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LETTERS TO THE EDITOR

NON-SMOOTH SPATIO-TEMPORAL TRANSFORMATION FOR IMPULSIVELY FORCED OSCILLATORS WITH RIGID BARRIERS

V. N. Pilipchuk

Mechanical Engineering, Wayne State University, Detroi, MI 48202, *U.S.A.*
E-mail: *pylyp@mel.eng.wayne.edu*

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A composition of two different implementations of the idea of non-smooth transformation of variables, space and time, is proposed in this note. The resulting transformation is illustrated on the impulsively forced non-linear oscillator between two absolutely rigid perfectly elastic barriers. It is shown that the space component of the transformation eliminates the barriers/constraints, whereas the time component removes external δ-pulses. As a result, the manifold of periodic solutions is described by a boundary-value problem with no space- or time-dependent δ-type singular terms.

The method of non-smooth transformation of the configuration space of moving particles under absolutely rigid barriers/constraints condition was proposed in references [1, 2], (see also reference [3]). On the other hand, special non-smooth/sawtooth transformation of *time* can be introduced for a class of strongly non-linear vibrating systems with *no barriers* [4]. The temporal implementation generates algebraic structures of "hyperbolic numbers" and requires some special mathematical tool [5] (see also references [6, 9] for physical applications and details). The result of this note is based on a complementary/opposite character of the two methods. All necessary features of their composition can be shown by considering the forced non-linear oscillator between the two absolutely rigid barriers.

Let us suppose that the restitution coefficient is equal to one, however the energy dissipation is taken into account by linear damping. The related differential equation of motion between the barriers is of the form

$$\ddot{x} + 2\zeta\dot{x} + f(x, \omega t) = pS''(\omega t) \tag{1}$$

and the constraints/barriers condition is

$$-1 \leqslant x(t) \leqslant 1, \tag{2}$$

where ζ, ω and p are constant parameters, the function $f(x, \omega t)$ is supposed to be periodic with respect to ωt, and the period is equal to four. Such normalization is convenient for transformations below, and it is dictated by special features of the saw-tooth, piece-wise linear sine, $S(l)$, with the unit amplitude and the period which is equal to four providing the unit slope, so that the regular part of the derivative (for almost all l) gives

$$[S'(l)]^2 = 1. \tag{3}$$

A unit-form analytic expression can be represented in the form $S(l) = (2/\pi)$ arcsin $[\sin(\pi l/2)]$.

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In equation (1) and below, overdot means time derivative, whereas prime denotes differentiation with respect to whole argument shown with the related function. For example, second order generalized derivative on the right-hand side of equation (1) is

$$S''(\omega t) = 2 \sum_{k=-\infty}^{\infty} [\delta(\omega t + 1 - 4k) - \delta(\omega t - 1 - 4k)]. \tag{4}$$

By the idea of non-smooth transformation of space, constraints (2) are eliminated by the change of the co-ordinate $x(t) \to l(t)$, such that

$$x = S(l). \tag{5}$$

When substituting transformation (5) into equation (1), one faces the singular term, $S''(l)$ $(\mathrm{d}l/\mathrm{d}t)^2$. Following references [1, 2], this term should be simply omitted together with the constraints, since this is the only term that can be caused by the rigid barriers.† Then, multiplying the equation by $S'(l)$ and taking into account equation (3), the final result of the transformation is represented in the form

$$\frac{\mathrm{d}^2 l}{\mathrm{d}t^2} + 2\zeta \frac{\mathrm{d}l}{\mathrm{d}t} + S'(l) f[S(l), \omega t] = pS''(\omega t) S'(l), \tag{6}$$

where $l(t) \in (-\infty, \infty)$, is the new-co-ordinate of the moving particle.

Starting next step, note that the sawtooth time transformation can be directly applied to the class of periodic motions only, unless some special modification has been done. Thus, the regimes, $l(t)$, are assumed to be periodic of the same period, as the external force. The sawtooth oscillating time, τ, is introduced as [5]

$$\tau = S(\omega t), \qquad e = S'(\omega t) \quad \text{and} \quad l = X(\tau) + Y(\tau) e. \tag{7}$$

Substituting equation (7) into equation (6), one obtains

$$\begin{aligned} &\omega^2 X''(\tau) + 2\omega\zeta Y'(\tau) + R(X, Y, \tau) \\ &\quad + [\omega^2 Y''(\tau) + 2\omega\zeta X'(\tau) + I(X, Y, \tau)] e \\ &\quad = [pR_S(X, Y) + pI_S(X, Y) e - \omega^2 X'(\tau)] S''(\omega t). \end{aligned} \tag{8}$$

$$\begin{Bmatrix} R \\ I \end{Bmatrix} = \frac{1}{2} \{S'(X+Y) f[S(X+Y), \tau] \pm S'(X-Y) f[S(X-Y), 2-\tau]\},$$

$$\begin{Bmatrix} R_S \\ I_S \end{Bmatrix} = \frac{1}{2} \{S'(X+Y) \pm S'(X-Y)\},$$

under the necessary condition of continuity for $l(t)$,

$$Y(\pm 1) = 0. \tag{9}$$

† Originally, some "impact terms" depending on the co-ordinate and velocity were placed on the right-hand side of the equation of motion instead of the constraints conditions. These terms are thought to be of the same structure as those produced by the differentiation of $S(l(t))$.

In equation (8), R and I components can be verified by direct substitution of explicit expressions for different pieces of the functions τ and e over the whole of their period [5].

The periodic singular term on the right-hand side is removed by the additional boundary condition

$$\left[X'(\tau) - \frac{p}{\omega^2} S'(X)\right]\Bigg|_{\tau = \pm 1} = 0 \tag{10}$$

since the δ-impulses (4) are located at those time instances, where $S(\omega t) = \pm 1$. In the same way, condition (9) was obtained, when taking first derivative of representation (7).

After the singular term has been eliminated, the two components of expression (8) must be separately set to zero. As a result, one obtains the differential equations

$$\omega^2 X''(\tau) + 2\omega\zeta Y'(\tau) + R(X, Y, \tau) = 0, \qquad \omega^2 Y''(\tau) + 2\omega\zeta X'(\tau) + I(X, Y, \tau) = 0 \tag{11}$$

in addition to the boundary conditions (9) and (10).

After solution of the boundary value problem (9)–(11) has been obtained, the solution of the original system (1) and (2) is given by composition of the two transformations (5) and (7) as follows:

$$x(t) = S(X(S(\omega t)) + Y(S(\omega t))S'(\omega t)). \tag{12}$$

In many particular cases, the above boundary-value problem can be simplified. Let us consider the example

$$\ddot{x} + \omega_0^2 x = pS''(\omega t), \quad -1 \leqslant x(t) \leqslant 1. \tag{13}$$

In this case, the temporal transformation with zero Y-component can be applied:

$$x = S(X(\tau)), \qquad \tau = S(\omega t). \tag{14}$$

The first two derivatives of expression (14) are found as

$$\dot{x} = S'(X)X'(\tau)S'(\omega t)\omega,$$

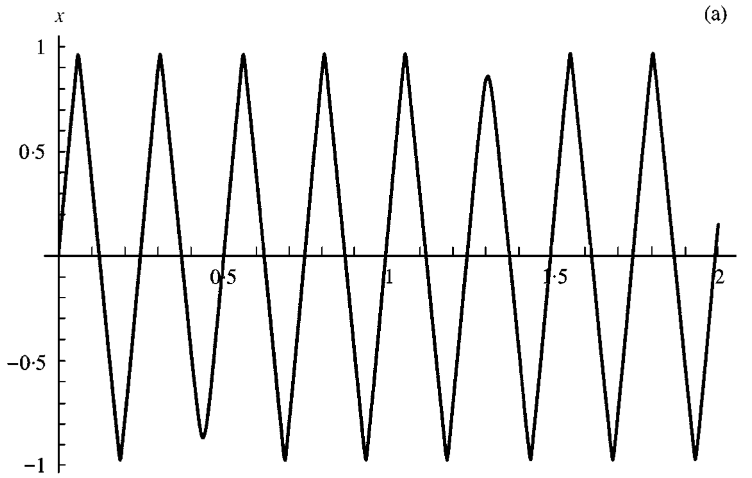
$$\ddot{x} = S''(X)[X'(\tau)]^2\omega^2 + S'(X)X''(\tau)\omega^2 + S'(X)X'(\tau)S''(\omega t)\omega^2, \tag{15}$$

where the relation $[S'(\omega t)]^2 = 1$ has been taken into account.

It is seen that the acceleration $\ddot{x}$ includes both space- and time-dependent singular terms associated with second derivatives $S''(X)$ and $S''(\omega t)$. The first singular term is due to non-smooth transformation of space and must be removed together with the constraints condition in equation (13) according to basic idea of the non-smooth transformation of space. The last term is caused by the non-smooth transformation of time and is going to compensate the external impulses on the right-hand side of equation (13). The latter is provided by condition (10). The rest of the equation is multiplied by $S'(X)$, and gives

$$X''(\tau) + \left(\frac{\omega_0}{\omega}\right)^2 S'(X)S(X) = 0. \tag{16}$$

Since the resulting boundary value problem, equations (10) and (16), does not include δ-type of singularities any more, the related standard numerical codes can be implemented. Figure 1 shows typical pair periodic solutions, obtained for parameters $\lambda = (\omega_0/\omega)^2 = 7{\cdot}5$



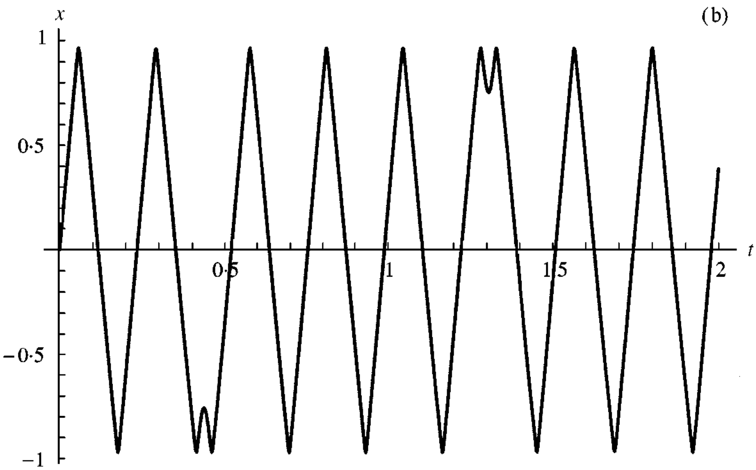


Figure 1. Typical temporal mode shapes of a now couple of periodic solutions corresponding to different initial slopes, (a) $X'(0) = 7{\cdot}24119$ and (b) $X'(0) = 7{\cdot}63019$. The system parameters are $\lambda = 7{\cdot}5$ and $P = 1{\cdot}0$.

and $P = p/\omega_0^2 = 1{\cdot}0$, where the circular frequency of the oscillator was taken as $\omega_0 = 2\pi$. (Note that the symmetric case, $X(-\tau) = -X(\tau)$, is considered only.) These solutions correspond to different initial slopes, (a) $X'(0) = 7{\cdot}24119$ and (b) $X'(0) = 7{\cdot}63019$. Under the above parameters, the number of periodic solution is 9. The number of periodic solutions is growing, when the parameters λ and/or P are increased. New pairs of solutions, as shown in Figure 1, will appear step by step. The small deviation of the initial slopes in (a) and (b) indicates that the related solutions, in Figure 1, just appeared. In this connection, we note work [7], were a complex sequence of transitions due to discontinuities was found in a harmonically forced oscillator under a "perfectly plastic" constraint.

To this end, it must be noted that the above composition of two transformations employs their different physical and also mathematical meaning, and became possible due to the complementary character of them. Note that in cases of spatially periodic structures, the temporal argument can be re-denoted in such a manner that it plays the role of the spacial co-ordinate of the continuous structure (see for example reference [8]). This idea, however, does not mix the methods, which are distinguished by their "mathematical targets", at least. Namely, the first method deals with a function/image, whereas another one transforms an argument/pre-image.

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