OUT–OF–PLANE VIBRATIONS OF CURVED NON–UNIFORM BEAMS OF CONSTANT RADIUS

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The governing differential equations for out-of-plane vibrations of curved non-uniform beams of constant radius are derived. Two physical parameters are introduced to simplify the analysis. The explicit relations between the flexural displacement, its first three order derivatives and the torsional displacement are derived. With these explicit relations, the two coupled governing characteristic differential equations can be decoupled and reduced to a sixth order ordinary differential equation with variable coefficients in the torsional displacement. It is shown that if the material and geometric properties of the beam are in arbitrary polynomial forms of spatial variable, then exact solutions for the out-of-plane vibrations of the beam can be obtained. The derived explicit relations can also be used to reduce the difficulty in experimental measurements. Finally, the influence of taper ratio, center angle and arc length on the first two natural frequencies of the beams is illustrated.

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1. INTRODUCTION

Curved beam structures have been used in many mechanical, aerospace and civil engineering applications such as spring design, curved wires in missile-guidance floated gyroscopes, curved girder bridges, brake shoes within drum brakes, tire dynamics, stiffeners in aircraft structures and turbo-machinery blades. It can also be used as a simplified model of a shell structure. Research in this area can be traced back to the 19th century [1, 2]. An interesting review can be found in the review papers by Markus and Nanasi [3], Laura and Maurizi [4], Chidamparam and Leissa [5] and Auciello and De Rosa [6].

In general, the out-of-plane and the in-plane vibrations of a general plane curved beam are coupled. However, based on the Bernoulli—Euler hypothesis, if the cross-section of the curved beam is uniform and doubly symmetric, then the out-of-plane and the in-plane vibrations are independent [7].

Out-of-plane vibrations of curved beams have been studied by many investigators [3–6]. The associated governing differential equations are two coupled differential equations in terms of the out-of-plane flexural displacement and the torsional displacement. It is known that if the beam is uniform, then the coefficients of the differential equations are constants. After some simple arithmetic operations, the two coupled differential equations can be reduced into one sixth order ordinary differential equation [2, 7]. The problem was solved by different analytical methods and the exact solutions can be obtained [1, 2, 7, 8]. However, it is not the case for the non-uniform beams. Due to the complexity in the coefficients of the governing differential equations, the two coupled differential equations have never been reduced into one sixth order ordinary differential equation before. Exact solutions for the curved non-uniform beam problem are only found in the work by

Suzuki *et al.* [9], who gave an exact series solution to the beams with the same boundary conditions at both the ends. Nevertheless, their method has difficulty in handing the problems with other kinds of boundary conditions. Hence, curved non-uniform beam problems were studied mainly by approximate methods such as the Rayleigh–Ritz method [10], the lumped mass approach [11], the transfer matrix method [12], the finite element method and the discrete Green function method [13].

In this paper, one considers a non-uniform beam with doubly symmetric cross-section. It is assumed that the thickness of the beam is small in comparison with the radius of the beam. Without considering the shear deformation, the rotary inertia and the warping effects, the governing differential equations for the out-of-plane vibrations of curved non-uniform beams of constant radius are derived via the Hamilton principle. By introducing two physical parameters, the analysis is simplified and it is found that the torsional displacement and its derivative can be explicitly expressed in terms of the flexural displacement. With these explicit relations, the two coupled governing characteristic differential equations are decoupled and reduced to one sixth order ordinary differential equation with variable coefficients in the torsional displacement. It can be shown that if the material and geometric properties of the beam are in arbitrary polynomial forms of spatial variable, exact solutions for the out-of-plane vibrations of non-niform curved beams can be obtained. It is worthwhile to mention that by employing the explicit relations, one only has to measure the torsional displacement instead of measuring the torsional and the flexural displacements simultaneously in the experimental study of the cruved beam. Hence, it greatly reduces the difficulty in experimental measurement. Finally, the influence of taper ratio, center angle and arc length on the first two natural frequencies of the beams is studied.

2. COUPLED GOVERNING DIFFERENTIAL EQUATIONS

Consider the out-of-plane and in-plane vibrations of a non-uniform curved beam of radius R, as shown in Figure 1. If the thickness of the beam is small in comparison with the radius of the beam, without considering the shear deformation, the rotary inertia and the warping effects, the displacement fields of the curved beam in cylindrical co-ordinates are [5]

$$u_r(r, s, z, t) = v(s, t) + z\phi(s, t),$$

$$u_\theta(r, s, z, t) = u(s, t) - r\frac{\partial v(s, t)}{\partial s} - z\frac{\partial w(s, t)}{\partial s},$$

$$(1)$$

$$u_z(r, s, z, t) = w(s, t) - r\phi(s, t),$$

where u_r, u_θ and u_z denote the displacements of the beam in the r, θ and z directions respectively. $s = R\theta$ and ϕ is the torsion angle. v, u and w are the neutral axis displacements of the beam in the r, θ and z directions respectively. Substituting equation (1) into the strain–displacement relations in the cylindrical co-ordinate [14] yields

$$\varepsilon_{\theta\theta} = -r\frac{\partial^2 v}{\partial s^2} + z\left(\frac{\phi}{R} - \frac{\partial^2 w}{\partial s^2}\right) + \left(\frac{\partial u}{\partial s} + \frac{v}{R}\right),$$

$$\gamma_{r\theta} = z\left(\frac{\partial \phi}{\partial s} + \frac{1}{R}\frac{\partial w}{\partial s}\right),$$

$$(2)$$

$$\gamma_{\theta z} = -r\left(\frac{\partial \phi}{\partial s} + \frac{1}{R}\frac{\partial w}{\partial s}\right),$$

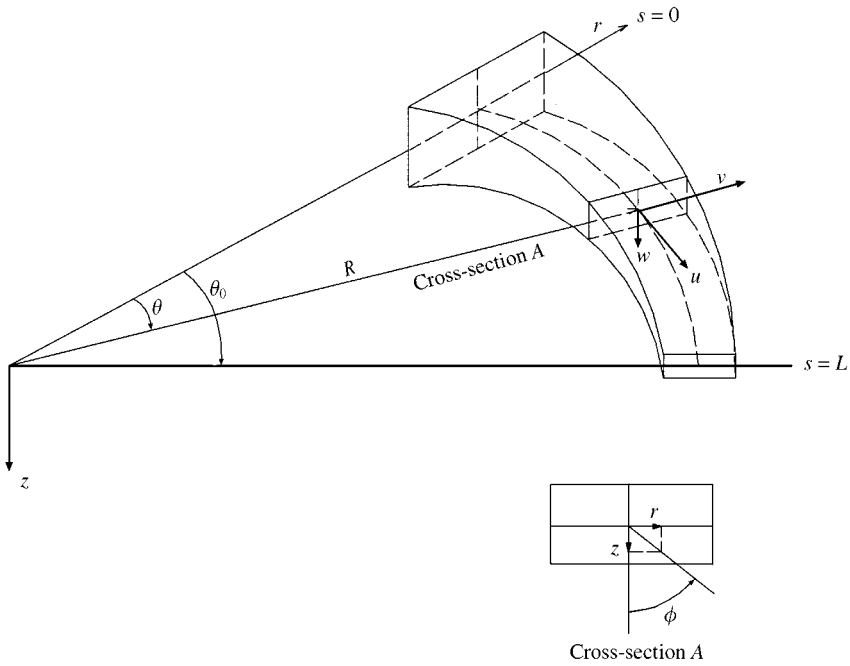$$\varepsilon_{rr} = \varepsilon_{zz} = \varepsilon_{rz} = 0.$$

Figure 1. Geometry and co-ordinate system of a curved non-uniform beam with constant radius.

The potential energy and the kinetic energy of the beam are

$$
V = \frac{1}{2} \int_0^L \int_A (E\varepsilon_{\theta\theta}^2 + G\gamma_{r\theta}^2 + G\gamma_{\theta z}^2)\, \mathrm{d}A\, \mathrm{d}s
$$

$$
= \int_0^L E\left[-\alpha_1 \frac{\partial^2 v}{\partial s^2}\left(\frac{\phi}{R} - \frac{\partial^2 w}{\partial s^2}\right) + \alpha_2 \left(\frac{\partial u}{\partial s} + \frac{v}{R}\right)\left(\frac{\phi}{R} - \frac{\partial^2 w}{\partial s^2}\right) \right.
$$

$$
\left. - \delta_1 \frac{\partial^2 v}{\partial s^2}\left(\frac{\partial u}{\partial s} + \frac{v}{R}\right) \right]\mathrm{d}s + \frac{1}{2}\int_0^L \left[EI_z\left(\frac{\partial^2 v}{\partial s^2}\right)^2 + EI_r\left(\frac{\phi}{R} - \frac{\partial^2 w}{\partial s^2}\right)^2 + EA \right.
$$

$$
\left. \times \left(\frac{\partial u}{\partial s} + \frac{v}{R}\right)^2 \right]\mathrm{d}s + \frac{1}{2}\int_0^L GI_\theta\left(\frac{\partial \phi}{\partial s} - \frac{1}{R}\frac{\partial w}{\partial s}\right)^2 \mathrm{d}s \tag{3}
$$

and

$$
T = \frac{1}{2}\int_0^L \left\{ \rho A\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] + \rho I_\theta\left(\frac{\partial \phi}{\partial t}\right)^2 \right\}\mathrm{d}s \tag{4}
$$

respectively. $E(s)$ is the Young's modulus, $G(s)$ is the shear modulus, $A(s)$ is the cross-sectional area, L is the length of the neutral axis and $\rho(s)$ is the mass per unit volume of the beam. $I_r(s)$ and $I_z(s)$ denote the area moments of inertia of the beam section about the r and

z axes respectively. $I_\theta(s)$ is the polar moment of inertia about the θ-axis. The terms $\alpha_1 \sim \alpha_2$ and δ_1 in equation (3) are given in Appendix A.

Via the Hamilton principle, the governing differential equations and the associated boundary conditions for the system can be derived. It can be shown that if the cross-section of the beam is doubly symmetric about the r and z axes, then the terms $\alpha_1 \sim \alpha_2$ will vanish. As a result, the in-plane and the out-of-plane vibrations of curved non-uniform beams are independent. This conclusion is consistent with that for the curved uniform beam theory [7].

The governing differential equations for the out-of-plane vibrations are the two coupled differential equations

$$\left[EI_r \left(\frac{1}{R}\phi - w'' \right) \right]'' + \left[GI_\theta \left(\frac{1}{R}\phi' + \frac{1}{R^2}w' \right) \right]' = \rho A \ddot{w},$$

$$\left[GI_\theta \left(\phi + \frac{1}{R}w' \right) \right]' - EI_r \left(\frac{1}{R^2}\phi - \frac{1}{R}w'' \right) = \rho I_\theta \ddot{\phi}, \tag{5}$$

where the primes denote differentiation with respect to s. When the beam is uniform, then the governing differential equations (5) can be reduced to those of the uniform beam [15]. For time-harmonic vibrations with angular frequency Ω, one assumes

$$\phi(s, t) = \Phi(s)\mathrm{e}^{\mathrm{i}\Omega t},$$

$$w(s, t) = W(s)\mathrm{e}^{\mathrm{i}\Omega t}. \tag{6}$$

Consequently, the two coupled governing characteristic differential equations of the beam are

$$\left[EI_r \left(\frac{1}{R}\Phi - W'' \right) \right]'' + \left[GI_\theta \left(\frac{1}{R}\Phi' + \frac{1}{R^2}W' \right) \right]' + \rho A\Omega^2 W = 0, \tag{7}$$

$$\left[GI_\theta \left(\Phi' + \frac{1}{R}W' \right) \right]' - EI_r \left(\frac{1}{R^2}\Phi - \frac{1}{R}W'' \right) + \rho I_\theta \Omega^2 \Phi = 0. \tag{8}$$

The associated boundary conditions are at $s = 0$ and L:

$$\left[EI_r \left(\frac{1}{R}\Phi - W'' \right) \right]' + GI_\theta \left(\frac{1}{R}\Phi' + \frac{1}{R^2}W' \right) = 0 \quad \text{or} \quad W = 0, \tag{9}$$

$$EI_r \left(\frac{1}{R}\Phi - W'' \right) = 0 \quad \text{or} \quad W' = 0, \tag{10}$$

$$GI_\theta \left(\Phi' + \frac{1}{R}W' \right) = 0 \quad \text{or} \quad \Phi = 0. \tag{11}$$

If the beam is clamped at the boundary, then the boundary conditions are $W = 0$, $W' = 0$ and $\Phi = 0$. If the beam is free at the boundary, then the other identities in equations (9)–(11) are specified.

3. UNCOUPLED GOVERNING DIFFERENTIAL EQUATION IN THE TORSIONAL ANGLE PARAMETER Φ

3.1. CURVED NON-UNIFORM BEAMS

To uncouple the governing differential equations (7) and (8), one defines two physical parameters

$$T_\Phi = EI_r \frac{\Phi}{R^2} - (GI_\theta \Phi')' - \rho I_\theta \Omega^2 \Phi, \tag{12}$$

$$F_{z\Phi} = \left(EI_r \frac{\Phi}{R} \right)'' + \left(GI_\theta \frac{\Phi'}{R} \right)', \tag{13}$$

where T_Φ and $F_{z\Phi}$ are the torque per unit arc length and the force per unit arc length in the z direction, caused by the torsional angle Φ respectively. Differentiating equation (8), it yields

$$\frac{1}{R} W''' = \frac{a_5}{R} W' + \left(\frac{T_\Phi}{p_w} \right)' - \frac{(GI_\theta)'}{p_w^2} T_\Phi, \tag{14}$$

where a_5 is given in Appendix B. Substituting equation (8) into the equation resulting from differentiating equation (14), one obtains

$$\frac{1}{R} W'''' = \frac{a_6}{R} W' + \left(\frac{T_\Phi}{p_w} \right)'' - \left[\frac{(GI_\theta)'}{p_w^2} T_\Phi \right]' + \frac{a_5}{p_w} T_\Phi, \tag{15}$$

where a_6 is given in Appendix B. Substituting equations (8), (14) and (15) into equation (7) leads to

$$\frac{1}{R} W = \frac{a_7}{R} W' + a_8 T_\Phi + \frac{EI_r}{\rho A \Omega^2} \left\{ \left(\frac{T_\Phi}{p_w} \right)'' - \left[\frac{(GI_\theta)'}{p_w^2} T_\Phi \right]' \right\} + \frac{1}{\rho A \Omega^2} \left[2(EI_r)' \right.$$

$$\times \left. \left(\frac{T_\Phi}{p_w} \right)' - \frac{F_{z\Phi}}{R} \right], \tag{16}$$

where a_7 and a_8 are listed in Appendix B. After differentiating equation (16) once and combining it with equation (8) and substituting the results into equations (8), (14) and (15), one can explicitly express W, W', W'', W''' in terms of Φ

$$\frac{1}{R} W' = \frac{q_\Phi}{a_9} + \frac{a_7}{p_w a_9} T_\Phi, \tag{17}$$

$$\frac{1}{R} W'' = -\frac{(GI_\theta)'}{p_w a_9} q_\Phi + \frac{1}{p_w} \left[1 - \frac{a_7 (GI_\theta)'}{p_w a_9} \right] T_\Phi, \tag{18}$$

$$\frac{1}{R} W''' = \frac{a_5}{a_9} q_\Phi + \frac{1}{p_w} \left[\frac{a_5 a_7}{a_9} - \frac{(GI_\theta)'}{p_w} \right] T_\Phi + \left(\frac{T_\Phi}{p_w} \right)', \tag{19}$$

$$\frac{1}{R}W = \frac{a_7}{a_9}q_\Phi + \left(a_8 + \frac{a_7^2}{p_w a_9}\right)T_\Phi + \frac{EI_r}{\rho A\Omega^2}\left\{\left(\frac{T_\Phi}{p_w}\right)'' - \left[\frac{(GI_\theta)'}{p_w^2}T_\Phi\right]'\right\}$$

$$+ \frac{1}{\rho A\Omega^2}\left[2(EI_r)'\left(\frac{T_\Phi}{p_w}\right) - \frac{F_{z\Phi}}{R}\right], \tag{20}$$

where

$$q_\Phi = \left\{\frac{EI_r}{\rho A\Omega^2}\left\{\left(\frac{T_\Phi}{p_w}\right)'' - \left[\frac{(GI_\theta)'}{p_w^2}T_\Phi\right]'\right\} + a_8 T_\Phi + \frac{1}{\rho A\Omega^2}\left[2(EI_r)'\left(\frac{T_\Phi}{p_w}\right)' - \frac{F_{z\Phi}}{R}\right]\right\}' \tag{21}$$

and a_9 is given in Appendix B. After substituting equations (17)–(20) into equation (7), one obtains an uncoupled sixth order ordinary differential equation in the torsional angle parameter Φ

$$\left\{\frac{a_{10}}{a_9}\left\{\frac{EI_r}{\rho A\Omega^2}\left\{\left(\frac{T_\Phi}{p_w}\right)'' - \left[\frac{(GI_\theta)'}{p_w^2}T_\Phi\right]'\right\} + a_8 T_\Phi + \frac{1}{\rho A\Omega^2}\left[2(EI_r)'\left(\frac{T_\Phi}{p_w}\right)' - \frac{F_{z\Phi}}{R}\right]\right\}'\right.$$

$$+ \frac{1}{p_w}\left[\frac{a_7 a_{10}}{a_9} + \frac{EI_r(GI_\theta)'}{p_w} - EI_r\right]T_\Phi - EI_r\left(\frac{T_\Phi}{p_w}\right)'\right\} + \rho A\Omega^2\left\{\frac{a_7}{a_9}\left\{\left[2(EI_r)'\left(\frac{T_\Phi}{p_w}\right)'\right.\right.\right.$$

$$\left. - \frac{F_{z\Phi}}{R}\right]\frac{1}{\rho A\Omega^2} + a_8 T_\Phi + \frac{EI_r}{\rho A\Omega^2}\left\{\left(\frac{T_\Phi}{p_w}\right)'' - \left[\frac{(GI_\theta)'}{p_w^2}T_\Phi\right]'\right\}\right\}' + \left(a_8 + \frac{a_7^2}{p_w a_9}\right)T_\Phi$$

$$\left. + \frac{EI_r}{\rho A\Omega^2}\left\{\left(\frac{T_\Phi}{p_w}\right)'' - \left[\frac{(GI_\theta)'}{p_w^2}T_\Phi\right]'\right\} + \frac{1}{\rho A\Omega^2}\left[2(EI_r)'\left(\frac{T_\Phi}{p_w}\right) - \frac{F_{z\Phi}}{R}\right]\right\}\left(\frac{EI_r}{R^2}\Phi\right)''$$

$$+ \left(\frac{GI_\theta}{R^2}\Phi'\right)' = 0, \tag{22}$$

where a_{10} is given in Appendix B. The boundary conditions (9)–(11) in the torsional angle parameter Φ now are

$$\left\{EI_r\left\{\frac{\Phi}{R^2} + \frac{(GI_\theta)'}{p_w a_9}\left\{\frac{EI_r}{\rho A\Omega^2}\left\{\left(\frac{T_\Phi}{p_w}\right)'' - \left[\frac{(GI_\theta)'}{p_w^2}T_\Phi\right]'\right\} + a_8 T_\Phi + \frac{1}{\rho A\Omega^2}\left[2(EI_r)'\right.\right.\right.\right.$$

$$\left.\left(\frac{T_\Phi}{p_w}\right)' - \frac{F_{z\Phi}}{R}\right]\right\}' - \frac{1}{p_w}\left[1 - \frac{a_7(GI_\theta)'}{p_w a_9}\right]T_\Phi\right\}\right\}' + \frac{GI_\theta}{R^2}\left\{\Phi' + \frac{1}{a_9}\left\{\frac{EI_r}{\rho A\Omega^2}\left\{\left(\frac{T_\Phi}{p_w}\right)''\right.\right.\right.$$

$$\left. - \left[\frac{(GI_\theta)'}{p_w^2}T_\Phi\right]'\right\} + a_8 T_\Phi + \frac{1}{\rho A\Omega^2}\left[2(EI_r)'\left(\frac{T_\Phi}{p_w}\right)' - \frac{F_{z\Phi}}{R}\right]\right\}' + \frac{a_7}{p_w a_9}T_\Phi\right\} = 0 \tag{23}$$

or

$$\frac{a_7}{a_9}\left\{\frac{EI_r}{\rho A\Omega^2}\left\{\left(\frac{T_\Phi}{p_w}\right)'' - \left[\frac{(GI_\theta)'}{p_w^2}T_\Phi\right]'\right\} + \frac{1}{\rho A\Omega^2}\left[2(EI_r)'\left(\frac{T_\Phi}{p_w}\right)' - \frac{F_{z\Phi}}{R}\right]\right\}'$$

$$+ a_8 T_\Phi + \left(a_8 + \frac{a_7^2}{p_w a_9}\right)T_\Phi + \frac{EI_r}{\rho A\Omega^2}\left\{\left(\frac{T_\Phi}{p_w}\right)'' - \left[\frac{(GI_\theta)'}{p_w^2}T_\Phi\right]'\right\} + \frac{1}{\rho A\Omega^2}$$

$$\left[2(EI_r)'\left(\frac{T_\Phi}{p_w}\right)' - \frac{F_{z\Phi}}{R}\right] = 0, \tag{24}$$

$$\frac{\Phi}{R^2} + \frac{(GI_\theta)'}{p_w a_9}\left\{\frac{EI_r}{\rho A \Omega^2}\left\{\left(\frac{T_\Phi}{p_w}\right)'' - \left[\frac{(GI_\theta)'}{p_w^2}T_\Phi\right]'\right\} + a_8 T_\Phi + \frac{1}{\rho A \Omega^2}\left[2(EI_r)'\right.\right.$$

$$\left.\left.\left(\frac{T_\Phi}{p_w}\right)' - \frac{F_{z\Phi}}{R}\right]\right\} - \frac{1}{p_w}\left[1 - \frac{a_7(GI_\theta)'}{p_w a_9}\right]T_\Phi = 0 \qquad (25)$$

or

$$\frac{a_7}{p_w a_9}T_\Phi + \frac{1}{a_9}\left\{\frac{EI_r}{\rho A \Omega^2}\left\{\left(\frac{T_\Phi}{p_w}\right)'' - \left[\frac{(GI_\theta)'}{p_w^2}T_\Phi\right]'\right\} + a_8 T_\Phi + \frac{1}{\rho A \Omega^2}\left[2(EI_r)'\left(\frac{T_\Phi}{p_w}\right)' - \frac{F_{z\Phi}}{R}\right]\right\}' = 0, \qquad (26)$$

$$\Phi' + \frac{1}{a_9}\left\{\frac{EI_r}{\rho A \Omega^2}\left\{\left(\frac{T_\Phi}{p_w}\right)'' - \left[\frac{(GI_\theta)'}{p_w^2}T_\Phi\right]'\right\} + a_8 T_\Phi + \frac{1}{\rho A \Omega^2}\left[2(EI_r)'\left(\frac{T_\Phi}{p_w}\right)'\right.\right.$$

$$\left.\left. - \frac{F_{z\Phi}}{R}\right]\right\}' + \frac{a_7}{p_w a_9}T_\Phi = 0 \qquad (27)$$

or

$$\Phi = 0. \qquad (28)$$

3.2. CURVED UNIFORM BEAMS

For curved uniform beams,

$$a_5 = a_6 = a_7 = 0, \quad a_8 = -\frac{GI_\theta}{\rho A \Omega^2 R^2 p_w}, \quad a_9 = 1 \text{ and } a_{10} = \frac{GI_\theta}{R^2}. \qquad (29)$$

T_Φ and $F_{z\Phi}$ are reduced to

$$T_\Phi = EI_r \frac{\Phi}{R^2} - GI_\theta \Phi'' - \rho I_\theta \Omega^2 \Phi, \qquad (30)$$

$$F_{z\Phi} = EI_r \frac{\Phi''}{R} + GI_\theta \frac{\Phi''}{R}. \qquad (31)$$

The governing equation (22) becomes

$$T_\Phi'''' + \frac{\rho A \Omega^2 p_w R^2}{GI_\theta EI_r}\left(\frac{a_8 GI_\theta}{R^2} - \frac{EI_r}{p_w^2}\right)T_\Phi'' - \frac{p_w}{REI_r}F_{z\Phi}'' + \frac{\rho A \Omega^2 p_w^2}{GI_\theta EI_r}\Phi''$$

$$+ \frac{a_8(\rho A \Omega^2)^2 p_w R^2}{GI_\theta EI_r}T_\Phi = 0. \qquad (32)$$

Substituting equations (30) and (31) into equation (32), the governing equation can be written as

$$\Phi^{(6)} + \left(\frac{2}{R^2} + \frac{\rho \Omega^2}{G}\right)\Phi'''' + \left[\frac{1}{R^4} - \frac{\rho}{E}\Omega^2\left(\frac{A}{I_r} + \frac{1}{R^2}\frac{I_\theta}{I_r}\right)\right]\Phi''$$

$$+ \frac{\rho A \Omega^2}{GI_\theta}\left(\frac{1}{R^2} - \frac{I_\theta}{I_r}\frac{\rho \Omega^2}{E}\right)\Phi = 0, \qquad (33)$$

where $\Phi^{(6)}$ indicates the sixth derivative of Φ with respect to s. Equation (33) is the uncoupled differential equation of a curved uniform beam and is the same as that given by Lee [16].

3.3. EXACT FUNDAMENTAL SOLUTIONS

The decoupled governing characteristic differential equation, in terms of Φ, for the out-of-plane vibration of a curved non-uniform beams is a sixth order ordinary differential equation with variable coefficients in the form of

$$e_0(x)\frac{d^6 Y(x)}{dx^6} + e_1(s)\frac{d^5 Y(x)}{dx^5} + \cdots + e_5(x)\frac{dY(x)}{dx} + e_6(x)Y(x) = 0, \quad x \in (0, 1). \quad (34)$$

If all of the coefficients are in the polynomial forms, i.e.,

$$e_p(x) = \sum_{j=0}^{m_p} a_{p,j} x^j, \quad p = 0\text{--}6, \quad (35)$$

where m_p, $p = 0\text{--}6$, are integers representing the number of terms in the series, then one can assume the six fundamental solutions of the differential equation to be in the form of [17]

$$Y_i(x) = \frac{1}{i!}x^i + \sum_{q=n}^{\infty} A_{q,i} x^q, \quad i = 0\text{--}5, \quad n = 6. \quad (36)$$

Substituting equations (35) and (36) into equation (34), collecting the coefficients of like powers, the following recurrence formula can be obtained:

$$A_{q,i} = -\frac{(q-n)!}{(q!)a_{0,0}}\left\{ \sum_{k=1}^{q-n} \frac{(q-k)!}{(q-k-n)!} a_{0,k} A_{q-k,i} + \sum_{m=1}^{n}\left[\frac{1}{(i-n+m)!} a_{m,q-m-i} \right.\right.$$

$$\left.\left. + \sum_{k=m}^{q-n} \frac{(q-k)!}{(q-k-n+m)!} a_{m,k-m} A_{q-k,i} \right]\right\}, \quad q \geqslant n, n = 6. \quad (37)$$

With this recurrence formula, one can generate the six exact fundamental solutions of the governing characteristic differential equation. They satisfy the following normalization condition at the origion of the co-ordinate system.

$$\begin{bmatrix} Y_0(0) & Y_0'(0) & Y_0''(0) & Y_0'''(0) & Y_0''''(0) & Y_0^{(5)}(0) \\ Y_1(0) & Y_1'(0) & Y_1''(0) & Y_1'''(0) & Y_1''''(0) & Y_1^{(5)}(0) \\ Y_2(0) & Y_2'(0) & Y_2''(0) & Y_2'''(0) & Y_2''''(0) & Y_2^{(5)}(0) \\ Y_3(0) & Y_3'(0) & Y_3''(0) & Y_3'''(0) & Y_3''''(0) & Y_3^{(5)}(0) \\ Y_4(0) & Y_4'(0) & Y_4''(0) & Y_4'''(0) & Y_4''''(0) & Y_4^{(5)}(0) \\ Y_5(0) & Y_5'(0) & Y_5''(0) & Y_5'''(0) & Y_5''''(0) & Y_5^{(5)}(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (38)$$

where $Y_i^{(5)}$, $i = 0\text{--}5$, indicates the fifth derivative of Y_i with respect to s.

After substituting the homogeneous solution which is a linear combination of these fundamental solutions into the associated boundary conditions, the frequency equation and natural frequencies of the beams are obtained, consequently.

4. NUMERICAL RESULTS

To illustrate the previous analysis, the out-of-plane vibrations of curved non-uniform beams of circular cross-section are studies. The following dimensionless parameters will be used in the following numerical analysis:

$$s^* = \frac{s}{L},$$

$$R_\theta^* = \frac{R}{\gamma_\theta(0)},$$

$$m(s^*) = \frac{\rho(s)A(s)}{\rho(0)A(0)},$$

$$I(s^*) = \frac{\rho(s)I_\theta(s)}{\rho(0)I_\theta(0)},$$

$$b_\theta(s^*) = \frac{G(s)I_\theta(s)}{G(0)I_\theta(0)},$$

$$b_{r\theta}(s^*) = \frac{E(s)I_r(s)}{G(0)I_\theta(0)},$$

$$L_\theta = R_\theta^*\theta_0,$$

$$\Lambda_\theta = \left[\Omega \Big/ \sqrt{\frac{G(0)}{\rho(0)L^2}}\right]^2.$$

(39)

Here, γ_z is the radius of gyration about the z-axis and θ_0 is the center angle.

In Table 1, the first out-of-plane natural frequencies of clamped–clamped curved beams are compared with those given in the existing literature [18]. The comparison shows that the results are very consistent.

TABLE 1

The first out-of-plane natural frequencies of curved uniform beams with clamped–clamped ends

θ_0		$b_{r\theta}(0) = 0\cdot615$		$b_{r\theta}(0) = 1\cdot0$		$b_{r\theta}(0) = 2\cdot0$	
(deg)	R	(1)	(2)	(1)	(2)	(1)	(2)
0	∞	—	22·373	—	22·373	—	22·373
90	50	—	20·840	—	20·694	—	20·363
180	50	18·379	18·361	18·132	18·128	17·564	17·560
270	50	17·767	17·765	16·877	16·875	15·343	15·342

Note: (1), $\alpha^2\sqrt{k\lambda_n}$ in Volterra and Morell [18]; (2), $c\sqrt{\Lambda_\theta}$ shown in the present paper, where $c = L_\theta/\sqrt{b_{r\theta}(0)}$.
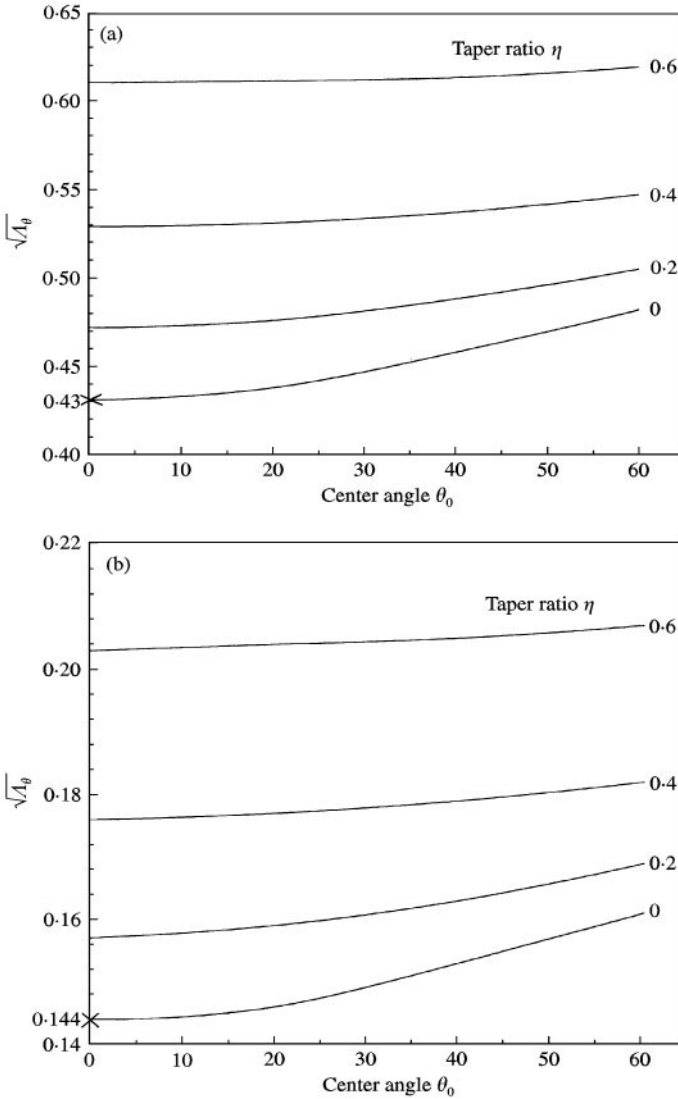
Figure 2. The influence of the center angle on the first dimensionless natural frequencies $\sqrt{\Lambda_\theta}$ of the beams with various taper ratios ($b_{r\theta}(0) = 1\cdot5$). L_θ values: (a) $L_\theta = 10$; (b) $L_\theta = 30$.

In the following, the natural frequencies of linearly tapered curved beams of circular cross-section with clamped–free ends are studied. The material and geometric properties of the beams with taper ratio η are $m(s^*) = (1 - \eta s^*)^2$, $b_\theta(s^*) = (1 - \eta s^*)^2$ and $b_{r\theta}(s^*) = b_{r\theta}(0)$ $(1 - \eta s^*)^4$ respectively.

In Figure 2, the influence of the center angle on the first dimensionless natural frequency $\sqrt{\Lambda_\theta}$ of the beams with various taper ratios and two different dimensionless arc lengths $L_\theta = 10$ and 30 is shown. For the beam with the center angle θ_0 being zero, the radius of the beam is infinite. It represents a straight beam. The ones in the figures with cross mark denote the dimensionless natural frequency of a straight uniform beam and are consistent with the exact solutions given by Meirovitch [19]. From the figure, it can be observed that
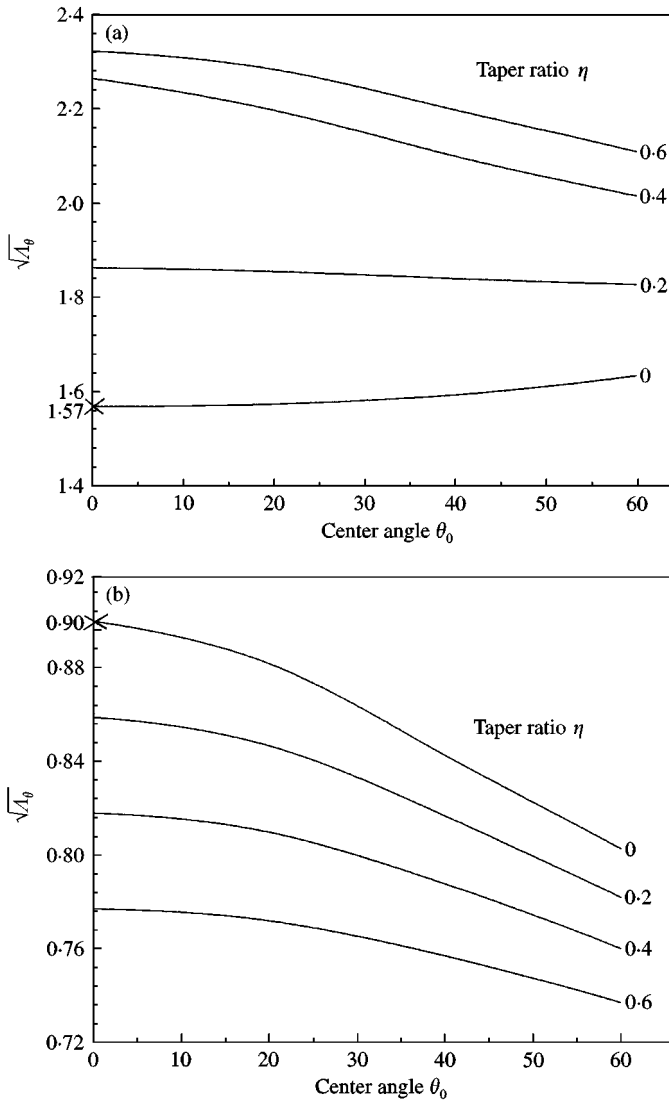
Figure 3. The influence of the center angle on the second dimensionless natural frequencies $\sqrt{\Lambda_\theta}$ of the beams with various taper ratios ($b_{r\theta}(0) = 1\cdot5$). L_θ values: (a) $L_\theta = 10$; (b) $L_\theta = 30$.

the first natural frequencies of the beams with the same taper ratio incresase as the center angle is increased. Those of the beams with the same taper ratio increase as the center angle is increased. However, the influence of the center angle on the natural frequencies of the beams with low taper ratio is more significant than those with high taper ratio.

Figure 3, the influence of the center angle θ_0 on the second dimensionless natural frequency of the beams with various taper ratios and two different dimensionless arc lengths $L_\theta = 10$ and 30 is shown. It is found that under different dimensionless arc lengths, the influence of the taper ratio and that of the curvature on the second natural frequencies are quite different and also different from that on the first natural frequencies. When $L_\theta = 10$, the second natural frequencies of the beams with the same center angle increase as the taper ratio is increased. However, the second natural frequencies of the beams with the same
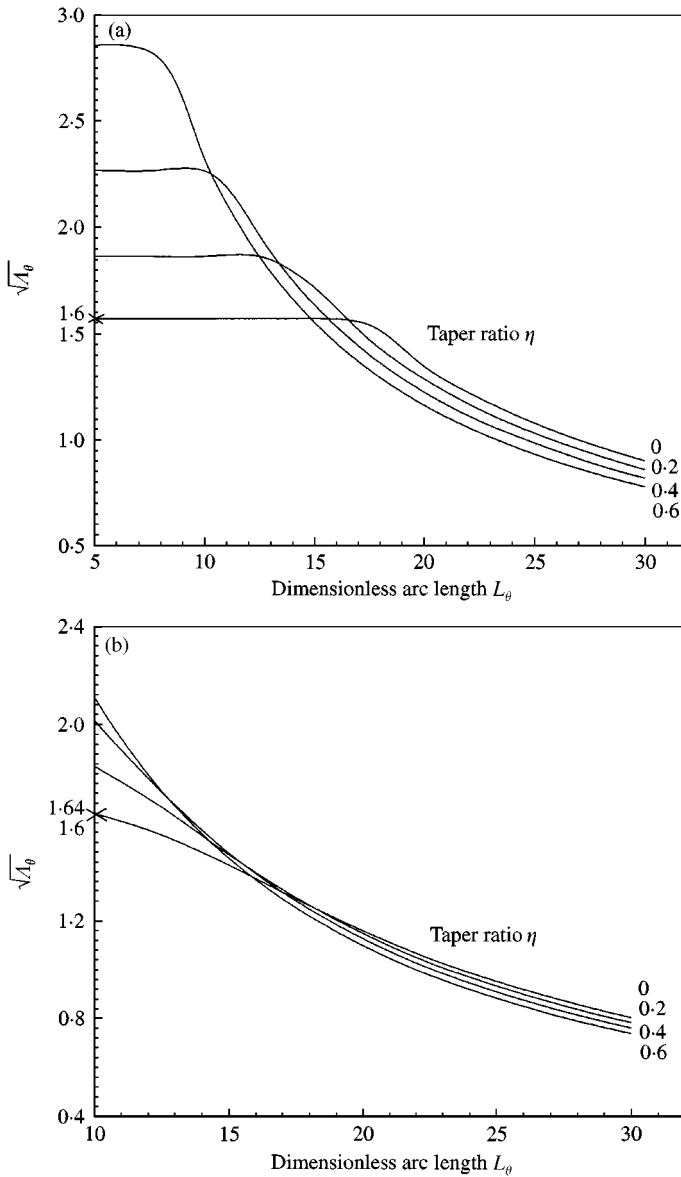
Figure 4. The influence of the dimensionless arc length L_θ on the second dimensionless natural frequencies $\sqrt{\Lambda_\theta}$ of the beams with various taper ratios ($b_{r\theta}(0) = 1 \cdot 5$). θ_0 values: (a) $\theta_0 = 0°$; (b) $\theta_0 = 60°$.

center angle decrease as the taper ratio is increased as $L_\theta = 30$. This phenomenon is well explained in Figure 4 in which the influence of the dimensionless arc length on the second dimensionless natural frequencies $\sqrt{\Lambda_\theta}$ of the beams with various taper ratios is illustrated. It is found that for the beams with different center angles, there exists a transient dimensionless arc length zone. When L_θ is below this transient zone the second natural frequencies of the beams will increase as the taper ratio is increased. When L_θ is above the transient zone the second natural frequencies of the beams will decrease as the taper ratio is increased.
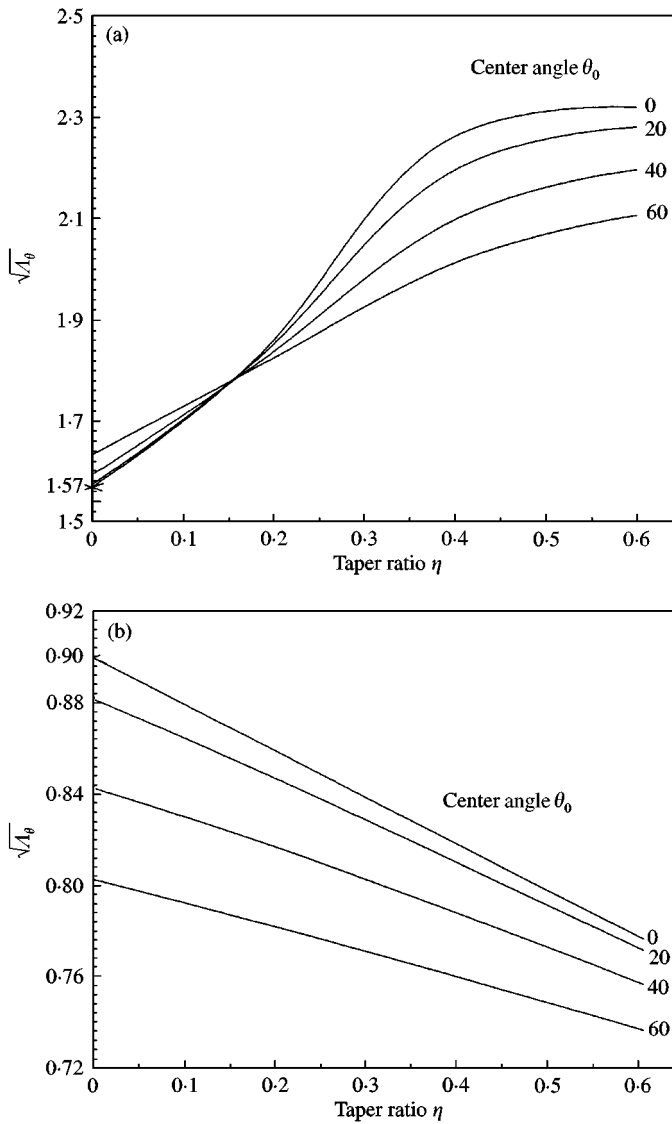
Figure 5. The influence of the taper ratio on the second dimensionless natural frequencies $\sqrt{\Lambda_\theta}$ of the beams with various center angles ($b_{r\theta}(0) = 1 \cdot 5$). L_θ values: (a) $L_\theta = 10$; (b) $L_\theta = 30$.

From Figure 3(a) one can also observe that for the beams with $L_\theta = 10$, when the taper ratio η is zero, the second natural frequencies of the beams increase as the center angle is increased. Otherwise, when the taper ratio is equal to or greater than $0 \cdot 2$, the second natural frequencies of the beams decrease as the center angle is increased. For the beams with $L_\theta = 30$ and various taper ratios, the second natural frequencies of the beams all decrease as the center angle is increased. This phenomenon is well illustrated in Figure 5. From Figure 5(a) it can be observed that for the beams with $L_\theta = 10$, when the taper ratio η is less than the critical value $\eta_c = 0 \cdot 156$ the second natural frequencies of the beams increase as the center angle is increased. Otherwise, the second natural frequencies decrease as the center angle is increased.

5. CONCLUSIONS

In this paper, two physical parameters are introduced to simplify the analysis of out-of-plane vibrations of curved non-uniform beams of constant radius. The explicit relations between the flexural displacement, its first three order derivatives and the torsional displacement for the out-of-plane vibrations of curved non-uniform beams of constant radius are established. The two coupled governing characteristic differential equations of the system are thus uncoupled and reduced to a sixth order ordinary differential equation with variable coefficients in the out-of-plane flexural displacement and the torsional displacement, respectively. With this development, the exact out-of-plane vibration solutions of curved non-uniform beams with material and geometric properties in arbitrary polynomial forms of spatial variable are provided. Numerical analysis shows that the taper ratio, the center angle and the arc length have significant influence on the first two natural frequencies of the beams.

ACKNOWLEDGMENT

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APPENDIX A

The terms α_1, α_2 and δ_1 are defined as

$$\alpha_1 = \int_A rz \, dA,$$

$$\alpha_2 = \int_A z \, dA,$$

$$\delta_1 = \int_A r \, dA,$$

APPENDIX B

The forms of p_w and a_1–a_{10} are

$$p_w = EI_r + GI_\theta,$$

$$g_1 = \frac{(EI_r)''}{p_w} + \frac{EI_r}{R^2 GI_\theta} - \frac{\rho\Omega^2}{G},$$

$$a_1 = -\frac{GI_\theta}{p_w g_1 EI_r}\left(\frac{E^2 I_r^2}{GI_\theta}\right)',$$

$$a_2 = \frac{1}{p_w g_1}\left(\frac{E^2 I_r^2}{GI_\theta}\right)'\left(\frac{\rho I_\theta \Omega^2}{EI_r} - \frac{1}{R^2}\right) - \frac{(GI_\theta)'}{GI_\theta},$$

$$a_3 = \frac{EI_r}{R^2 GI_\theta} - \frac{\rho\Omega^2}{G},$$

$$a_4 = 1 - \left[\frac{(GI_\theta)'}{GI_\theta} - \frac{(p_w + EI_r)'}{p_w}\right]\left[\left(\frac{1}{g_1}\right)' + \frac{1}{p_w g_1^2}\left(\frac{E^2 I_r^2}{GI_\theta}\right)\left(\frac{\rho I_\theta \Omega^2}{EI_r} - \frac{1}{R^2}\right)\right]$$

$$- \frac{(GI_\theta)'}{g_1 GI_\theta} - \frac{1}{g_1}\left[\frac{(GI_\theta)'}{GI_\theta} - \frac{(p_w + EI_r)'}{p_w}\right]',$$

$$a_5 = \left[\frac{(GI_\theta)'}{p_w}\right]^2\left[\frac{(GI_\theta)'}{p_w}\right]',$$

$$a_6 = -\frac{(GI_\theta)'}{p_w^3}\left\{\left\{1 + \left[\frac{(GI_\theta)'}{p_w}\right]'\right\}(GI_\theta)'(GI_\theta + p_w)' - z_w(GI_\theta + p_w)''\right\}\frac{(GI_\theta)'''}{p_w},$$

$$a_7 = \left\{-\left[\frac{EI_r(GI_\theta)'}{p_w}\right]'' - \frac{(GI_\theta)'}{R^2}\left(1 - \frac{GI_\theta}{p_w}\right) + \frac{(GI_\theta)'}{p_w}\left\{2\left[\frac{EI_r(GI_\theta)'}{p_w}\right]' + EI_r\right.\right.$$

$$\left.\left.\left\{\left[\frac{(GI_\theta)'}{p_w}\right]' - \frac{(GI_\theta)'}{p_w}\right\}\right\}\right\}\frac{1}{\rho A\Omega^2},$$

$$a_8 = \frac{1}{\rho A\Omega^2 p_w}\left\{(EI_r)'' - \frac{GI_\theta}{R^2} + EI_r\left\{\left[\frac{(GI_\theta)'}{p_w}\right]^2 - \left[\frac{(GI_\theta)'}{p_w}\right]'\right\} - \frac{2(EI_r)'(GI_\theta)'}{p_w}\right\},$$

$$a_9 = \left\{\left[\frac{EI_r(GI_\theta)'}{p_w}\right]'' - \frac{(GI_\theta)'}{R^2}\left(1 - \frac{GI_\theta}{p_w}\right) - \frac{(GI_\theta)'}{p_w}\left\{2\left[\frac{EI_r(GI_\theta)'}{p_w}\right]' + EI_r\right.\right.$$

$$\left.\left.\left\{\left[\frac{(GI_\theta)'}{p_w}\right]' - \frac{(GI_\theta)'}{p_w}\right\}\right\}\right\}\left[\left(\frac{1}{\rho A\Omega^2}\right)' + \frac{(GI_\theta)'}{p_w\rho A\Omega^2}\right] + 1 + \frac{1}{\rho A\Omega^2 R}$$

$$\left\{-\frac{(GI_\theta)'}{p_w}\left\{2\left\{\frac{EI_r(GI_\theta)'}{p_w}\right]' + EI_r\left\{\left[\frac{(GI_\theta)'}{p_w}\right]' - \frac{(GI_\theta)'}{p_w}\right\}\right\} + \left[\frac{EI_r(GI_\theta)'}{p_w}\right]''\right.$$

$$\left.- \frac{(GI_\theta)'}{R^2}\left(1 - \frac{GI_\theta}{p_w}\right)\right\}'\right\},$$

$$a_{10} = \frac{GI_\theta}{R^2} + \left[EI_r\frac{(GI_\theta)'}{p_w}\right]' - EI_r\left[\frac{(GI_\theta)'}{p_w}\right]^2.$$