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CLOSED-FORM SOLUTIONS FOR NATURAL FREQUENCY FOR INHOMOGENEOUS BEAMS WITH ONE SLIDING SUPPORT AND THE OTHER CLAMPED[1]

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1. INTRODUCTION

This paper closely follows the recent studies by Elishakoff and Rollot [1], Elishakoff and Candan [2] and Elishakoff and Becquet [3]. It deals with the closed-form solution for beam eigenvalues; buckling is considered in reference [1], while vibration is studied in references [2, 3]. References [2, 3] treat, respectively, pinned–pinned beams and sliding–pinned beams. The paper by Elishakoff and Candan [4] deals with three other boundary conditions: pinned–clamped, clamped–free and clamped–clamped beam. Here, we complete these cases with studying the vibration of a clamped–sliding beam. Note that sliding boundary condition was studied by Bokaian [5]. Like our previous studies, this investigation is posed as an inverse vibration problem. The first step is to postulate the mode shape of the vibrating beam, which is represented by a polynomial function satisfying all boundary conditions. We ought to note that in all the cases, we obtain the same expression of the natural frequency. As we treat the inhomogeneous beam (Young's modulus and the density are given by polynomial functions), the problem of the engineer is to accurately know the material density in order to obtain the Young's modulus that corresponds to the selected mode shape.

In this paper, a clamped–sliding beam is studied. Moreover, we treat two specific cases, which are associated with constant and linear variations of the density. The fundamental natural frequency is given for all cases of variation of the material density.

2. FORMULATION OF THE PROBLEM

The dynamic behavior of a beam, with a constant cross-sectional area A and a constant moment of inertia I, is given by

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left[E(\xi)\frac{\mathrm{d}^2 w(\xi)}{\mathrm{d}\xi^2}\right] - kL^{\ 4}\rho(\xi)w(\xi) = 0, \tag{1}$$

[1] This paper is dedicated to the 65th Anniversary of Professor Yehuda Stavsky's birth.

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where $E(\xi)$, $w(\xi)$ and $\rho(\xi)$ are, respectively, the Youngs modulus, the mode shape and the density, L is the length of the beam and k contains the fundamental natural frequency,

$$k = \omega^2 A/I. \tag{2}$$

In equation (1) the modulus of elasticity and the material density are taken as functions of the axial co-ordinate. The independent parameter of equation (1) is the non-dimensional axial co-ordinate $\xi = x/L$ (x being the dimensional co-ordinate). We assume that the properties of the inhomogeneous beam are

$$\rho(\xi) = \sum_{i=0}^{m} a_i \xi^i, \qquad E(\xi) = \sum_{i=0}^{n} b_i \xi^i, \qquad w(\xi) = \sum_{i=0}^{p} w_i \xi^i. \tag{3–5}$$

Consequently, m, n and p (the degrees of the polynomial functions $E(\xi)$, $w(\xi)$ and $\rho(\xi)$) are linked by equation (1), i.e., $n - m = 4$.

3. BOUNDARY CONDITIONS

We treat the case of a clamped–pinned beam. The boundary conditions read

$$w(0) = 0, \qquad w'(0) = 0, \qquad w'(1) = 0, \qquad w'(1) = 0. \tag{6–9}$$

The degree of the mode shape $w(\xi)$, being a polynomial function, must be at least 4 for it to satisfy all boundary conditions. The mode shape reads as

$$w(\xi) = \xi^2 - \xi^3 + \tfrac{1}{4}\xi^4. \tag{10}$$

4. SOLUTION OF THE DIFFERENTIAL EQUATION

Equation (1) is expanded by substituting equations (3)–(5). We obtain

$$\begin{aligned}
&2\sum_{i=0}^{m+2}(i+1)(i+2)b_{i+2}\xi^i - 6\sum_{i=1}^{m+3} i(i+1)b_{i+1}\xi^i + 3\sum_{i=2}^{m+4} i(i-1)b_i\xi^i \\
&\quad - 12\sum_{i=0}^{m+3}(i+1)b_{i+1}\xi^i + 12\sum_{i=1}^{m+4} ib_i\xi^i + 6\sum_{i=0}^{m+4} b_i\xi^i \\
&\quad - kL^4\sum_{i=2}^{m+2} a_{i-2}\xi^i + kL^4\sum_{i=3}^{m+3} a_{i-3}\xi^i \\
&\quad - \frac{1}{4}kL\sum_{i=4}^{m+4} a_{i-4}\xi^i = 0
\end{aligned} \tag{11}$$

Equation (11) is valid for any ξ, so for each coefficient in front of ξ^i, $0 \leqslant i \leqslant m+4$, we have a single equation. These read as

$$\xi^0: \quad 4b_2 - 12b_1 + 6b_0 = 0, \tag{12}$$

$$\xi^1: \quad 12b_3 - 36b_2 + 18b_1 = 0 \tag{13}$$

$$\xi^2\colon \quad 24b_4 - 72b_3 + 36b_2 - kL^4 a_0 = 0, \tag{14}$$

$$\xi^3\colon \quad 40b_5 - 120b_4 + 60b_3 + kL^4(a_0 - a_1) = 0 \tag{15}$$

$$\vdots$$

$$\xi^i\colon \quad (i+1)(i+2)(2b_{i+2} - 6b_{i+1} + 3b_i) + kL^4(a_{i-3} - a_{i-2} - \tfrac{1}{4}a_{i-4}) = 0$$

$$\text{for } 4 \leqslant i \leqslant m+2, \tag{16}$$

$$\vdots$$

$$\xi^{m+3}\colon \quad (m+4)(m+5)(-6b_{m+4} + 3b_{m+3}) + kL^4(a_m - \tfrac{1}{4}a_{m-1}) = 0 \tag{17}$$

$$\xi^{m+4}\colon \quad 3(m+5)(m+6)b_{m+4} - \tfrac{1}{4}kL^4 a_m = 0. \tag{18}$$

We obtained a system of $m+5$ equations $(0, \ldots i, \ldots, m+4)$; equation (16) is a recursive equation. We are looking for the unknown k. Thus, coefficients a_i and b_i are linked by other relations, given at a later stage. These relations are valid for the general case for $m \geqslant 2$. The case $m \leqslant 1$ is given in section 5. The general case is treated in section 6..

5. CASE OF THE UNIFORM AND LINEAR DENSITIES

5.1. UNIFORM DENSITY

$E(\xi)$ and $\rho(\xi)$ are given by

$$\rho(\xi) = a_0, \qquad E(\xi) = \sum_{i=0}^{4} b_i \xi^i. \tag{19}$$

Equation (1), with the above expressions of $E(\xi)$ and $\rho(\xi)$, leads to

$$4b_2 - 12b_1 + 6b_0 = 0, \qquad -36b_2 + 12b_3 + 18b_1 = 0, \tag{20, 21}$$

$$36b_2 - 72b_3 + 24b_4 - kL^4 a_0 = 0, \quad 60b_3 - 120b_4 + kL^4 a_0 = 0, \qquad 90b_4 - \tfrac{1}{4}kL^4 a_0 = 0 \tag{22–24}$$

We have six unknowns, b_0, b_1, b_2, b_3, b_4 and k, and five equations, given by equations (20)–(24). Thus, b_0, b_1, b_2, b_3 and k are calculated in terms of b_4. These coefficients and k are obtained as:

$$b_0 = 88b_4/9, \qquad b_1 = 16b_4/3, \qquad b_2 = 4b_4/3, \qquad b_3 = -4b_4, \qquad k = 360b_4/L^4 a_0 \tag{25–29}$$

From equation (29), the fundamental natural frequency is derived as

$$\omega^2 = 360Ib_4/AL^4 a_0. \tag{30}$$

Figure 1 represents $E(\xi)/b_4$ for the specific case $a_0 = 1$.

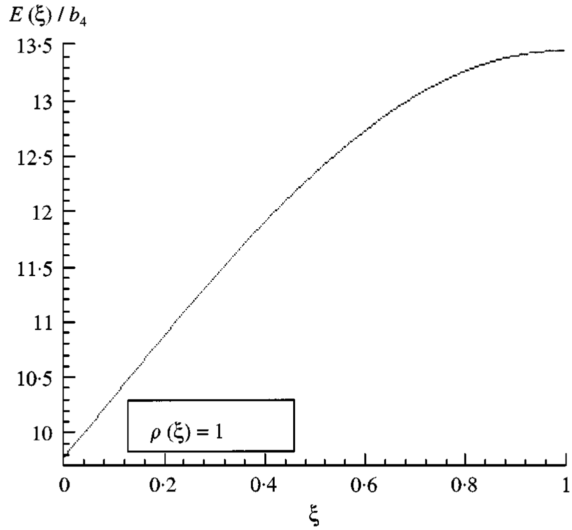

Figure 1. Variation of $E(\xi)/b_4$, $\xi \in [0; 1]$, for constant density.

5.2. LINEAR DENSITY

In this second case, $E(\xi)$ and $\rho(\xi)$ are

$$\rho(\xi) = a_0 + a_1\xi, \qquad E(\xi) = \sum_{i=0}^{5} b_i \xi^i. \tag{31}$$

Equation (1), valid for every ξ, imposes

$$4b_2 - 12b_1 + 6b_0 = 0, \qquad -36b_2 + 12b_3 + 18b_1 = 0, \tag{32, 33}$$

$$36b_2 - 72b_3 + 24b_4 - kL^4 a_0 = 0, \qquad 60b_3 - 120b_4 + 40b_5 + kL^4(a_0 - a_1) = 0,$$

$$90b_4 - 180b_5 + kL^4(a_1 - \tfrac{1}{4}a_0) = 0, \qquad 126b_5 - \tfrac{1}{4}kL^4 a_1 = 0. \tag{34–37}$$

The six equations in system (32)–(37) are a system of seven unknowns. Hence, b_5 is taken as an arbitrary coefficient. The unknowns b_0, b_1, b_2, b_3, b_4 and k read as

$$b_0 = 8b_5(77a_0 + 61a_1)/45a_1, \quad b_1 = 8b_5(42a_0 + 37a_1)/45a_1, \tag{38, 39}$$

$$b_2 = 4b_5(7a_0 + 13a_1)/15a_1, \quad b_3 = -4b_5(21a_0 - 2a_1)/15a_1 \tag{40, 41}$$

$$b_4 = b_5(7a_0 - 18a_1)/5a_1, \quad k = 504b_5/L^4 a_1 \tag{42, 43}$$

leading to the natural frequency:

$$\omega^2 = 504Ib_5/AL^4 a_1. \tag{44}$$

The dependence $E(\xi)/b_5$ with ξ is shown in Figure 2 for the particular case $\rho(\xi) = 1 + 2\xi$, $a_0 = 1$, $a_2 = 1$.

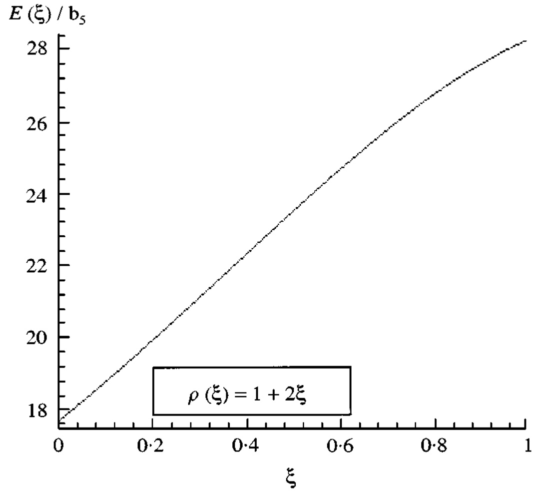

Figure 2. Variation of $E(\xi)/b_5$, $\xi \in [0; 1]$, for linear variation of the density.

6. GENERAL CASE: COMPATIBILITY CONDITION

From equations (12)–(18), we calculate k. Obviously, these different expressions of k are equal to each other. Thus

$$k = 12(2b_4 - 6b_3 + 3b_2)/L^4 a_0, \tag{45}$$

$$k = 20(2b_5 - 6b_4 + 3b_3)/L^4(a_1 - a_0) \tag{46}$$

$$\vdots$$

$$k = 4(i+1)(i+2)(2b_{i+2} - 6b_{i+1} + 3b_i)/L^4(-4a_{i-3} + 4a_{i-2} + a_{i-4}) \tag{47}$$

$$\vdots$$

$$k = -12(m+4)(m+5)(2b_{m+4} - b_{m+3})/L^4(-4a_m + a_{m-1}) \tag{48}$$

$$k = 12(m+5)(m+6)b_{m+4}/L^4 a_m. \tag{49}$$

Let us assume that the material density coefficients a_i are known. The above expressions of k allow us to obtain the material Young's modulus coefficients b_i.

As material density coefficients are known, equations (45)–(49) lead to the knowledge of coefficients b_i ($i = 2, \ldots, m+4$). The coefficients b_0 and b_1 are calculated by equations (12) and (13).

Yet, k is also unknown: Thus, we have $m+5$ equations (equations (12), (13), (45)–(49)) with $m+6$ unknowns (b_i, $i = 0, \ldots, m+4$ and k). We need to fix one of the coefficients b_j in order to compute the other coefficients b_i, $i \neq j$ and k. Here, we choose b_{m+4} to be specified.

Equation (48) yields

$$b_{m+3} = \frac{(m+6)a_{m-1} - (2m+16)a_m}{(m+4)a_m} b_{m+4}. \tag{50}$$

Equation (47), with $i = m + 2$, results in

$$b_{m+2} = \frac{(3m^2 + 33m + 90)\,a_{m-2} - (6m^2 + 78m + 252)\,a_{m-1} - (2m^2 + 14m - 48)\,a_m}{3(m+3)(m+4)a_m} b_{m+4}. \tag{51}$$

With equation (47), we have the general expression of b_i, for $4 \leqslant i \leqslant m + 1$. Hence,

$$\begin{aligned} b_i = &- \frac{[(6 + 2i)\,a_{i-4} + (24 + 8i)(a_{i-2} - a_{i-3})]\,b_{i+3}}{3(i+1)(4a_{i-2} - 4a_{i-1} - a_{i-3})} \\ &- \frac{[-(18 + 6i)\,a_{i-4} + (70 + 22i)\,a_{i-3} - (64 + 16i)\,a_{i-2} - 8(i+1)\,a_{i-1}]\,b_{i+2}}{3(i+1)(4a_{i-2} - 4a_{i-1} - a_{i-3})} \\ &- \frac{[(9 + 3i)\,a_{i-4} - (30 + 6i)\,a_{i-3} - 12(i-1)\,a_{i-2} + 24(i+1)\,a_{i-1}]\,b_{i+1}}{3(i+1)(4a_{i-2} - 4a_{i-1} - a_{i-3})}. \end{aligned} \tag{52}$$

Equation (46) yields

$$b_3 = -2\,\frac{6(a_0 - a_1)\,b_6 + (14a_1 + 4a_2 - 17a_0)\,b_5 + (3a_1 - 12a_2 + 6a_0)\,b_4}{3(-4a_1 + 4a_2 + a_0)}. \tag{53}$$

Equation (45) results in

$$b_2 = \frac{(24a_0 + 6a_1)\,b_4 + (3a_0 - 18a_1)\,b_3 - 10a_0 b_5}{9(a_0 - a_1)} \tag{54}$$

From equations (13) and (12), we obtain

$$b_1 = -\tfrac{2}{3}\,b_3 + 2b_2, \qquad b_0 = -\tfrac{2}{3}\,b_2 + 2b_1. \tag{55}$$

Equation (49) leads to the natural frequency ω,

$$\omega^2 = 12(m+5)(m+6)\,Ib_{m+4}/AL^4 a_m. \tag{56}$$

In Figure 3, we present the dependence $E(\xi)/b_{16}$.

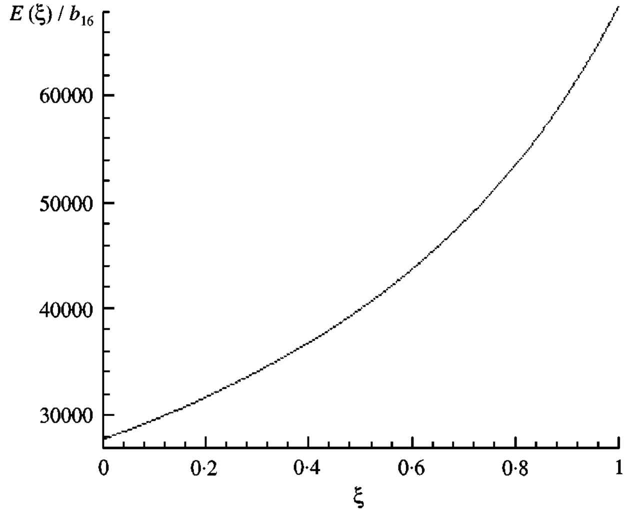

Figure 3. Variation of $E(\xi)/b_{16}$, $\xi \in [0; 1]$, $\rho(\xi) = \sum_{i=0,\; i \neq 4, i \neq 12}^{15} (16 - i)\,\xi^i + 3\xi^4 + 8\xi^{12}$.

7. CONCLUSION

It is remarkable that the expression for the natural frequency obtained within the present formulation for the clamped–sliding beam coincides with its counterparts irrespective of the boundary conditions. It should be stressed, however, that the stiffness coefficients *depend* upon boundary condition. This implies that for a given material density expression, in order for beams under *differing* boundary conditions to have the same frequency, they must have *differing* expressions for stiffness. This feature is in line with one's anticipation, since even uniform beams with different boundary conditions may possess the same natural frequency if the stiffness is properly 'adjusted'. The present formulation screens, as it were, the beams that have the same frequencies under differing boundary conditions.

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