analyze the fully symmetric modes of vibration only. The three governing differential equations to be satisfied are reproduced in dimensionless form below:

$$\frac{\partial^2 W}{\partial \xi^2} + \frac{1}{\phi^2}\frac{\partial^2 W}{\partial \eta^2} + \frac{\partial \psi_\xi}{\partial \xi} + \frac{1}{\phi}\frac{\partial \psi_\eta}{\partial \eta} + \frac{\lambda^4 \phi_h^2}{\nu_3} W = 0, \tag{14}$$

$$\frac{\partial^2 \psi_\xi}{\partial \xi^2} + \frac{\nu_1}{\phi^2}\frac{\partial^2 \psi_\xi}{\partial \eta^2} + \frac{\nu_2}{\phi}\frac{\partial^2 \psi_\eta}{\partial \eta\,\partial \xi} - \frac{\nu_3}{\phi_h^2}\left(\psi_\xi + \frac{\partial W}{\partial \xi}\right) + \frac{\lambda^4 \phi_h^2}{12}\psi_\xi = 0, \tag{15}$$

$$\frac{\partial^2\psi_\eta}{\partial\xi^2}+\frac{1}{\phi^2 v_1}\frac{\partial^2\psi_\eta}{\partial\eta^2}+\frac{v_2}{\phi v_1}\frac{\partial^2\psi_\eta}{\partial\xi\partial\eta}-\frac{v_3}{\phi_h^2 v_1}\left(\psi_\eta+\frac{1}{\phi}\frac{\partial W}{\partial\eta}\right)+\frac{\lambda^4\phi_h^2}{12v_1}\psi_\eta=0, \tag{16}$$

where $\phi_h = h/a$, $v_l = (1-v)/2$, $v_2 = (1+v)/2$ and $v_3 = 6\kappa^2(1-v)$.

Shear forces, bending moments and twisting moments may be written as

$$Q_\xi=\psi_\xi+\frac{\partial W}{\partial\xi},\quad Q_\eta=\psi_\eta+\frac{1}{\phi}\frac{\partial W}{\partial\eta},$$

$$M_\xi=\frac{\partial\psi_\xi}{\partial\xi}+\frac{v}{\phi}\frac{\partial\psi_\eta}{\partial\eta},\quad M_\eta=\frac{\partial\psi_\eta}{\partial\eta}+v\phi\frac{\partial\psi_\xi}{\partial\xi},\quad M_{\xi\eta}=\frac{\partial\psi_\eta}{\partial\xi}+\frac{1}{\phi}\frac{\partial\psi_\xi}{\partial\eta}.$$

The building blocks employed are similar to those of the traditional approach, as shown in Figure 1. The major difference is that now three boundary conditions are to be enforced along each edge. The first building block is driven by a distributed harmonic bending moment. The driven edge is free of transverse shear force and twisting moment. The amplitude of the driving moment is expressed as

$$\frac{\partial\psi_\eta}{\partial\eta}+v\phi\frac{\partial\psi_\xi}{\partial\xi}=\sum_{m=0,1}^{\infty}E_m\cos(m\pi\xi). \tag{17}$$

The remaining edges have slip-shear conditions imposed. This is again indicated by two adjacent small circles. For this building block, slip-shear conditions imply zero rotation of the cross-section running along the edge, as well as zero transverse shear force and zero twisting moment.

The dimensionless plate lateral displacement and cross-sectional rotations are expressed as

$$W(\xi,\eta)=\sum_{m=0,\,1}^{\infty}X_m(\eta)\cos(m\pi\xi),\quad \psi_\xi(\xi,\eta)=\sum_{m=1,2}^{\infty}Y_m(\eta)\sin(m\pi\xi),$$

$$\psi_\eta(\xi,\eta)=\sum_{m=0,\,1}^{\infty}Z_m(\eta)\cos(m\pi\xi). \tag{18}$$

It is found advantageous to compute the response of the building block separately for the case where '*m*' equals zero, and '*m*' is greater than zero. We begin with computing response of the first building block with $m \geqslant 1$.

Substituting equations (18) into the differential equations it is found that the variables are separable and we obtain a set of three coupled ordinary homogeneous differential equations relating the functions $X_m(\eta)$, $Y_m(\eta)$, and $Z_m(\eta)$. These equations are expressed in matrix form as

$$\begin{Bmatrix}X_m''\\Y_m''\\Z_m''\end{Bmatrix}+\begin{bmatrix}0&0&a_{m1}\\0&0&a_{m2}\\a_{m3}&a_{m4}&0\end{bmatrix}\begin{Bmatrix}X_m'\\Y_m'\\Z_m'\end{Bmatrix}+\begin{bmatrix}b_{m1}&b_{m2}&0\\b_{m3}&b_{m4}&0\\0&0&b_{m5}\end{bmatrix}\begin{Bmatrix}X_m\\Y_m\\Z_m\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}. \tag{19}$$

where coefficients a_{m1}, a_{m2}, etc., are defined in reference [5].

It is found that through operating on the above set of equations with judiciously selected operators, and then conducting a process of addition and subtraction of the resulting equations, the individual quantities $X_m(\eta)$, etc., can be separated into sixth order ordinary differential equations involving these individual quantities only. For each value of '*m*' six unknowns will appear, however, three are eliminated immediately in view of symmetry about the ξ-axis. Other unknowns are evaluated by means of boundary conditions enforced along the driven edge. Four forms of solution are found possible with $m \geqslant 1$.

The reader will already appreciate that conducting the analysis as described above will entail a vast amount of work. It will be seen shortly that utilization of the superposition-Galerkin method obviates the need for the above computational efforts.

We turn next to the situation when 'm' = 0. We are now handling what is essentially a thick strip problem and computational work is considerably reduced. Only two differential equations, which govern the functions $W(\xi, \eta)$, and $Y_\eta(\eta, \xi)$, apply. The computational procedure is essentially the same as that described above and it is found that only two possible forms of solution exist. Solution for the second building block is again obtained through an interchange of axes, and eigenvalues and mode shapes are obtained, after superposition of the two building blocks, by established procedures as described in reference [5]. It is important to note that a solution for the response of the above building blocks cannot be obtained by the Galerkin method due to mixed derivatives appearing in the formulation of boundary conditions along their driven edge.

2.2.2. *Analysis by the superposition-Galerkin method*

In order to conduct this analysis we employ the building blocks of Figure 3. The first three building blocks of this set replace the first building block of the previous set.

We begin by focusing attention on the first building block of the set. All non-driven edges are given slip-shear boundary conditions as defined above. Lateral displacement and cross-section rotations are again expressed as in equation (18). We begin by examining the case with $m > 0$. Bending moments, shear forces and twisting moments along the driven edge are expressed as

$$M_\eta = \frac{\partial \psi_\eta}{\partial \eta} + \nu\phi \frac{\partial \psi_\xi}{\partial \xi}, \quad Q_\eta = \psi_\eta + \frac{1}{\phi}\frac{\partial W}{\partial \eta}, \quad M_{\xi\eta} = \frac{\partial \psi_\eta}{\partial \xi} + \frac{1}{\phi}\frac{\partial \psi_\xi}{\partial \eta}. \tag{20–22}$$

The driving moment along the edge, $\eta = 1$, is controlled by prescribing the first term on the right-hand side of equation (20), thus

$$\frac{\partial \psi_\eta}{\partial \eta} = \sum_{m=0,1}^{\infty} E_m \cos(m\pi\xi). \tag{23}$$

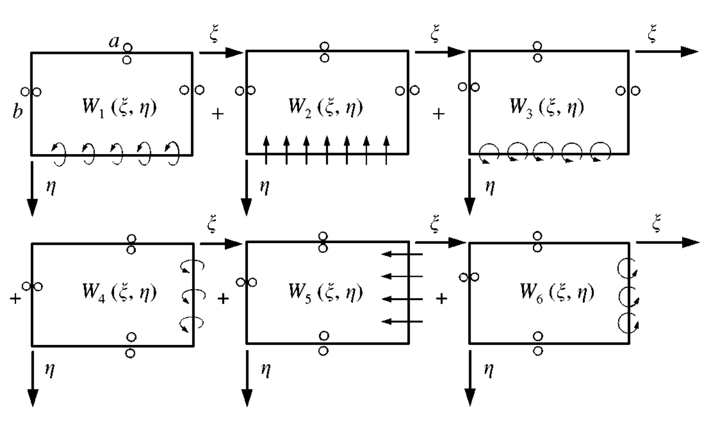

Figure 3. Building blocks utilized in analysis of completely free Mindlin plate by the superposition-Galerkin method.

Instead of setting Q_η and $M_{\xi\eta}$ of equations (21) and (22) equal to zero we demand only that $\partial W/\partial\eta$ and $\partial\Psi_\xi/\partial\eta$ of equations (21) and (22), respectively, should vanish. There will obviously be a residual shear force and twisting moment left along the driven edge. This will be taken out later by the other two building blocks. It is to be noted that the boundary conditions prescribed above contain no mixed derivatives. A Galerkin-type solution is therefore achievable for the first building block.

The series representations given immediately below have been utilized to permit a Galerkin type solution for response of the first building block. The reader may wish to verify that each term of each series satisfies exactly the required boundary conditions.

$$X_m(\eta) = \sum_{i=1,2} E_i \cos(i-1)\pi\eta, \quad Y_m(\eta) = \sum_{j=1,2} E_j \cos(j-1)\pi\eta \tag{24, 25}$$

and

$$Z_m(\eta) = \sum_{l=1,2} E_l \sin(2l-1)\frac{\pi}{2}\eta + E_m\eta. \tag{26}$$

The same components of the boundary condition expressions prescribed for the first building block are also prescribed for the other two building blocks of the set.

For the second building block the quantity $\partial W/\partial\eta$ is prescribed in series form identical to that of equation (23). For the third building block the quantity $\partial\Psi_\xi/\partial\eta$ is prescribed as

$$\frac{\partial\psi_\xi}{\partial\eta} = \sum_{m=1,2}^{\infty} E_m \sin(m\pi\xi). \tag{27}$$

The remaining prescribed quantities are each set equal to zero.

The following series representations utilized in connection with the second building block response meet the Galerkin requirements,

$$X_m(\eta) = \sum_{i=1,2}^{\infty} E_i \cos(i-1)\pi\eta + E_m\frac{\eta^2}{2}, \tag{28}$$

$$Y_m(\eta) = \sum_{j=1,2}^{\infty} E_j \cos(j-1)\pi\eta \tag{29}$$

and

$$Z_m(\eta) = \sum_{l=1,2}^{\infty} E_l \sin(2l-1)\frac{\pi}{2}\eta. \tag{30}$$

Series representations used in connection with the third building block are

$$X_m(\eta) = \sum_{i=1,2}^{\infty} E_i \cos(i-1)\pi\eta, \tag{31}$$

$$Y_m(\eta) = \sum_{j=1,2}^{\infty} E_j \cos(j-1)\pi\eta + E_m\frac{\eta^2}{2} \tag{32}$$

and

$$Z_m(\eta) = \sum_{l=1,2}^{\infty} E_l \sin(2l-1)\frac{\pi}{2}\eta. \tag{33}$$

Computation of the building block response for any of the three building blocks, and any value of 'm' is now easily achieved following established Galerkin procedures. The series related to each building block are differentiated, as required, and substituted into the governing differential equations. We may designate the three quantities thereby obtained as Q_{m1}, Q_{m2}, and Q_{m3}. Each quantity is a function of η. Each of these quantities is then expanded in an appropriate series of kk terms. We can minimize work by choosing series which permit us to take advantage of orthogonality. In the work reported here the quantities Q_{m1} and Q_{m2} have been expanded in cosine series of the form, $\cos(n-1)\pi\eta$, $n = 1, 2, \ldots$, while the quantity Q_{m3} was expanded in a sine series of the form, $\sin(n\pi\eta)$, $n = 1, 2, \ldots$.

Setting the coefficients of these new series equal to zero we thereby obtain, for each building block, a set of 3 kk non-homogeneous algebraic equations relating the 3 kk unknowns, E_i, E_j, etc., and the driving coefficient E_m. Solving these equations by standard computer techniques we obtain a linear relationship between each of the unknowns and the driving coefficient, E_m. The response of each building block to each excitation term is thus available in series form. The reader may wish to examine reference [1] where a similar solution technique was employed, and described in detail, for the fully clamped Mindlin plate. It is pointed out that the main co-efficient matrix for the left-hand side of the above sets of non-homogeneous equations is identical for all three building blocks. It need be computed only once. The off-diagonal elements in many of the natural segments into which the above main matrix is divided are zero.

Finally, examining equation (18), it is seen that when the subscript 'm' takes on the value zero, only two building blocks are involved. The moment and shear force acting along the driven edge are expressed as

$$M_\eta = \frac{\partial \psi_\eta}{\partial \eta}, \quad Q_\eta = \psi_\eta + \frac{1}{\phi}\frac{\partial W}{\partial \eta}. \tag{34, 35}$$

For the first building block we prescribe the quantity $\partial \Psi_\eta/\partial \eta$ as given by equation (22), along the driven edge. The quantity $\partial W/\partial \eta$ of equation (35) is set equal to zero. All boundary condition requirements are satisfied through expressing the quantities $X_m(\eta)$ and $Z_m(\eta)$ as

$$X_m(\eta) = \sum_{i=1,2}^{\infty} E_i \cos(i-1)\pi\eta \tag{36}$$

and

$$Z_m(\eta) = \sum_{j=1,2}^{\infty} E_j \sin(2j-1)\frac{\pi}{2}\eta + E_m\eta. \tag{37}$$

For the second building block the quantity $\partial W/\partial \eta$ is expressed by a series identical to that of equation (23). The quantity $\partial \Psi_\eta/\partial \eta$ is set equal to zero. The following series are utilized in representing the functions of equations (36) and (37),

$$X_m(\eta) = \sum_{i=1,2}^{\infty} E_i \cos(i-1)\pi\eta + E_m \frac{\eta^2}{2} \tag{38}$$

and

$$Z_m(\eta) = \sum_{j=1,2}^{\infty} E_j \sin(2j-1)\frac{\pi}{2}\eta. \tag{39}$$

A linear relationship between the individual coefficients, E_i, E_j, and the driving coefficients E_m is obtained following the Galerkin procedure as described above.

In all solutions for fully symmetric mode building blocks it will be apparent that functions $W(\xi, \eta)$ and $\Psi_\xi(\xi, \eta)$ must possess symmetry with respect to the ξ-axis, while the function $\Psi_\eta(\xi, \eta)$ must possess anti-symmetry. These conditions are seen to be satisfied by all solutions discussed above.

Solutions for the final three building blocks are again extracted from the first three through interchange of co-ordinate axes. The six building block solutions are available, they are superimposed and conditions of zero net bending moment, transverse shear force, and twisting moment, are imposed along the edges $\eta = 1$, and $\xi = 1$, of the superimposed set. One thus arrives at the required eigenvalue matrix.

It will be obvious that exact solutions could also be obtained for the building blocks of Figure 3. In fact, this has been done for verification purposes. Very good agreement was obtained when eigenvalues obtained in this rather laborious fashion were compared with those obtained earlier.

3. COMPARISON OF COMPUTED RESULTS

It is necessary to choose a value for 'k', the number of terms in the building block solutions required to give the desired convergence whether one employs the traditional superposition method or the superposition-Galerkin method. With this latter method we must also choose an appropriate value for kk, the number of terms utilized in expansions related to the Galerkin procedures. In all cases the objective has been to choose values such that their increase would not change the fourth significant digit in computed eigenvalues. A value of five for the quantity 'k' was found to be adequate for all computations reported here. Values of 'kk' will be discussed later.

3.1. THE THIN ORTHOTROPIC PLATE

In computations related to this plate a value of 100 was found to be adequate for the parameter kk. It is possible that lower values would have given satisfactory results.

In Table 1 the first four eigenvalues are tabulated for the fully symmetric modes of a thin square completely free plate. Only one-quarter of the plate was analyzed as discussed earlier. Two sets of values for the orthotropic parameters were employed. The first set, with DHX = DHY = 1, corresponds to the isotropic plate. In the second set both parameters were set equal to $\frac{1}{2}$. Comparison is made between eigenvalues obtained by means of the

TABLE 1

Comparison of eigenvalues obtained for fully symmetric modes of the thin orthotropic square plate (A — traditional superposition method, B—superposition-Galerkin method)

	$DHX = DHY = 1{\cdot}0$		$DHX = DHY = \frac{1}{2}$	
Mode	A	B	A	B
1	4·806	4·806	4·800	4·800
2	6·106	6·106	6·048	6·048
3	15·69	15·69	10·74	10·74
4	29·11	29·11	28·46	28·46

TABLE 2

Comparison of eigenvalues obtained for fully symmetric modes of the square Mindlin plate (A—traditional superposition method, B—superposition-Galerkin method)

	$\Phi_h = 0{\cdot}02$		$\Phi_h = 0{\cdot}2$	
Mode	A	B	A	B
1	4·805	4·805	4·648	4·646
2	6·103	6·104	5·863	5·863
3	15·60	15·66	13·73	13·73
4	29·05	29·08	24·40	24·40

conventional superposition method [4], and the present superposition-Galerkin method. It is seen that agreement between eigenvalues obtained by the two different methods is excellent.

3.2. THE THICK MINDLIN PLATE

Effects of varying the parameter kk for this problem were investigated in two different ways. In one approach its effect on elements of the printed-out eigenvalue matrix were studied. The second approach involved studying its effect on computed eigenvalues.

As a result of studies in connection with the first approach it was concluded that matrix elements related to the third building block were much more sensitive to variations in 'kk' than those of the first two building blocks. These convergence studies led to the adoption of $kk = 60$ for the first two building blocks while the value was increased to 300 for the third building block. Results obtained were found to be stable and to agree quite well with those of reference [5].

Eigenvalues computed earlier by the traditional superposition method are compared with those computed here by the superposition-Galerkin method in Table 2. These eigenvalues pertain to fully symmetric modes of a square Mindlin plate. Two different thickness ratios were utilized. It will be appreciated that ratios of 0·02 and 0·2, as utilized in the present work, where one quarter of the plate is analyzed, correspond to values of 0·01 and 0·1, of the earlier analysis where the full plate was analyzed.

It is seen that agreement between the two sets of computed eigenvalues is very good, with slight differences in the fourth digits in some cases only.

4. DISCUSSION AND CONCLUSIONS

It is found that accurate free vibration analysis of the completely free rectangular plate can be conducted by the superposition-Galerkin method as described here. It will be obvious to the reader that by coupling the present sets of building blocks with similar sets driven along the edges $\eta = 0$, and $\xi = 0$, a general analysis of all possible modes of the completely free plate is achievable. Very little additional work will be required to achieve this extension in capabilities as most of the elements of the larger eigenvalue matrix associated with the general problem can be inferred from those obtained above.

One might be concerned about the fact that for thin plates the new method requires replacing individual building blocks with a pair. In the case of the Mindlin plate individual blocks of the earlier procedure are replaced with a set of three. It turns out, in fact, that the additional building blocks do not significantly increase the work load. As noted earlier, for example, in the case of Mindlin plate analysis the same main coefficicent matrix is utilized for obtaining the Galerkin solution for response of all three building blocks. It need to be computed only once.

The advantages of the superposition-Galerkin method over the traditional superposition method have been enumerated in reference [1] and may be consulted by the reader. Summarized briefly, it may be stated that the entire problem of handling the various possible forms of solution for a fourth order ordinary differential equation is eliminated when analyzing the thin isotropic, or orthotropic, plate. The possible forms of solution depend on the input parameters.

In the case of the Mindlin plate problem the obtaining of various possible forms of solution to a set of three simultaneous ordinary coupled homogeneous differential equations is completely avoided. Furthermore, problems related to computer overflow, or underflow, sometimes experienced in the traditional approach, are eliminated as hyperbolic functions no longer appear in the analysis.

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APPENDIX A: NOMENCLATURE

a, b	edge lengths of quarter plate
D	flexural rigidity of Mindlin plate

D_x, D_y	flexural rigidities associated with x- and y-directions, respectively, of orthotropic plate
D_τ	torsional rigidity of orthotropic plate
DHX	$= H/D_x$
DHY	$= H/D_y$
DXY	$= D_x/D_y$
h	plate thickness
H	$2H = v_x D_x + v_y D_y + 4 D_\tau$
M	bending moment
M_ξ, M_η	dimensionless bending moments associated with ξ and η directions, respectively, of Mindlin plate
W	plate lateral displacement divided by "a"
xy	distances along plate edges in ξ and η directions
ξ	$= x/a$
η	$= y/b$
λ^2	eigenvalue $= \omega a^2 \sqrt{\rho/D_x}$ for orthotropic plate, $\omega a^2 \sqrt{\rho/D}$ for Mindlin plate
ω	circular frequency of vibration
ρ	mass of plate per unit area
v	Poisson's ratio
v^*	$= (2 - v)$
κ^2	Mindlin shear correction factor
v_x, v_y	Poisson ratios associated with x and y directions, respectively, of orthotropic plate
ϕ	$= b/a$
ϕ_h	$= h/a$
ψ_ξ, ψ_η	plate cross-section rotations associated with ξ and η directions respectively