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APPLICATION OF A THREE-DIMENSIONAL SHELL THEORY TO THE FREE VIBRATION OF SHELLS ARBITRARILY DEEP IN ONE DIRECTION

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A three-dimensional shell theory is presented which is applicable to doubly curved thick open shells which are arbitrarily deep (have a large side-length to radius of curvature ratio) in one principal direction but are shallow in the other direction. The strain–displacement equations for the proposed "deep-shallow" shell theory are expressed in Cartesian co-ordinates and the limits of applicability of these equations are discussed. These equations are then used in a Ritz variational formulation with algebraic polynomials as trial functions to solve for the natural frequencies of a number of doubly curved shell problems. A novel approach is also proposed in which penalty functions are introduced to enforce continuity of displacements at two opposite ends of a shell of rectangular platform, increasing the range of problems which can be treated to include closed shells, such as cylinders, barrels, cooling-tower-type structures, toroids, rings, etc. (a sub-class of shells of revolution).

1. INTRODUCTION

Marguerre [1], Reissner [2–4] and Vlasov [5] are credited with the initial development of shallow shell theory (see Leissa [6]). Work on the free vibration of shallow shells has been discussed in a monograph on the vibration of shells by Leissa [6], in a book by Soedel [7], as well as in a more recent review article by Qatu [8]. The free vibration behaviour of deep cylindrical shell panels based on the deep shell theory of Novozilov [9] has been studied by Mizusawa [10] using the thin strip method. Lee *et al.* [11] compared results obtained using a deep shell theory with results obtained using shallow shell theory for cantilevered cylindrically curved panels. The free vibration of thin closed cylindrical shells has been extensively studied and is reviewed in a paper by Koga [12] as well as in Leissa's monograph [6]. The free vibration of laminated barrel shells has been treated in a recently published paper by Qatu [13] where extensive numerical results are presented.

In the present paper, a three-dimensional shell theory applicable to a class of doubly curved thick open shells which are arbitrarily deep (have a large side-length to radius of curvature ratio) in one principal direction but which are shallow in the other direction is proposed. The strain–displacement equations of this "deep-shallow" shell theory are expressed in Cartesian co-ordinates. Simple algebraic polynomials which satisfy the boundary conditions on the six faces of a parallelepiped are used as trial functions in a Ritz approach to obtain an eigenvalue equation based on the proposed strain–displacement equations. (The use of simple polynomials as admissible functions in a Ritz approach to treat plates, shells and solids has been discussed in a number of previous papers [14–16].

A novel technique is also proposed in which penalty functions are introduced to enforce continuity of displacements at two opposite ends of a shell of rectangular platform (rather than to enforce boundary conditions [17] or continuity between elements [18]). This simple device increases the range of problems which can be treated using the "deep–shallow" shell equations to closed shells, such as cylinders, barrels, co oling-tower-type structures, toroids, rings, etc. (a sub-class of shells of revolution).

In order to validate the proposed approach, as well as to explore the limits of applicability, natural frequencies are obtained for a range of problems including cylindrically curved panels, thick and thin cylindrical shells and barrel shells and are compared, where possible, with values published in the open literature.

2. THEORY

2.1. STRAIN–DISPLACEMENT EQUATIONS

Consider a homogeneous isotropic shell described in orthogonal curvilinear co-ordinates α_1, α_2 lying along the neutral surface and a co-ordinate α_3 normal to the neutral surface as shown in Figure 1. If it is assumed that the principal curvatures R_1 and R_2 are constant and lie along the co-ordinates α_1 and α_2, the three-dimensional linear strain–displacement equations are given by (cf. Soedel [7, pp. 25–26] and Leissa [6, p. 7])

$$\varepsilon_{11} = \frac{1}{A_1(1+\alpha_3/R_1)}\left[\frac{\partial U_1}{\partial \alpha_1} + \frac{U_2}{A_2}\frac{\partial A_1}{\partial \alpha_2} + U_3\frac{A_1}{R_1}\right],$$

$$\varepsilon_{12} = \frac{A_1(1+\alpha_3/R_1)}{A_2(1+\alpha_3/R_2)}\frac{\partial}{\partial \alpha_2}\left[\frac{U_1}{A_1(1+\alpha_3/R_1)}\right] + \frac{A_2(1+\alpha_3/R_2)}{A_1(1+\alpha_3/R_1)}\frac{\partial}{\partial \alpha_1}\left[\frac{U_2}{A_2(1+\alpha_3/R_2)}\right]$$

$$\varepsilon_{22} = \frac{1}{A_2(1+\alpha_3/R_2)}\left[\frac{\partial U_2}{\partial \alpha_2} + \frac{U_1}{A_1}\frac{\partial A_2}{\partial \alpha_1} + U_3\frac{A_2}{R_2}\right],$$

$$\varepsilon_{13} = A_1\left(1+\frac{\alpha_3}{R_1}\right)\frac{\partial}{\partial \alpha_3}\left(\frac{U_1}{A_1(1+\alpha_3/R_1)}\right) + \frac{1}{A_1(1+\alpha_3/R_1)}\frac{\partial U_3}{\partial \alpha_1}. \tag{1}$$

$$\varepsilon_{33} = \frac{\partial U_3}{\partial \alpha_3}, \quad \varepsilon_{23} = A_2\left(1+\frac{\alpha_3}{R_2}\right)\frac{\partial}{\partial \alpha_3}\left(\frac{U_2}{A_2(1+\alpha_3/R_2)}\right) + \frac{1}{A_2(1+\alpha_3/R_2)}\frac{\partial U_3}{\partial \alpha_2}.$$

where ε_{11}, ε_{22}, ε_{33} are the normal strains, ε_{12}, ε_{13} and ε_{23} the shear strains, and U_1, U_2 and U_3 the displacements in the α_1, α_2 and α_3 directions, respectively, and where A_1 and A_2 are the first fundamental quantities or Lamé parameters.

The strain–displacement equations given by equation (1) can be simplified if it is further assumed that: (i) *The ratios of thickness a_3 to radius of curvature are small*, that is a_3/R_1 and $a_3/R_2 \ll 1$. It can then be assumed that $1+\alpha_3/R_1$ and $1+\alpha_3/R_2$ in equations (1) are approximately equal to 1. (ii) *Either the ratio of sidelength to radius of curvature a_1/R_1 or a_2/R_2 is small* ($\ll 1$), or in other words the shell is shallow in either the α_1 or α_2 direction (in the following analysis it will be assumed that $a_2/R_2 \ll 1$). The shell can then be approximately described in Cartesian co-ordinates by letting $x = R_1\,\mathrm{d}\theta = \alpha_1$, $y = \alpha_2$, $z = \alpha_3$, $R_x = R_1$ and $R_y = R_2$ from which sidelengths $a = R_1, \theta = a_1, b = 2R_2\sin(\phi/2) \cong a_2$ and thickness $c = a_3$ and the Lamé parameters A_1 and A_2 are equal to 1 as shown in Figure 2.

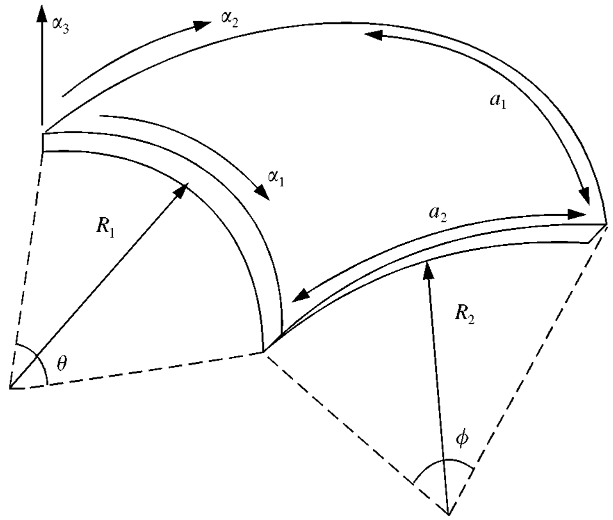

Figure 1. Doubly curved shell in orthogonal curvilinear co-ordinates.

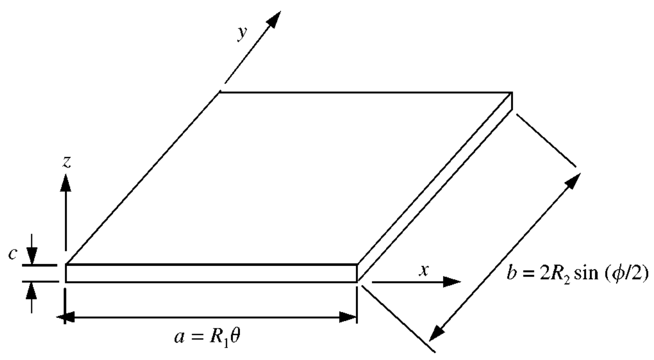

Figure 2. Cartesian representation of doubly curved shell.

The strain–displacement equations can be expressed in terms of the displacements u, v and w in the x, y and z Cartesian co-ordinates, respectively, as

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{w}{R_x}, \quad \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{w}{R_y}, \quad \varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} - \frac{u}{R_x},$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z}, \quad \varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} - \frac{v}{R_y}. \tag{2}$$

The shell is effectively "unwrapped" or "unravelled" in the deep direction as shown in Figure 2 (instead of using the more typical projection onto the x–y plane used in shallow shell theory). This very simple technique extends the applicability of the strain–displacement equations (2) to shells which are arbitrarily deep in one direction. In fact, shells may even be wrapped end to end ($\theta = 2\Pi$; $a = 2\Pi R_x$) or coiled ($\theta > 2\Pi$; $a > 2\Pi R_x$) (although for coiled shells the interference of one surface on another is not taken

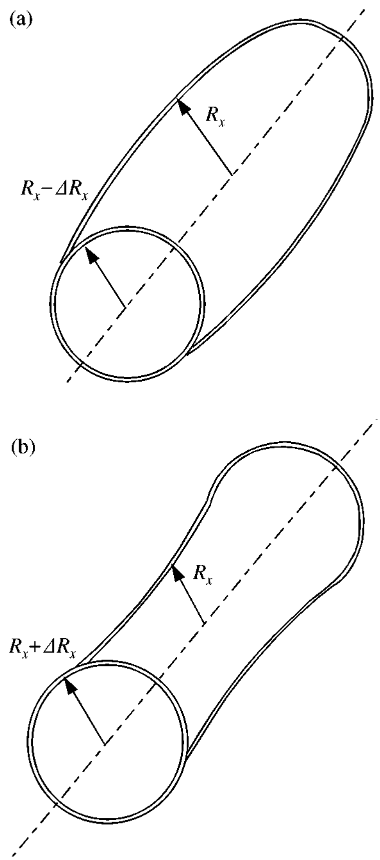

Figure 3. (a) Closed barrel shell (constant positive meridional curvature R_y); (b) closed cooling tower-shaped shell (constant negative meridional curvature R_y).

into account). It should be noted that the same approach can be used to extend the applicability of classical thin-shallow shell theory to deep–shallow shells (all that is needed is a correction in the sidelength used for the "deeper" direction).

Shells of revolution can be modelled in Cartesian co-ordinates by using the unwrapped planform ($a = 2\Pi R_x$) and by enforcing continuity at $x = 0$ and a. However, for these shells the radius of curvature R_x will not, in general, be constant but will vary along the axial length b as is shown in Figure 3 (except for the special case of a cylinder where $1/R_y = 0$). It can be shown that, in addition to the previously stated assumptions, the proposed equations will only apply for closed shells provided $b^2/(8R_xR_y) \ll 1$ (see Appendix A).

It should be noted that: (i) If a/R_x and $b/R_y = 0$ the strain–displacement equations (2) reduce to the three-dimensional elasticity equations for a solid in Cartesian co-ordinates and are valid for arbitrary thickness c; and (ii) if one of the curvatures is infinite (e.g., $b/R_y = 0$), that is if the shell is cylindrically curved, then the only limitation of the proposed equations is that $c/R_x \ll 1$.

2.2. FREE VIBRATIONS USING RITZ APPROACH

If simple harmonic motion at radian natural frequency ω is assumed, the displacements u, v and w in the x, y and z directions, respectively, can be expressed as $u(x, y, z, t) = U(x, y, z)\sin\omega t$; $v(x, y, z, t) = V(x, y, z)\sin\omega t$ and $w(x, y, z, t) = W(x, y, z)\sin\omega t$. The displacements U, V and W for a thick shell of rectangular planform can be approximated using algebraic polynomials as follows:

$$U(x, y, z) = \sum_{i=0}^{n_x} \sum_{j=0}^{n_y} \sum_{k=0}^{n_z} A_{ijk} x^{i+l^u_{x=0}} y^{j+l^u_{y=0}} z^{k+l^u_{z=0}} (a-x)^{l^u_{x=a}} (b-y)^{l^u_{y=b}} (c-z)^{l^u_{z=c}},$$

$$V(x, y, z) = \sum_{i=0}^{n_x} \sum_{j=0}^{n_y} \sum_{k=0}^{n_z} B_{ijk} x^{i+l^v_{x=0}} y^{j+l^v_{y=0}} z^{k+l^v_{z=0}} (a-x)^{l^v_{x=a}} (b-y)^{l^v_{y=b}} (c-z)^{l^v_{z=c}}, \quad (3)$$

$$W(x, y, z) = \sum_{i=0}^{n_x} \sum_{j=0}^{n_y} \sum_{k=0}^{n_z} C_{ijk} x^{i+l^w_{x=0}} y^{j+l^w_{y=0}} z^{k+l^w_{z=0}} (a-x)^{l^w_{x=a}} (b-y)^{l^w_{y=b}} (c-z)^{l^w_{z=c}},$$

where A_{ijk}, B_{ijk} and C_{ijk} are as yet undetermined linear coefficients. The index $l^u_{x=0}$ depends on the geometric boundary conditions on the surface $x = 0$ for the U displacement and takes the value 0 for no restraint ($U \neq 0$) and 1 for full restraint ($U = 0$). Similarly, indices $l^V_{x=0}$ and $l^W_{x=0}$ depend upon the restraints imposed upon V and W in the $x = 0$ plane and indices $(l^U_{x=a}, l^V_{x=a}, l^W_{x=a})$, $(l^U_{y=0}, l^V_{y=0}, l^W_{y=0})$, $(l^U_{y=b}, l^V_{y=b}, l^W_{y=b})$, $(l^U_{z=0}, l^V_{z=0}, l^W_{z=0})$ and $(l^U_{z=c}, l^V_{z=c}, l^W_{z=c})$ depend on the restraints in the U, V and W directions on the other five faces of the solid shell (at $x = a$, $y = 0$, $y = b$, $z = 0$ and $z = c$).

The maximum strain energy V_{max} can be expressed straightforwardly in terms of the normal and shear strains:

$$V_{max} = \frac{1}{2} \iiint [\lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})^2 + 2G(\varepsilon^2_{xx} + \varepsilon^2_{yy} + \varepsilon^2_{zz}) + G(\gamma^2_{xy} + \gamma^2_{yz} + \gamma^2_{zx})] \, dx \, dy \, dz, \quad (4)$$

where $\lambda = vE/(1 + v)(1 - 2v)$ and $G = E/2(1 + v)$ are the Lamé parameters.

By substituting equations (2) into equation (4), V_{max} can be expressed in terms of the displacements integrated over the volume of the solid.

The maximum kinetic energy T_{max} can also be expressed in terms of the displacements as

$$T_{max} = \left(\frac{\rho\omega^2}{2}\right) \iiint (U^2 + V^2 + W^2) \, dx \, dy \, dz, \quad (5)$$

where ρ is the density of the element material and the integration is again performed over the volume of the solid.

Finally, the trial function series (3) are substituted for U, V and W in the maximum kinetic and strain energy expressions and the Lagrangian functional $L_{max} = (T_{max} - V_{max})$ is minimized with respect to the undetermined linear coefficients A_{ijk}, B_{ijk} and C_{ijk} to give a homogeneous linear system of equations. Eigenvalues and the corresponding eigenvectors can then be obtained by a number of methods (in the present paper using subspace iteration). It should be noted that since the polynomial trial function series described by equations (3) form a mathematically complete set of functions, the results obtained from the Ritz minimization process will converge monotonically from above to the exact frequencies (of the approximated problem) as the number of terms in each series tends to infinity.

2.3. ENFORCEMENT OF CONTINUITY CONDITIONS TO MODEL SHELLS OF REVOLUTION

The approach described above can be used to treat cylindrically shaped closed shells by creating a fictitious seam or cut in the shell along the axial length. The shell can then be unwraped into the Cartesian co-ordinate system so long as continuity of geometric boundary conditions along the cut edge is ensured. For a closed cylindrical shell with axial co-ordinate in the y direction and with $x = 2\Pi R_x = a$, these continuity conditions are given by $U_{x=0} = U_{x=a}$, $V_{x=0} = V_{x=a}$ and $W_{x=0} = W_{x=a}$. In the present paper, the continuity conditions are satisfied by using connecting springs of very high stiffness value K_U, K_V and K_W to enforce continuity of displacements U, V, and W at $x = 0$ and a. The strain energy contribution V_U, V_V and V_W of these springs is then simply added to the strain energy V_{max} of the shell

$$V_u = \frac{1}{2} K_u \iint (U_{x=0} - U_{x=a})^2 \, \mathrm{d}y \, \mathrm{d}z,$$

$$V_v = \frac{1}{2} K_v \iint (V_{x=0} - V_{x=a})^2 \, \mathrm{d}y \, \mathrm{d}z, \tag{6}$$

$$V_w = \frac{1}{2} K_w \iint (W_{x=0} - W_{x=a})^2 \, \mathrm{d}y \, \mathrm{d}z.$$

The use of penalty functions or artificial springs to enforce continuity could usefully be applied even when cylindircal co-ordinates are used (for example, polynomial functions could be used instead of the more usual Fourier series). In general, the technique of using artificial springs to enforce continuity of geometric, and if required, natural boundary conditions at two points along a non-periodic function, extends the range of functions which can be used as trial functions in a number of problems.

3. NUMERICAL RESULTS

In order to validate the proposed approach as well as to explore the limits of applicability, numerical results have been generated for a range of problems and are compared with results published in the open literature or with results generated by the author using the finite element method. For all cases considered, Poisson's ratio was taken as 0·3.

In Table 1, the lowest five frequency parameters $\Omega = \omega\sqrt{(\rho/E)}$ as obtained using the present approach are given for a fully simply supported cylindrical panel for $c/R_x = 0{\cdot}01$, $b/R_x = 0{\cdot}52356$ and for a/R_x ranging from 0·2618 ($\theta = 15^\circ$) to 2·0942 ($\theta = 120^\circ$). Comparison results obtained by Mizusawa [10] using a spline strip approach in a deep thin shell theory based on the equations proposed by Novozhilov [9] are also given. For the cases considered here, the only assumptions made in using the present approach are (i) that the planform can be approximated as a parallepiped (rather than actual truncated wedge) and (ii) that the radius of curvature R_x is constant through the thickness c of the panel (both of which are very closely approximated since the shell is thin). Agreement between the present results and those obtained by Mizusawa can be seen to be excellent with at most a 1·3% difference in the results for all cases except for $\theta = 120^\circ$ where at most a 5% discrepancy is observed. From the brief convergence study given, the present results can be seen to converge rapidly and in many cases to 4 significant figures. It might also be noted that in all cases but one (Mode 4, 90°), the present results are lower than those obtained by Mizusawa and are therefore more accurate as the Ritz approach leads to upper bounds on the natural frequencies (the author believes the results obtained for the 120° case by Mizusawa would converge closer to the present results if a larger number of strips were used in the solution).

TABLE 1

Frequency parameters $\Omega = \omega\sqrt{(\rho/E)}$ for a fully supported cylindrical panel for $b/R_x = 0{\cdot}52356$, $c/R_x = 0{\cdot}01$, and $R_y/R_x = 0$

a/R_x	θ	$n_x \times n_y \times n_z$	Mode number 1	2	3	4	5
0·2618	15°	$8 \times 8 \times 4$	0·3002	0·5214	0·8178	0·9575	1·129
		$10 \times 10 \times 4$	0·3002	0·5214	0·8177	0·9574	1·129
		Mizusawa [10]	0·3009	0·5233	0·8232	0·9675	1·143
0·5236	30°	$8 \times 8 \times 4$	0·2821	0·3003	0·5043	0·5214	1·5669
		$10 \times 10 \times 4$	0·2821	0·3002	0·5043	0·5214	1·5667
		Mizusawa [10]	0·2821	0·3009	0·5050	0·5233	1·5701
0·7853	45°	$8 \times 8 \times 4$	0·2432	0·3005	0·3674	0·4868	0·4912
		$10 \times 10 \times 4$	0·2432	0·3002	0·3674	0·4632	0·4868
		Mizusawa [10]	0·2433	0·3010	0·3674	0·4645	0·4877
1·5707	90°	$10 \times 10 \times 4$	0·2437	0·2556	0·2821	0·3128	0·3674
		$12 \times 6 \times 4$	0·2432	0·2531	0·2821	0·3067	0·3674
		Mizusawa [10]	0·2443	0·2552	0·2825	0·3042	0·3675
2·0942	120°	$10 \times 10 \times 4$	0·2425	0·2481	0·2619	0·2821	0·3013
		$12 \times 6 \times 4$	0·2425	0·2481	0·2619	0·2821	0·3003
		Mizusawa [10]	0·2452	0·2492	0·2682	0·2826	0·3148

TABLE 2

Frequency parameters $\Omega^* = \omega a^2\sqrt{(12(1-\nu^2)\rho/(Ec^2))}$ for a cylindrical panel cantilevered at $y = 0$ with $b/a = 1$, $R_y/R_x = \infty$

			Symmetric modes				Antisymmetric modes			
a/R_x	a/c		$S1$	$S2$	$S3$	$S4$	$A1$	$A2$	$A3$	$A4$
0·7	40	Present $8 \times 8 \times 4$	11·24	26·20	40·33	63·30	9·097	34·86	61·33	77·36
		Deep [11]	11·24	26·29	40·64	63·55	9·150	35·09	61·88	79·49
		Shallow [11]	10·81	27·35	40·45	64·11	9·219	35·11	64·69	79·61
1·6	40	Present $8 \times 8 \times 4$	17·73	33·63	47·14	88·59	12·55	45·35	51·03	86·92
		Deep [11]	17·91	33·47	47·69	89·74	12·63	45·64	51·61	87·33
		Shallow [11]	18·69	31·96	49·55	97·12	11·42	46·22	64·96	87·24
0·7	500	Present $8 \times 8 \times 4$	30·26	93·48	122·2	161·6	35·70	82·10	93·02	170·9
		Deep [11]	30·29	93·60	122·3	161·9	35·74	82·22	93·15	174·9
		Shallow [11]	30·34	93·62	127·1	156·5	35·42	82·55	93·22	174·9
1·6	500	Present $8 \times 8 \times 4$	49·92	124·2	133·9	225·5	47·33	132·2	158·1	182·0
		Deep [11]	49·98	124·4	134·0	225·4	47·36	131·8	157·3	182·3
		Shallow [11]	46·92	129·6	140·6	241·7	49·43	131·9	145·8	206·9

In Table 2, the frequency parameters $\Omega^* = \omega a^2\sqrt{(12(1-\nu^2)\rho/(Ec^2))}$ for the first four symmetric and antisymmetric modes of a cylindrical panel cantilevered at $y = 0$ as obtained using the present approach for both a very thin $c/a = 1/500$ and a thin $c/a = 1/40$ panel and for both a relatively shallow $a/R_x = 0{\cdot}7$ and a deeper $a/R_x = 1{\cdot}6$ curved panel (all combinations for a total of four cases) are compared with results published by Lee *et al.*

[11]. Again, for the cases considered, here, the only simplifying assumptions made in the present approach are (i) parallepiped platform and (ii) a constant radius of curvature R_x (again these are both very closely approximated since the shell is for all cases relatively thin). Lee *et al.* generated results using both a shallow-shell theory and a deep-shell theory and, as can be seen from Table 2, agreement of the present results with their published deep-shell theory based results is excellent with at most a 2·7% discrepancy. Again it should be noted that in the vast majority of cases, the present results are lower than those presented by Lee *et al.* and can therefore be considered more accurate.

In Table 3, the lowest seven non-dimensional frequency parameters $\Omega = \omega R_x \sqrt{(\rho/E)}$ for axial wave number 1 are given for a cylinder of thickness to radius ratio $c/R_x = 0{\cdot}3$ and thickness to length ratio $c/b = 0{\cdot}3$ with shear diaphragm boundary conditions on each end as obtained by modelling: (i) a quarter of the cylinder as a 90° shell panel and applying all distinct combinations of symmetry and antisymmetry conditions at $x = 0$ and at $x = \Pi/2\ R_x = a$; (ii) half the cylinder as a 180° shell panel applying all combinations of symmetry and antisymmetry conditions at $x = 0$ and at $x = \Pi R_x = a$; (iii) a full cylinder ensuring continuity of displacements at $x = 0$ and at $x = 2\ \Pi R_x = a$ by using artificial springs (penalty functions) with stiffness parameters $K'_U = K_U/E$, $K_V = K'_V/E$ and $K'_W = K_W/E$ equal to 10^6. For the full cylinder case and for $12 \times 4 \times 4$ terms in the series a brief convergence study for penalty function stiffness $K'_U = K'_V = K'_W = 10^2$ and 10^4 is also included in the table. The present approach assumes (i) that the radius of curvature $R_x = 1$ is constant through the thickness c whereas R_x actually varies from 0·85 to 1·15 as shown in Figure 4(a) and (ii) that the unravelled planform of the cylinder is a parellelepiped, rather than a truncated wedge, as shown in Figure 4(b). In spite of these assumptions, excellent agreement is achieved with the results obtained by Armenakas *et al.* [20] using an exact solution (at most 1·5% difference in predicted non-dimensional frequency). As the circumferential wave number of a mode increases, more terms in the polynomial expansion in the circumferential direction are required to achieve convergence using the present approach particularly when symmetry is not exploited and the full shell is modelled.

TABLE 3

Frequency parameters $\Omega = \omega R_x \sqrt{(\rho/E)}$ for a cylindrical shell with shear diaphragm conditions at both ends of $c/R_x = 0{\cdot}3$ and $b/R_x = 1$

			Mode type (axial wave number, circumferential wave number)							
	$n_x \times n_y \times n_z$		(1, 2)	(1, 1)	(1, 0)	(1, 3)	(1, 4)	TORS	(1, 1)	(1, 5)
90°	$6 \times 6 \times 3$		1·173	1·185	1·247	1·353	1·711	1·948	2·094	2·176
	$9 \times 9 \times 6$		1·162	1·179	1·242	1·333	1·673	1·948	2·094	2·120
180°	$8 \times 8 \times 5$		1·162	1·179	1·242	1·333	1·714	1·948	2·094	2·223
	$12 \times 4 \times 4$		1·162	1·179	1·242	1·333	1·674	1·948	2·094	‡
	$n_x \times n_y \times n_z$	K'_u, K'_v, K'_w								
360°	$12 \times 4 \times 4$	10^2	1·162	1·179	1·241	1·338	1·918	1·947	2·085	‡
		10^4	1·162	1·179	1·242	1·341	1·923	1·948	2·093	‡
		10^6	1·162	1·179	1·242	1·341	1·923	1·948	2·094	‡
	$14 \times 4 \times 4$	10^6	1·162	1·179	1·242	1·334	1·715	1·948	2·094	‡
	Exact [20]		1·161	1·173	1·232	1·340	1·690	1·948	2·085	2·146§
%Error†			0·1	0·5	0·8	−0·5	−1·0	0·0	0·4	−1·2

† %Error = $100 \times (9 \times 9 \times 6$ result 90° − exact)/exact.
‡ Numerical problems experienced.
§ Results obtained by Soldatos *et al.* [19].

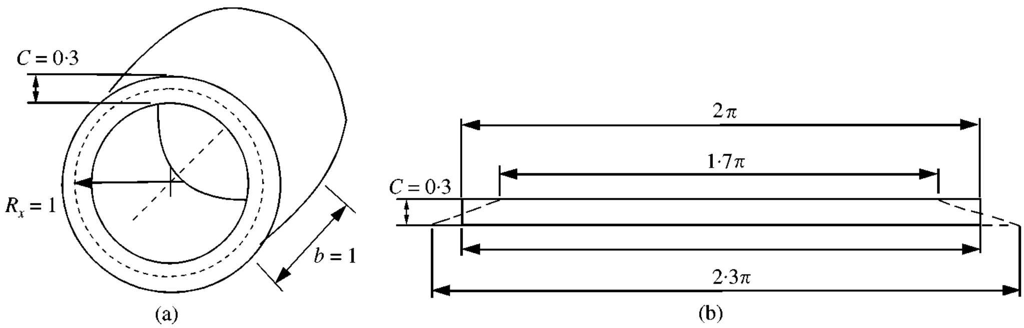

Figure 4. (a) Cylindrical shell of radius $R_x = 1$, axial length $b = 1$ and thickness $c = 0{\cdot}3$; (b) cylindrical shell of Figure 4(a) unravelled into Cartesian co-ordinates showing approximated parallelepiped cross-section used overlaid on actual truncated wedge cross-section.

In Table 4, the lowest five non-dimensional frequency parameters $\Omega = \omega R_x\sqrt{(\rho/E)}$ for a fully simply supported closed cylindrical shell with a non-dimensional thickness to radius of curvature ratio c/R_x ranging from 0·1 to 1·0, and a length to radius of curvature ratio $b/R_x = 1$ are compared with the exact results obtained by Amenakas *et al.* [20]. As can be seen in Table 4, agreement between the approximate and exact results becomes poorer as the shell becomes thicker. However, it is still remarkably good even for very thick shells (e.g., $c/R_x = 1$) where the simplifying assumptions ((i) planform approximated as a parellepiped, and (ii) R_x constant through thickness) of the present approach become significant.

In Table 5, the lowest five non-dimensional frequency parameters $\Omega = \omega R_x\sqrt{(\rho/E)}$ are given for thin ($c/R_x = 0{\cdot}01$) doubly curved closed shells similar to those shown in Figure 3(a) and 3(b), with axial length to radius of curvature ratio $b/R_x = 5$, and for $R_y/R_x = \infty$ (cylinder), $R_y/R_x = 40$, 20 and 10 (increasingly "barrelled" shells as shown in Figure 3(a)) and $R_y/R_x = -40$, -20 and -10 (increasingly "cooling tower" shaped shells as shown in Figure 3(b)). For all cases considered the shell is clamped at both ends. Results obtained using the present approach are compared with very accurate values obtained using the commercial finite element software package ANSYS 5.3 [21]. The most significant assumption made using the present approach is that the curvature R_x is constant along the axial length whereas, as is shown in Appendix A, it is in actual fact a function of R_y (except for the special case of the cylinder). A second, less significant, assumption is that the unravelled planform of the doubly curved shell is rectangular in planform (with side-length ratios $b/R_x = 5$ and $a/R_x = 2\Pi$). Very good agreement is obtained between the present results and the comparison based on an accurate three-dimensional finite element model of the problem (i.e., without the assumptions made in the present approach). As might be expected, as the shell becomes more curved along its axial length, and therefore deviates more significantly from the actual problem, the discrepancy between numerical results obtained using the present approach and the finite element results becomes larger.

4. CONCLUDING REMARKS

Natural frequencies using the proposed strain–displacement equations for "deep-shallow" shells have been shown to give remarkably accurate results when compared with values available in the open literature. It should again be stressed that although the full three-dimensional strain–displacement equations were derived, these could be further simplified for thin shells by introducing the usual Love–Kirchoff approximations to give a deep–shallow thin shell theory.

Table 4

Frequency parameters $\Omega = \omega R_x \sqrt{(\rho/E)}$ for cylindrical shells with shear diaphragm conditions at both ends, with $b/R_x = 1$, and c/R_x varying from 0·1 to 1

c/R_x	$n_x \times n_y \times n_z$	MODE type (axial wave number, circumferential wave number)								%Error	
0·1		(1, 3)	(1, 2)	(1, 1)	(1, 4)	(1, 5)	(1, 0)	(1, ?)	(1, ?)	‡	†
	8 × 8 × 5	0·7121†	0·7758	0·9332	0·7900	0·9845	1·033	1·258	1·590	/	0·3
	Exact [20]	0·7098	0·7739	0·9317	0·7880	0·9842§	1·031	/	/		
0·2		(1, 2)	(1, 1)	(1, 3)	(1, 0)	(1, 4)	(1, 5)	TORS	(1, 1)	‡	†
	8 × 8 × 5	0·9689	1·047	1·051	1·127†	1·303	1·670‡	1·948	2·088	0·5	0·5
	Exact [20]	0·9656	1·043	1·050	1·121	1·306	1·679§	1·948	2·084		
0·3		(1, 2)	(1, 1)	(1, 0)	(1, 3)	(1, 4)	TORS	(1, 1)	(1, 5)		
	8 × 8 × 5	1·162	1·179	1·242†	1·333	1·673	1·948	2·094	2·120‡	1·2	0·8
	Exact [20]	1·161	1·173	1·232	1·340	1·690	1·948¶	2·085	2·146§		
0·4		(1, 1)	(1, 2)	(1, 0)	(1, 3)	(1, 4)	TORS	(1, 1)	(1, 2)	‡	†
	8 × 8 × 5	1·300	1·320	1·355	1·539‡	1·922	1·948	2·102	2·423†	1·2	1·4
	Exact [20]	1·303	1·325	1·341	1·558	1·957	1·948	2·087	2·390		
0·5		(1, 1)	(1, 2)	(1, 0)	(1, 3)	TORS	(1, 4)	(1, 1)	(1, 2)	‡	†
	8 × 8 × 5	1·402	1·455	1·441	1·686	1·948	2·090‡	2·112	2·441†	2·5	2·0
	Exact [20]	1·398	1·439	1·454	1·719	1·948¶	2·143	2·090	2·393		
0·7		(1, 1)	(1, 2)	(1, 0)	(1, 3)	TORS	(1, 1)	(1, 4)	(1, 2)	‡	†
	8 × 8 × 5	1·548	1·600	1·611	1·865	1·948	2·134	2·284	2·477†	3·6	3·2
	Exact [20]	1·552	1·631	1·591	1·925	1·948¶	2·095	2·360	2·392		
0·9		(1, 1)	(1, 2)	(1, 0)	(1, 3)	TORS	(1, 1)	(1, 2)	(1, 4)	‡	†
	7 × 7 × 7	1·638	1·692	1·718	1·961‡	1·948	2·154†	2·383	2·501	3·7	2·7
	Exact [20]	1·651	1·739	1·698	2·036	1·948¶	2·097	2·379	2·433		
1·0		(1, 1)	(1, 2)	(1, 0)	TORS	(1, 3)	(1, 1)	(1, 2)	(1, 4)	‡	†
	7 × 7 × 7	1·669	1·723	1·757	1·948	1·993‡	2·160†	2·415	2·502	3·5	3·0
	Exact [20]	1·687	1·776	1·738	1·948§	2·066	2·097	2·368	2·430		

† Max positive error = 100 × (present − exact)/exact.
‡ Max negative error = 100 × (exact − present)/exact.
§ Results obtained by Soldatos *et al.* [19].
¶ Results obtained by the author.

TABLE 5

Frequency parameters $\Omega = \omega R_x \sqrt{(\rho/E)}$ for closed barrel shells clamped at both ends with $c/R_x = 0{\cdot}01$ and $b/R_x = 5$

R_y/R_x	$\%\Delta R_x/R_x \cong b^2/(8R_xR_y)\times 100$		MODE type (axial wave number, circumferential wave number)					% difference	
∞	0%	$n_x\times n_y\times n_z$	(1, 4)	(1, 5)	(1, 3)	(1, 6)	(2, 5)	−	+
		$6\times 6\times 3$	0·0681	0·0818	0·0842	0·186	0·130		
		$8\times 8\times 5$	0·0664	0·0796	0·0810	0·109	0·113		
		$8\times 14\times 3^{\dagger}$	0·0659	0·0795	0·0800	0·109	0·112	0·9	0·0
		FE‡	0·0661	0·0799	0·0801	0·110	0·113		
40	8%	$n_x\times n_y\times n_z$	(1, 4)	(1, 3)	(1, 5)	(1, 6)	(2, 5)	−	+
		$6\times 6\times 3$	0·0792	0·0918	0·0937	0·123	0·129		
		$8\times 8\times 5$	0·0776	0·0876	0·0926	0·114	0·126		
		$8\times 14\times 3^{\dagger}$	0·0707	0·0874	0·0915	0·113	0·125	6·2	4·4
		FE‡	0·0751	0·0867	0·0875	0·113	0·124		
−40	8%	$n_x\times n_y\times n_z$	(1, 4)	(1, 5)	(1, 3)	(2, 5)	(1, 6)	−	+
		$6\times 6\times 3$	0·0638	0·0749	0·0828	0·107	0·118		
		$8\times 8\times 5$	0·0620	0·0742	0·0782	0·103	0·109		
		$8\times 14\times 3^{\dagger}$	0·0617	0·0742	0·0782	0·103	0·108	4·2	1·9
		FE‡	0·0630	0·0773	0·0777	0·103	0·106		
20	16%	$n_x\times n_y\times n_z$	(1, 4)	(1, 3)	(1, 5)	(1, 6)	(2, 5)	−	+
		$6\times 6\times 3$	0·0948	0·104	0·109	0·132	0·144		
		$8\times 8\times 5$	0·0933	0·101	0·108	0·123	0·141		
		$8\times 14\times 3^{\dagger}$	0·0927	0·100	0·107	0·123	0·140	2·4	5·6
		FE‡	0·0893	0·0985	0·101	0·126	0·138		
−20	16%	$n_x\times n_y\times n_z$	(1, 4)	(1, 3)	(1, 5)	(2, 5)	(2, 4)	−	+
		$6\times 6\times 3$	0·0673	0·0740	0·0878	0·102	0·108		
		$8\times 8\times 5$	0·0656	0·0736	0·0834	0·0975	0·106		
		$8\times 14\times 3^{\dagger}$	0·0655	0·0730	0·0834	0·0971	0·106	10·0	1·4
		FE‡	0·0674	0·0803	0·0822	0·0975	0·110		
10	32%	$n_x\times n_y\times n_z$	(1, 4)	(1, 3)	(1, 5)	(1, 2)	(1, 6)	−	+
		$6\times 6\times 3$	0·133	0·139	0·145	0·160	0·184		
		$8\times 8\times 5$	0·132	0·136	0·144	0·153	0·179		
		$8\times 14\times 3^{\dagger}$	0·131	0·136	0·143	0·152	0·180	0·0	14·4
		FE‡	0·124	0·128	0·134	0·152	0·154		
−10	32%	$n_x\times n_y\times n_z$	(1, 4)	(1, 3)	(1, 5)	(2, 5)	(2, 4)	−	+
		$6\times 6\times 3$	0·0906	0·0891	0·104	0·109	0·100		
		$8\times 8\times 5$	0·0891	0·0889	0·0996	0·105	0·0986		
		$8\times 14\times 3^{\dagger}$	0·0890	0·0888	0·0994	0·105	0·0982	11·0	1·0
		FE‡	0·0908	0·0983	0·100	0·104	0·109		

† Lowest values obtained using $14\times 8\times 3$ or $8\times 14\times 3$ terms. ‡ Finite element results using 8000 brick elements.

A technique for extending the range of problems which can be treated using the "deep-shallow" strain displacement equations in a Ritz approach to closed shells by enforcing continuity at the two connected ends of the shell was also proposed and the applicability of this approach was demonstrated quite conclusively by comparing results obtained with exact values for a hollow circular cylinder.

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REFERENCES

1. K. MARGUERRE 1938 *Proceedings of the Fifth International Congress of Applied Mechanics*, 93–101. Zur Theorie der Gekrummten Platte Grosser Formanderung.
2. E. REISSNER 1946 *Journal of Mathematics and Physics*, Part I **25**, 80–85. Stresses and displacements of shallow spherical shells.
3. E. REISSNER 1947 *Journal of Mathematics and Physics*, Part II **26**, 279–300. Stresses and displacements of shallow spherical shells.
4. E. REISSNER 1959 *Journal of Mathematics and Physics* **38**, 16–35. On the determination of stresses and displacements for unsymmetrical deformations of a shallow spherical shell.
5. V. S. VLASOV 1964 General Theory of Shells and Its Application in Engineering. (translated from the Russian). *NASA TT F*-99.
6. A. LEISSA 1969 *Vibration of Shells*. U.S. Government Printing Office (reprinted in 1993 by The Acoustical Society of America).
7. W. SOEDEL 1981 *Vibrations of Shells and Plates*. New York: Marcel Dekker.
8. M. S. QATU 1992 *Shock and Vibration Digest* **24**, 3–15. Review of shallow shell vibration research.
9. V. V. NOVOZHILOV 1959 *The Theory of Thin Shells*. Netherlands: Noordhoff Groninge.
10. T. MIZUSAWA 1988 *International Journal of Numerical Methods in Engineering* **26**, 663–676. Application of spline strip method to analyse vibration of open cylindrical shell.
11. J. K. LEE, A. W. LEISSA and A. J. WANG 1983 *International Journal of Mechanical Science* **25**, 361–383. Vibrations of cantilevered circular cylindircal shells: shallow vs. deep shell theory.
12. T. KOGA 1989 *Transactions of the Japan Society of Mechanical Engineering*, *Series I*, **32**, 311–319. Free vibrations of circular cylindrical shells.
13. M. S. QATU 1999 *Journal of Vibration and Control* **5**, 851–889. Theory and Vibration Analysis of Laminated Barrel Thin Shells.
14. C. S. KIM, P. G. YOUNG and S. M. DICKINSON 1990 *Journal of Sound and Vibration* **143**, 379–394. On the flexural vibration of rectangular plates approached by using simple polynomials in the Rayleigh–Ritz method.
15. P. G. YOUNG and S. M. DICKINSON 1995 *Journal of Sound and Vibration* **181**, 203–214. Vibration of a class of shallow shells bounded by edges described by polynomials. Part I: theoretical approach and validation.
16. P. G. YOUNG and S. M. DICKINSON 1995 *Journal of Applied Mechanics* **62**, 707–708. Free vibration of a class homogeneous isotropic solids.
17. P. G. YOUNG and S. M. DICKINSON 1994 *Journal of Sound and Vibration* **177**, 93–109. Further studies on the vibration of plates with curved edges, including complicating effects.
18. J. YUAN and S. M. DICKINSON 1992 *Journal of Sound and Vibration* **159**, 39–56. The flexural vibration of rectangular plate systems approached by using artificial springs in the Rayleigh–Ritz method.
19. K. P. SOLDATOS and V. P. HADJIGEORGIOU 1990 *Journal of Sound and Vibration* **137**, 369–384. Three dimensional solution of the free vibration problem of homogeneous isotropic shells and panels.
20. A. E. ARMENAKAS, D. C. GAZIS and G. HERRMANN 1969 *Free Vibrations of Circular Cylindrical Shells*. Oxford: Pergamon Press.
21. Ansys Reference Manual 1996. *Release* 5.3, SAS IP inc.

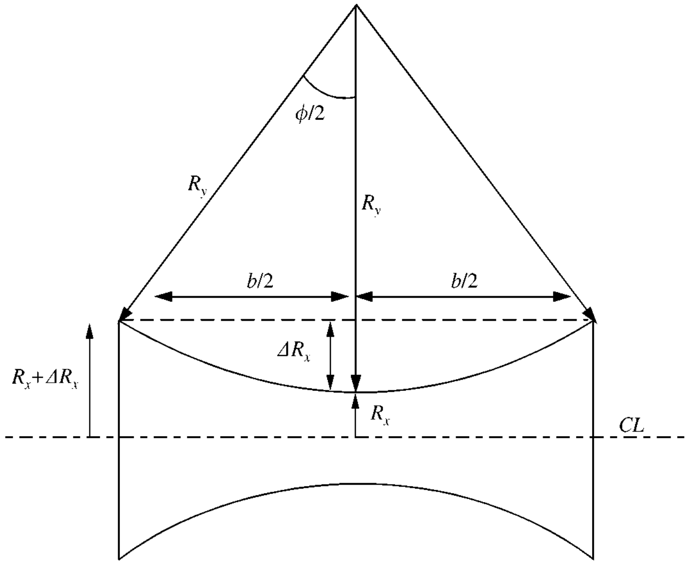

Figure A1. Closed cooling tower-shaped shell (with constant negative meridional curvature R_y).

APPENDIX A

As is shown in Figure A1, a closed shell with constant meridional curvature R_y will have a radial curvature R_x which is a function of the axial position (in Cartesian co-ordinates a function of y).

To approximate R_x as a constant it is necessary that $\Delta R_x/R_x \ll 1$.

Now from Figure A1,

$$\Delta R_x/R_x = R_y(1 - \cos(\theta))/R_x \text{ and } \sin(\theta) = (b/2)/R_y.$$

If θ is assumed small (as is implied by the requirement $b/R_y \ll 1$, or in other words the requirement that the shell is shallow in the y direction) then

$$\theta \cong \sin(\theta) = b/(2Ry) \text{ and } \cos(\theta) \cong 1 - \theta^2/2 = 1 - b^2/(8R_y^2)$$

and $\Delta R_x/R_x$ can now be expressed in terms of the axial length b and the curvature R_x and R_y as $\Delta R_x/R_x \cong b^2/(8R_yR_x) \ll 1$.