Journal of Sound and Vibration (2000) **238**(3), 529–539
doi:10.1006/jsvi.2000.3009, available online at http://www.idealibrary.com on IDEAL®

LETTERS TO THE EDITOR

CLOSED-FORM SOLUTIONS FOR NATURAL FREQUENCY FOR INHOMOGENEOUS BEAMS WITH ONE SLIDING SUPPORT AND THE OTHER PINNED

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(*Received* 23 *March* 2000)

1. INTRODUCTION

Closed-form solutions for non-homogeneous beams have been obtained recently by Elishakoff and Rollot [1] and Elishakoff and Candan [2]. In particular, reference [1] dealt with stability of inhomogeneous columns, whereas reference [2] was devoted to their vibration. Reference [2] contained both deterministic and probabilistic formulations, with deterministic relationship serving as a transfer function for the probabilistic calculations. In both cases, polynomial representation of the mode shape was *postulated*, and a closed-form solution was obtained by formulating an inverse vibration problem. In this study, we deal with vibrations of a beam that has sliding support on the left end and pinned support on the right end. Here we demand a function that satisfies all boundary conditions, to serve as a *mode shape* of the vibrating beam. We then construct an inhomogeneous beam that has the postulated function as the mode shape. It is shown, remarkably, that the expression of the natural frequency of the sliding – pinned beam *coalesces* with that of the pinned–pinned beam, the latter being determined in reference [2]. Specific cases of variations of material density are given, for constant, linear, parabolic, cubic and quartic variations. These constitute particular cases. The general case is also treated, when the variation is at least quintic. The closed-form rational expressions for fundamental natural frequencies are derived for all above cases.

2. FORMULATION, OF THE PROBLEM

The governing differential equation of the dynamic behavior of a beam (assuming that the cross-sectional area A of beam is constant, as well as the moment of inertia I) is

$$\frac{d^2}{d\xi^2}\left[E(\xi)\frac{d^2w(\xi)}{d\xi^2}\right] - kL^4\rho(\xi)w(\xi) = 0 \qquad (1)$$

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where $w(\xi)$ is the mode shape, ξ the non-dimensional co-ordinate ($\xi = x/L$), L the length, $E(\xi)$ the Young's modulus, and $\rho(\xi)$ the density. Moreover,

$$k = \omega^2 A/I \tag{2}$$

is the frequency coefficient where ω^2 is the sought natural frequency. In this study, we assume that $\rho(\xi)$, $E(\xi)$ and $w(\xi)$ are polynomial functions, given by:

$$\rho(\xi) = \sum_{i=0}^{m} a_i \xi^i, \qquad E(\xi) = \sum_{i=0}^{n} b_i \xi^i, \qquad w(\xi) = \sum_{i=0}^{p} w_i \xi^i, \tag{3–5}$$

where m, n and p are, respectively, the coefficients of $\rho(\xi)$, $E(\xi)$ and $w(\xi)$. These are linked by the orders of the derivatives of equation (1), namely $n - m = 4$.

3. BOUNDARY CONDITIONS

The boundary conditions are

$$w'(0) = 0, \qquad w'''(0) = 0, \qquad w(1) = 0, \qquad w''(1) = 0. \tag{6–9}$$

We have four boundary conditions, so we must choose at least $p = 4$. One can check that the following polynomial function agrees with the boundary conditions (6)–(9):

$$w(\xi) = 1 - \tfrac{6}{5}\xi^2 + \tfrac{1}{5}\xi^4 \tag{10}$$

4. SOLUTION OF THE DIFFERENTIAL EQUATION

Equation (1) with the polynomial functions $\rho(\xi)$, $E(\xi)$ and $w(\xi)$ yields

$$-\frac{12}{5}\sum_{i=0}^{m+2}(i+1)(i+2)b_{i+2}\xi^i + \frac{12}{5}\sum_{i=2}^{m+4} i(i-1)b_i\xi^i + \frac{48}{5}\sum_{i=1}^{m+4} ib_i\xi^i + \frac{24}{5}\sum_{i=0}^{m+4} b_i\xi^i$$
$$- kL^4\sum_{i=0}^{m} a_i\xi^i + \frac{6}{5}kL^4\sum_{i=2}^{m+2} a_{i-2}\xi^i - \frac{1}{5}kL^4\sum_{i=4}^{m+4} a_{i-4}\xi^i = 0. \tag{11}$$

The equations above must be satisfied for any ξ, so we have for each ith power of ξ the following equations:

$$\xi^0: \quad 24(b_0 - b_2) - 5k^4 a_0 = 0, \tag{12}$$

$$\xi^1: \quad 72(b_1 - b_3) - 5kL^4 a_1 = 0, \tag{13}$$

$$\xi^2: \quad 144(b_2 - b_4) + kL^4(6a_0 - 5a_2) = 0, \tag{14}$$

$$\xi^3: \quad 240(b_3 - b_5) + kL^4(6a_1 - 5a_3) = 0, \tag{15}$$

$$\vdots$$

$$\xi^i: \quad 12(i+1)(i+2)(b_i - b_{i+2}) + kL^4(6a_{i-2} - a_{i-4} - 5a_i) = 0, \text{ for } 4 \leqslant i \leqslant m, \tag{16}$$

$$\vdots$$

$$\xi^{m+1}: \quad 12(m+2)(m+3)(b_{m+1} - b_{m+3}) + kL^4(6a_{m-1} - a_{m-3}) = 0, \tag{17}$$

$$\xi^{m+2}: \quad 12(m+3)(m+4)(b_{m+2} - b_{m+4}) + kL^4(6a_m - a_{m-2}) = 0, \tag{18}$$

$$\xi^{m+3}: \quad 12(m+4)(m+5)b_{m+3} - kL^4 a_{m-1} = 0, \tag{19}$$

$$\xi^{m+4}: \quad 12(m+5)(m+6)b_{m+4} - kL^4 a_m = 0. \tag{20}$$

Note that equation (16) is valid only for $4 \leqslant i \leqslant m$. But, we must have $m \geqslant 5$ in order to employ the above general equations. The explanation of why general equations are valid for $m \geqslant 5$ will be given at a later stage.

From equations (12)–(20), we have $m + 5$ relations between the coefficients a_i and b_i, these having a recursive form between b_i and b_{i+2}. The sole unknown is the natural frequency coefficient k. Thus, there must be other relations between a_i and b_i to assure the compatibility of equations (12)–(20). These relations will be formulated at a later stage.

We first treat the cases in which $m < 5$. The general case will be treated in section 6.

5. THE DEGREE OF THE MATERIAL DENSITY POLYNOMIAL IS LESS THAN FIVE

5.1. UNIFORM DENSITY ($m = 0$)

In this sub-case, $E(\xi)$ and $\rho(\xi)$ read

$$\rho(\xi) = a_0, \qquad E(\xi) = \sum_{i=0}^{4} b_i \xi^i. \tag{21}$$

By the substitution of equation (21) into equation (1), we obtain

$$24(b_0 - b_2) - 5kL^4 a_0 = 0, \qquad b_1 = b_3, \tag{22, 23}$$

$$144(b_2 - b_4) + 6kL^4 a_0 = 0, \qquad b_3 = 0, \tag{24, 25}$$

$$360b_4 - kL^4 a_0 = 0. \tag{26}$$

We obtained five equations for six unknowns: b_0, b_1, b_2, b_3, b_4 and k. We take b_4 to be an arbitrary constant. The coefficients b_i then become

$$b_0 = 61b_4, \qquad b_1 = 0, \qquad b_2 = -14b_4, \qquad b_3 = 0, \qquad k = 360b_4/L^4 a_0. \tag{27–31}$$

The fundamental natural frequency is

$$\omega^2 = 360Ib_4/AL^4 a_0 \tag{32}$$

Figure 1 depicts the variation of $E(\xi)/b_4$.

5.2. LINEARLY VARYING DENSITY ($m = 1$)

Here, $E(\xi)$ and $\rho(\xi)$ are given by

$$\rho(\xi) = a_0 + a_1\xi, \qquad E(\xi) = \sum_{i=0}^{5} b_i \xi^i. \tag{33}$$

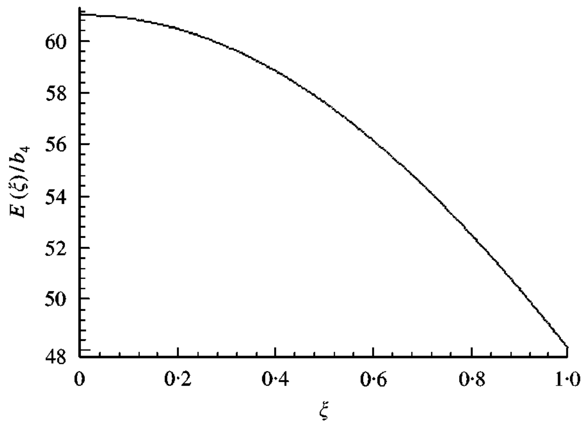

Figure 1. Variation of $E(\xi)/b_4$, $\xi \in [0; 1]$, for the constant density.

The substitution of equation (33) into equation (1) yields

$$24(b_0 - b_2) - 5kL^4 a_0 = 0, \qquad 72(b_1 - b_3) - 5kL^4 a_1 = 0, \tag{34, 35}$$

$$144(b_2 - b_4) + 6kL^4 a_0 = 0, \qquad 240(b_3 - b_5) + 6kL^4 a_1 = 0, \tag{36, 37}$$

$$360b_4 - kL^4 a_0 = 0, \qquad 504b_5 - kL^4 a_1 = 0. \tag{38, 39}$$

We have six equations with seven unknowns, b_0, b_1, b_2, b_3, b_4, b_5 and k. The coefficient b_5 is taken as arbitrary constant. Then the solution of the set (34)–(39) is expressed as

$$b_0 = 427a_0 b_5/5a_1, \qquad b_1 = 117b_5/5, \qquad b_2 = -\,98a_0 b_5/5a_1, \tag{40–42}$$

$$b_3 = -\,58b_5/5, \qquad b_4 = 7a_0 b_5/5a_1, \qquad k = 504b_5/L^4 a_1. \tag{43–45}$$

The fundamental natural frequency is expressed by the formula

$$\omega^2 = 504Ib_5/AL^4 a_1. \tag{46}$$

To illustrate this case, Figure 2 portrays the function $E(\xi)/b_5$ for $a_0 = a_1 = 1$.

5.3. PARABOLICALLY VARYING DENSITY ($m = 2$)

The density and the elastic modulus are expressed as

$$\rho(\xi) = \sum_{i=0}^{2} a_i \xi^i, \qquad E(\xi) = \sum_{i=0}^{6} b_i \xi^i. \tag{47}$$

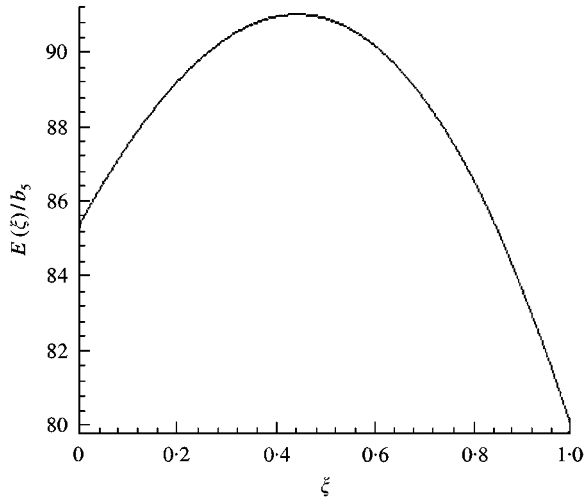

Figure 2. Variation of $E(\xi)/b_5$, $\xi \in [0; 1]$, for the linear variation of the density; $\rho(\xi) = 1 + \xi$.

The substitution of equation (47) into equation (1) results in

$$24(b_0 - b_2) - 5kL^4a_0 = 0, \qquad 72(b_1 - b_3) - 5kL^4a_1 = 0, \tag{48, 49}$$

$$144(b_2 - b_4) + kL^4(6a_0 - 5a_2) = 0, \qquad 240(b_3 - b_5) + 6kL^4a_1 = 0, \tag{50, 51}$$

$$360(b_4 - b_6) + kL^4(6a_2 - a_0) = 0, \qquad 504b_5 - kL^4a_1 = 0, \qquad 672b_6 - kL^4a_2 = 0. \tag{52–54}$$

We note that there are seven equations for eight unknowns, one of which, namely b_6 is taken here as an arbitrary constant. Hence,

$$b_0 = (1708a_0 + 197a_2)b_6/15a_2, \qquad b_1 = 156a_1b_6/5a_2, \tag{55, 56}$$

$$b_2 = -(392a_0 - 197a_2)b_6/15a_2, \qquad b_3 = -232a_1b_6/15a_2, \tag{57, 58}$$

$$b_4 = (28a_0 - 153a_2)b_6/15a_2, \qquad b_5 = 4a_1b_6/3a_2, \qquad k = 672b_6/L^4a_2. \tag{59–61}$$

leading to the fundamental natural frequency

$$\omega^2 = 672Ib_6/(AL^4a_2). \tag{62}$$

Figure 3 illustrates the dependence $E(\xi)/b_6$ for the specific case $a_0 = a_1 = a_2 = 1$.

5.4. MATERIAL DENSITY AS A CUBIC POLYNOMIAL ($m = 3$)

In this particular case, $E(\xi)$ and $\rho(\xi)$ are represented as the following polynomial functions:

$$\rho(\xi) = \sum_{i=0}^{3} a_i\xi^i, \qquad E(\xi) = \sum_{i=0}^{7} b_i\xi^i. \tag{63}$$

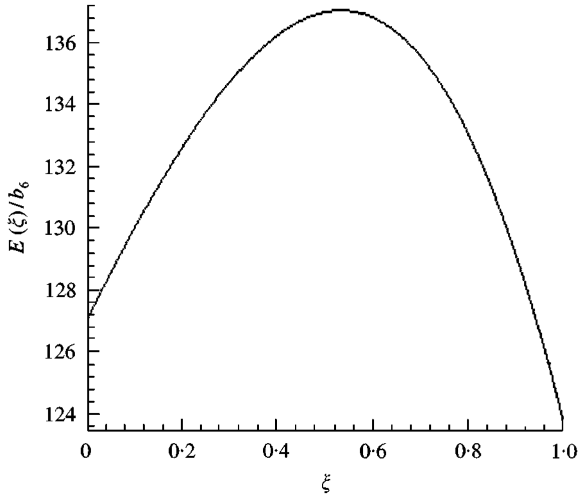

Figure 3. Variation of $E(\xi)/b_6$, $\xi \in [0; 1]$, for the parabolic variation of the density; $\rho(\xi) = 1 + \xi + \xi^2$.

The requirement that equation (1) is valid for every ξ imposes

$$24(b_0 - b_2) - 5kL^4a_0 = 0, \qquad 72(b_1 - b_3) - 5kL^4a_1 = 0, \qquad (64, 65)$$

$$144(b_2 - b_4) + kL^4(6a_0 - 5a_2) = 0, \qquad 240(b_3 - b_5) + kL^4(6a_1 - 5a_3) = 0, \quad (66, 67)$$

$$360(b_4 - b_6) + kL^4(6a_2 - a_0) = 0, \qquad 504(b_5 - b_7) + kL^4(6a_3 - a_1) = 0, \quad (68, 69)$$

$$672b_6 - kL^4a_2 = 0, \qquad 864b_7 - kL^4a_3 = 0. \qquad (70, 71)$$

The coefficients b_i, to assure the compatibility of equation (64)–(71), must satisfy the following relations, expressed in terms of b_7:

$$b_0 = 3(1708a_0 + 197a_2)b_7/35a_3, \qquad b_1 = (1404a_1 + 305a_3)b_7/35a_3, \qquad (72, 73)$$

$$b_2 = -3(392a_0 - 197a_2)b_7/35a_3, \qquad b_3 = (-696a_1 + 305a_3)b_7/35a_3, \quad (74, 75)$$

$$b_4 = 3(28a_0 - 153a_2)b_7/35a_3, \qquad b_5 = -(-12a_1 + 65a_3)b_7/7a_3, \qquad (76, 77)$$

$$b_6 = 9a_2b_7/7a_3, \qquad k = 864b_7/L^4a_3. \qquad (78–79)$$

The fundamental natural frequency is

$$\omega^2 = 864Ib_7/AL^4a_3. \qquad (80)$$

The dependence $E(\xi)/b_7$ for the specific case $a_j = 1$ versus ξ is shown in Figure 4.

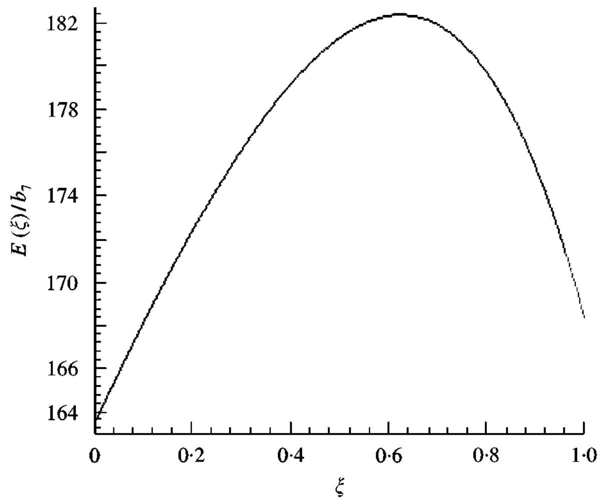

Figure 4. Variation of $E(\xi)/b_7$, $\xi \in [0; 1]$, for the cubic variation of the density; $\rho(\xi) = 1 + \xi + \xi^2 + \xi^3$.

5.5. MATERIAL DENSITY AS A QUARTIC POLYNIMIAL ($m = 4$)

In these circumstance, $E(\xi)$ and $\rho(\xi)$ are polynomial functions given by

$$\rho(\xi) = \sum_{i=0}^{4} a_i \xi^i, \qquad E(\xi) = \sum_{i=0}^{8} b_i \xi^i. \tag{81}$$

Substitution of the above expressions into the equation (1) results in

$$24(b_0 - b_2) - 5kL^4 a_0 = 0, \qquad 72(b_1 - b_3) - 5kL^4 a_1 = 0, \tag{82, 83}$$

$$144(b_2 - b_4) + kL^4(6a_0 - 5a_2) = 0, \qquad 240(b_3 - b_5) + kL^4(6a_1 - 5a_3) = 0, \tag{84, 85}$$

$$360(b_4 - b_6) + kL^4(6a_2 - a_0 - 5a_4) = 0, \qquad 504(b_5 - b_7) + kL^4(6a_3 - a_1) = 0, \tag{86, 87}$$

$$672(b_6 - b_8) + kL^4(6a_4 - a_2) = 0, \qquad 864b_7 - kL^4 a_3 = 0, \qquad 1080b_8 - kL^4 a_4 = 0. \tag{88–90}$$

We have nine equations for eight unknowns, b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8 and k. We express these unknowns in terms of b_8. Thus,

$$b_0 = (5124a_0 + 591a_2 + 178a_4)b_8/28a_4, \qquad b_1 = (1404a_1 + 305a_3)b_8/28a_4, \tag{91, 92}$$

$$b_2 = -(1176a_0 - 591a_2 - 178a_4)b_8/28a_4, \qquad b_3 = -(696a_1 - 305a_3)b_8/28a_4, \tag{93, 94}$$

$$b_4 = (84a_0 - 459a_2 + 178a_4)b_8/28a_4, \qquad b_5 = 5(12a_1 - 65a_3)b_8/28a_4, \tag{95, 96}$$

$$b_6 = (45a_2 - 242a_4)b_8/28a_4, \qquad b_7 = 5a_3b_8/4a_4, \qquad k = 1080b_8/L^4 a_4 \tag{97–99}$$

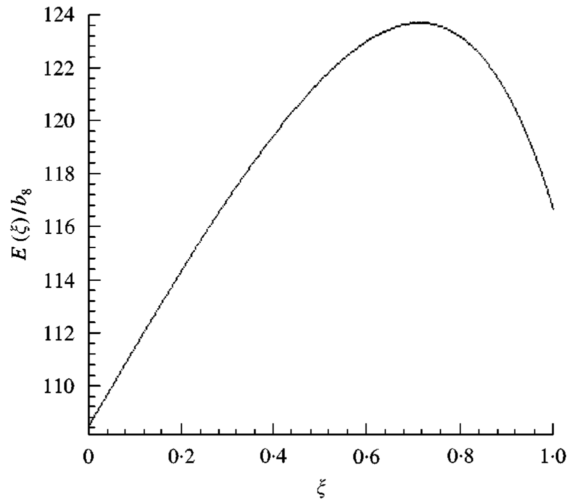

Figure 5. Variation of $E(\xi)/b_8$, $\xi \in [0; 1]$, for the quintic variation of the density; $\rho(\xi) = 1 + \xi + \xi^2 + \xi^3 + \xi^4$.

The fundamental natural frequency is given by

$$\omega^2 = 1080 I b_8 / A L^4 a_4. \quad (100)$$

Figure 5 presents the ratio $E(\xi)/b_8$ for $a_j = 1$. In the following, we present the general case, with $5 \leqslant i \leqslant m$.

6. GENERAL CASE: COMPATIBILITY CONDITIONS

For the unknown k, we have different expressions stemming from equations (12)–(20):

$$k = 24(b_0 - b_2)/5a_0 L^4, \qquad k = 72(b_1 - b_3)/5a_1 L^4, \quad (101, 102)$$

$$k = -144(b_2 - b_4)/(6a_0 - 5a_2)L^4, \qquad k = -240(b_3 - b_5)/(6a_1 - 5a_3)L^4, \quad (103, 104)$$

$$\vdots$$

$$k = -12(i+1)(i+2)(b_i - b_{i+2})/(6a_{i-2} - a_{i-4} - 5a_i)L^4, \qquad \text{for } 4 \leqslant i \leqslant m, \quad (105)$$

$$\vdots$$

$$k = -12(m+2)(m+3)(b_{m+1} - b_{m+3})/(6a_{m-1} - a_{m-3})L^4, \quad (106)$$

$$k = -12(m+3)(m+4)(b_{m+2} - b_{m+4})/(6a_m - a_{m-2})L^4, \quad (107)$$

$$k = 12(m+4)(m+5)b_{m+3}/a_{m-1}L^4, \qquad k = 12(m+5)(m+6)b_{m+4}/a_m L^4. \quad (108, 109)$$

Compatibility conditions demand that all these expressions, as representing the same natural frequency coefficient, to be equal.

Reference [2] dealt with two cases: (1) material density coefficients a_i were specified and elastic modulus coefficients were determined; and (2) elastic modulus coefficients b_i were specified, whereas material density coefficients have to be evaluated. In this paper, for simplicity, we treat only the first case. One can consult reference [2] for the details of the second case, for the pinned–pinned beam.

Equations (101)–(109), in conjunction with the knowledge of the coefficients a_i, permit us to obtain a closed-form solution of the natural frequency.

We assume the material density ($a_i = 0, \dots, m$) coefficients to be known. From the equations (101)–(109), we can compute the coefficient b_i. Firstly, let us observe equation (109). The knowledge of b_{m+4} leads to the natural frequency. Moreover, b_{m+4} and a_m have the same sign (due to the positivity of k). Secondly, we need only one coefficient b_i to determine all $b_j, j \neq i$. This is due to the recursive form of equation (101)–(109). We assume that the coefficient b_{m+4} is known. Then, we calculate the other coefficients b_i, $i = 0, \dots, m+3$. From equation (108) in conjunction with equation (109), we get

$$b_{m+3} = \frac{m+6}{m+4}\frac{a_{m-1}}{a_m} b_{m+4} \tag{110}$$

Equations (107) and (109) lead to

$$b_{m+2} = -\frac{(m+7)(5m+24)a_m - (m+5)(m+6)a_{m-2}}{(m+3)(m+4)a_m} b_{m+4}. \tag{111}$$

Analogously, equations (106) and (108) yield

$$b_{m+1} = -\frac{(m+6)((5m^2+49m+114)a_{m-1} - (m+4)(m+5)a_{m-3})}{(m+2)(m+3)(m+4)a_m} b_{m+4}. \tag{112}$$

Equation (105), with $i = m$ and equation (107) result in

$$\begin{aligned} b_m = {} & 8\,\frac{2m^3+30m^2+136m+183}{(m+1)(m+2)(m+3)(m+4)} b_{m+4} \\ & - \frac{(m+5)^2(m+6)(5m+14)}{(m+1)(m+2)(m+3)(m+4)}\frac{a_{m-2}}{a_m} b_{m+4} \\ & + \frac{(m+5)(m+6)}{(m+1)(m+2)}\frac{a_{m-4}}{a_m} b_{m+4}. \end{aligned} \tag{113}$$

Equation (105), with $i = m-1$, and equation (106) becomes

$$\begin{aligned} b_{m-1} = {} & 8\,\frac{(m+6)(2m^3+24m^2+82m+75)}{m(m+1)(m+2)(m+3)(m+4)}\frac{a_{m-1}}{a_m} b_{m+4} \\ & - \frac{(m+4)(m+5)(m+6)(5m+9)}{m(m+1)(m+2)(m+3)}\frac{a_{m-3}}{a_m} b_{m+4} \\ & + \frac{(m+5)(m+6)}{m(m+1)}\frac{a_{m-5}}{a_m} b_{m+4}. \end{aligned} \tag{114}$$

We need to calculate b_m and b_{m-1} in order to use the general expression of b_i for $4 \leqslant i \leqslant m-2$:

$$b_i = \left[\frac{(i+3)(i+4)}{(i+1)(i+2)}\frac{6a_{i-2}-a_{i-4}-5a_i}{6a_i-a_{i-2}-5a_{i+2}}+1\right]b_{i+2}-\left[\frac{(i+3)(i+4)}{(i+1)(i+2)}\frac{6a_{i-2}-a_{i-4}-5a_i}{6a_i-a_{i-2}-5a_{i+2}}\right]b_{i+4}. \tag{115}$$

Note that equation (115) is only valid for $i \leqslant m-2$ because of the coefficient a_{i+2}. Indeed, $i+2 < m+1$. Just as $m-1 > i > 3$ (due to the coefficient a_{i-4}), we must have $m > 4$ (this explains why cases $m \leqslant 4$ are particular cases). Now, we calculate the coefficients b_3, b_2, b_1 and b_0. Equation (104) leads to

$$b_3 = \frac{21}{10}\left(\frac{6a_1-5a_3}{6a_3-a_1-5a_5}+1\right)b_5 - \frac{21}{10}\left(\frac{6a_1-5a_3}{6a_3-a_1-5a_5}\right)b_7. \tag{116}$$

Equation (103) results in

$$b_2 = \frac{5}{2}\left(\frac{6a_0-5a_2}{6a_2-a_0-5a_4}+1\right)b_4 - \frac{5}{2}\left(\frac{6a_0-5a_2}{6a_2-a_0-5a_4}\right)b_6. \tag{117}$$

From equation (102), we obtain

$$b_1 = \frac{1}{10}\left(\frac{234a_1+45a_3+50a_5}{-6a_3+a_1+5a_5}\right)b_5 - \frac{1}{10}\left(\frac{224a_1+105a_3}{-6a_3+a_1+5a_5}\right)b_7. \tag{118}$$

Equation (101) gives

$$b_0 = \frac{1}{2}\left(\frac{122a_0+13a_2+10a_4}{-6a_2+a_0+5a_4}\right)b_4 - \frac{1}{2}\left(\frac{120a_0+25a_2}{-6a_2+a_0+5a_4}\right)b_6. \tag{119}$$

Figure 6 portrays the function, for $m = 15$, $E(\xi)/b_{16}$ with a_i specified as $16 - i$.

From equation (109) in view of equation (2), we deduce the natural frequency squared,

$$\omega^2 = 12(m+5)(m+6)b_{m+4}I/a_m AL^4 \tag{120}$$

which, remarkably, coincides with its counterpart for the pinned–pinned beam [2]. Still, the expression for coefficients b_j differ for these two cases. It is also notable that by formally substituting $m = 0, 1, 2, 3, 4$, we get the expressions derived in equations (32), (46), (62), (80), (100), respectively.

7. CONCLUSIONS

Apparently for the first time in the literature we obtained closed-form solutions for the natural frequencies of the inhomogeneous beams with one sliding support as well as the other pinned. We hope that this study, as well as its companions, references [1–3], will arouse

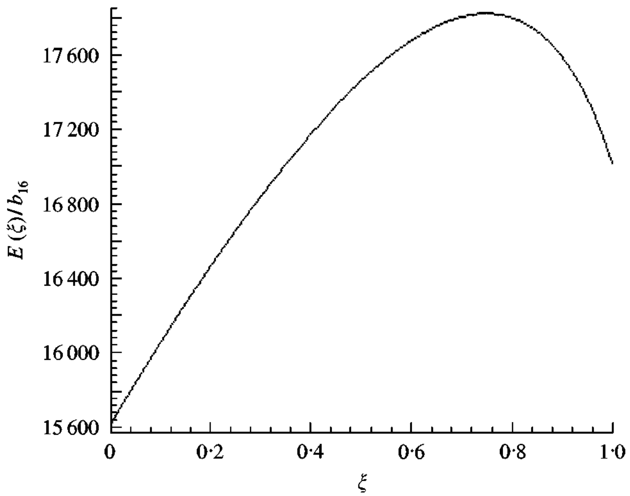

Figure 6. Variation of $E(\xi)/b_{16}$, $\xi \in [0; 1]$, $\rho(\xi) = \sum_0^{15}(16 - i)\xi^i$.

intensified search for additional closed-form solutions for inhomogeneous and/or non-uniform structures.

AKNOWLEDGMENT

This study was performed when Roland Becquet was a visiting scholar at the Florida Atlantic University.

REFERENCES

1. I. Elishakoff and O. Rollot 1999 *Journal of Sound and Vibration* **224**, 172/82. New closed-form solutions for buckling of a variable stiffness by Mathematica®
2. S. Candan and I. Elishakoff 1999 *Applications of Statistics and Probability* (R. E. Melchers and M. G. Stewart, editors), Vol. 2, pp. 1059–1067, Rotterdam: Balkema. Infinite number of closed-form solutions for reliability of inhomogeneous beams.
3. A. Bokayan 1988 *Journal of Sound and Vibration* **126**, 49–65 Natural frequences of beams under compressive axial loads.