Journal of Sound and Vibration (2001) **239**(1), 139–155
doi:10.1006/jsvi.2000.3211, available online at http://www.idealibrary.com on IDEAL®

MOMENT LYAPUNOV EXPONENTS OF A TWO-DIMENSIONAL SYSTEM UNDER REAL-NOISE EXCITATION

W.-C. Xie

Solid Mechanics Division, Faculty of Engineering, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

(*Received* 7 *March* 2000, *and in final form* 28 *June* 2000)

The moment Lyapunov exponents of a two-dimensional system under real-noise excitation, an Ornstein–Uhlenbeck process, are studied in this paper. The method of regular perturbation is applied to obtain the weak-noise expansions of the moment Lyapunov exponent, Lyapunov exponent, and stability index in terms of the small fluctuation parameter.

1. INTRODUCTION

The loadings imposed on the structures are quite often random forces, such as those arising from earthquakes, wind and ocean waves, which can be described satisfactorily only in probabilistic terms. Under the action of such loadings, the parameters that describe the motion of the structure will fluctuate in a stochastic manner. The response of the structure is governed by the stochastic differential equations, in which the parameters or coefficients are stochastic processes. Investigations of stability under parametric stochastic excitation have become increasingly important.

In this paper, the parametric stability of the following non-dimensional two-dimensional system under weak real-noise excitation is studied:

$$\frac{\mathrm{d}^2 q}{\mathrm{d}\tau^2} + 2\beta \frac{\mathrm{d}q}{\mathrm{d}\tau} + [\omega_0^2 - \varepsilon_0\, \xi(\tau)]\, q = 0, \tag{1}$$

where τ is the time variable, $q(\tau)$ the generalized co-ordinate, β the damping constant, ω_0 the circular natural frequency of the system, $\varepsilon_0 > 0$ a small fluctuation parameter, and $\xi(\tau)$ an Ornstein–Uhlenbeck process given by [1]

$$\mathrm{d}\xi(\tau) = -\,\alpha_0\, \xi(\tau)\, \mathrm{d}\tau + \sigma_0 \circ \mathrm{d}W(\tau), \tag{2}$$

where $W(\tau)$ is a standard Wiener process. Letting $\sigma_0 = \sqrt{2a_0}\,\alpha_0$, the correlation function and spectral density of the Ornstein–Uhlenbeck process $\xi(\tau)$ are

$$R(T) = a_0\, \alpha_0\, e^{-\alpha_0 |T|}, \qquad S(\Omega) = \frac{a_0}{1 + (\Omega/\alpha_0)^2},$$

in which the parameter α_0 characterizes the bandwidth of the noise and a_0 is related to the spectral density of the noise. When $\alpha_0 \to \infty$, the Ornstein–Uhlenbeck process $\xi(t)$ becomes a standard Gaussian white-noise process with constant spectral density a_0.

The sample or almost-sure stability of the trivial solution of system (1) is determined by the Lyapunov exponent, which characterizes the average exponential rate of growth of the solutions of system (1) for τ large, defined as

$$\lambda_{q(\tau)} = \lim_{\tau \to \infty} \frac{1}{\tau} \log \|\mathbf{q}(\tau)\|, \tag{3}$$

where $\mathbf{q}(\tau) = \{q(\tau), q'(\tau)\}^{\mathrm{T}}$ and $\|\cdot\|$ denotes the Euclidean vector norm. Depending on the initial conditions $q(0)$ and $q'(0)$, there are two Lyapunov exponents for system (1). The trivial solution of system (1) is stable with probability one (w.p.1) if the top Lyapunov exponent is negative, whereas it is unstable w.p.1 if the top Lyapunov exponent is positive.

On the other hand, the stability of the pth moment of the trivial solution of system (1), $E[\|\mathbf{q}(\tau)\|^p]$, is determined by the moment Lyapunov exponent

$$\Lambda_{q(\tau)}(p) = \lim_{\tau \to \infty} \frac{1}{\tau} \log E[\|\mathbf{q}(\tau)\|^p], \tag{4}$$

where $E[\,\cdot\,]$ denotes expected value. If $\Lambda_{q(\tau)}(p) < 0$, then $E[\|\mathbf{q}(t)\|^p] \to 0$ as $\tau \to \infty$.

The pth moment Lyapunov exponent $\Lambda_{q(\tau)}(p)$ is a convex analytic function in p with $\Lambda_{q(\tau)}(0) = 0$ and $\Lambda'_{q(\tau)}(0)$ is equal to the top Lyapunov exponent $\lambda_{q(\tau)}$. The non-trivial zero $\delta_{q(\tau)}$ of $\Lambda_{q(\tau)}(p)$, i.e. $\Lambda_{q(\tau)}(\delta_{q(\tau)}) = 0$, is called the stability index.

However, suppose the top Lyapunov exponent $\lambda_{q(\tau)}$ is negative, implying that system (1) is sample stable, the pth moment typically grows exponentially for large enough p, implying that the pth moment of system (1) is unstable. This can be explained by large deviation. Although the solution of the system $\|\mathbf{q}(\tau)\| \to 0$ as $\tau \to \infty$ w.p.1 at an exponential rate $\lambda_{q(\tau)}$, there is a small probability that $\|\mathbf{q}(\tau)\|$ is large, which makes the expected value $E[\|\mathbf{q}(\tau)\|^p]$ of this rare event large for large enough values of p, leading to the pth moment instability.

To have a complete picture of the dynamical stability of system (1), it is important to study both the sample and the moment stability and to determine both the top Lyapunov exponent and the pth moment Lyapunov exponent. The objective of this paper is to obtain weak-noise expansions of the top Lyapunov exponent, moment Lyapunov exponent, and the stability index of system (1) in terms of the small fluctuation parameter ε_0.

The damping term in system (1) can be removed by letting $q(\tau) = x(\tau)\mathrm{e}^{-\beta\tau}$ to yield

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} + [\omega^2 - \varepsilon_0\, \xi(\tau)]\, x = 0, \tag{5}$$

where $\omega^2 = \omega_0^2 - \beta^2$. The Lyapunov exponent and moment Lyapunov exponent of system (5) are related to those of system (1) as follows:

$$\lambda_{q(\tau)} = -\beta + \lambda_{x(\tau)}, \qquad \Lambda_{q(\tau)}(p) = -p\beta + \Lambda_{x(\tau)}(p). \tag{6}$$

System (5) may be further simplified by the time scaling $t = \omega\tau$ to

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + [1 - \varepsilon\zeta(t)]x = 0, \tag{7}$$

where $\varepsilon = \varepsilon_0/\omega > 0$ is a small fluctuation parameter, and $\zeta(t)$ an Ornstein–Uhlenbeck process given by

$$\mathrm{d}\zeta(t) = -\alpha\,\zeta(t)\,\mathrm{d}t + \sigma \circ \mathrm{d}W(t), \tag{8}$$

where $\alpha = \alpha_0/\omega$, $\sigma = \sigma_0/\sqrt{\omega}$.

The Lyapunov exponent and moment Lyapunov exponent of system (7) are related to those of system (5) by

$$\lambda_{x(\tau)} = \omega\lambda_{x(t)}, \qquad \Lambda_{x(\tau)}(p) = \omega\Lambda_{x(t)}(p). \tag{9}$$

Hence, from equations (6) and (9), one obtains

$$\lambda_{q(\tau)} = -\beta + \omega\lambda_{x(t)}, \qquad \Lambda_{q(\tau)}(p) = -p\beta + \omega\Lambda_{x(t)}(p). \tag{10}$$

Without loss of generality, the moment Lyapunov exponent of system (7) is studied in the remainder of this paper.

A systematic study of moment Lyapunov exponents is presented in reference [2] for linear Itô systems and reference [3] for linear stochastic systems under real-noise excitations. The connection between the moment Lyapunov exponents and the large deviation theory was studied by Baxendale [4], Arnold and Kliemann [5], Baxendale and Stroock [6]. A systematic presentation of the theory of random dynamical systems and a comprehensive list of references can be found in reference [7].

Although the moment Lyapunov exponents are important in the study of dynamic stability of stochastic systems, the actual evaluations of the moment Lyapunov exponents are very difficult. Very few results on the moment Lyapunov exponents have been published. Using the analytic property of the moment Lyapunov exponents, Arnold *et al.* [8] obtained expansions in terms of $\varepsilon_0 p$ under both white- and real-noise excitations. However, for system (1), moment instability normally occurs for large values of p. This makes the results obtained by Arnold *et al.* [8] inappropriate for determining the stability index. Khasminskii and Moshchuk [9] obtained an asymptotic expansion of the moment Lyapunov exponent of system (1) under white-noise parametric excitation in terms of the small-fluctuation parameter ε_0, from which the stability index was obtained.

In this paper, a procedure similar to that employed in Khasminskii and Moshchuk [9] is applied to obtain a weak-noise expansion of the moment Lyapunov exponent of system (7), or equivalently (1), under real-noise excitation in terms of the small fluctuation parameter ε. Expansions of the Lyapunov exponent and stability index are also obtained.

2. FORMULATION

Consider the two-dimensional system (7) under real-noise parametric excitation, i.e., an Ornstein–Uhlenbeck process $\zeta(t)$ given by equation (8). The generator of process $\zeta(t)$ is

$$G = \frac{\sigma^2}{2}\frac{\partial^2}{\partial\zeta^2} - \alpha\zeta\frac{\partial}{\partial\zeta}. \tag{11}$$

Letting $x_1 = x$, $x_2 = \dot{x}$, equation (7) may be written in the form of state equation

$$\begin{Bmatrix}\dot{x}_1\\ \dot{x}_2\end{Bmatrix} = \mathbf{A}(\zeta)\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}, \qquad \mathbf{A}(\zeta) = \begin{bmatrix}0 & 1\\ -1+\varepsilon\zeta & 0\end{bmatrix}. \tag{12}$$

Apply the Khasminskii transformation [10]:

$$s_1 = \frac{x_1}{a} = \cos\varphi, \qquad s_2 = \frac{x_2}{a} = \sin\varphi, \qquad a = \|\mathbf{x}\| = (x_1^2 + x_2^2)^{1/2}, \tag{13}$$

and denote $\mathbf{s} = \{s_1, s_2\}^{\mathrm{T}} = \{\cos\varphi, \sin\varphi\}^{\mathrm{T}}$. From the general theory of moment Lyapunov exponents [3] it is well known that, assuming that ζ is strongly elliptic, the moment Lyapunov exponent $\Lambda_{x(t)}(p)$ of system (12) is the principal simple eigenvalue of the infinitesimal operator $L(p)$

$$L(p)\,T(\zeta, \mathbf{s}) = \Lambda_{x(t)}(p)\,T(\zeta, \mathbf{s}), \quad L(p) = \mathscr{L} + pQ(\zeta, \mathbf{s}), \tag{14}$$

where

$$\mathscr{L} = G + \mathbf{h}^{\mathrm{T}}\frac{\partial}{\partial \mathbf{s}}, \quad Q(\zeta, \mathbf{s}) = \mathbf{s}^{\mathrm{T}}\mathbf{A}(\zeta)\mathbf{s} = \varepsilon\zeta\cos\varphi\sin\varphi,$$

$$\mathbf{h}(\zeta, \mathbf{s}) = (\mathbf{A}(\zeta) - Q(\zeta, \mathbf{s})\mathbf{I})\,\mathbf{s} = \begin{Bmatrix} -\varepsilon\zeta\cos^2\varphi\sin\varphi + \sin\varphi \\ (-1 + \varepsilon\zeta)\cos\varphi - \varepsilon\zeta\cos\varphi\sin^2\varphi \end{Bmatrix}.$$

Since

$$\frac{\partial}{\partial s_1} = -\sin\varphi\frac{\partial}{\partial\varphi}, \qquad \frac{\partial}{\partial s_2} = \cos\varphi\frac{\partial}{\partial\varphi},$$

one has

$$\mathbf{h}^{\mathrm{T}}\frac{\partial}{\partial \mathbf{s}} = h_1\frac{\partial}{\partial s_1} + h_2\frac{\partial}{\partial s_2} = (-1 + \varepsilon\zeta\cos^2\varphi)\frac{\partial}{\partial\varphi},$$

and the infinitesimal operator $L(p)$ is obtained as

$$L(p) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial\zeta^2} - \alpha\zeta\frac{\partial}{\partial\zeta} + (-1 + \varepsilon\zeta\cos^2\varphi)\frac{\partial}{\partial\varphi} + \varepsilon p\zeta\cos\varphi\sin\varphi. \tag{15}$$

The infinitesimal operator $L(p)$ of the eigenvalue problem (14) for the pth moment Lyapunov exponent can also be derived using a more straight-forward approach without resorting to the general theory of moment Lyapunov exponents. This approach was first applied by Wedig [11] to derive the eigenvalue problem for the moment Lyapunov exponent of a two-dimensional linear Itô stochastic system.

Equations (7) and (8) may be considered as a three-dimensional system

$$\mathrm{d}\begin{Bmatrix} x_1 \\ x_2 \\ \zeta \end{Bmatrix} = \begin{Bmatrix} x_2 \\ (-1 + \varepsilon\zeta)x_1 \\ -\alpha\zeta \end{Bmatrix}\mathrm{d}t + \begin{Bmatrix} 0 \\ 0 \\ \sigma \end{Bmatrix}\mathrm{d}W.$$

Apply the Khasminskii transformation (13) and define the pth norm $P = a^p$. The Itô equations for P and φ can be obtained by Itô's lemma:

$$\mathrm{d}P = \varepsilon pP\zeta\cos\varphi\sin\varphi\,\mathrm{d}t, \quad \mathrm{d}\varphi = (-1 + \varepsilon\zeta\cos^2\varphi)\,\mathrm{d}t. \tag{16}$$

Applying a linear stochastic transformation,

$$S = T(\zeta,\varphi)P, \quad P = T^{-1}(\zeta,\varphi)S, \quad -\infty < \zeta < \infty, \quad -\tfrac{1}{2}\pi \leqslant \varphi \leqslant \tfrac{1}{2}\pi,$$

the Itô equation for the new pth norm process S is given by, from Itô's lemma,

$$\mathrm{d}S = [\tfrac{1}{2}\sigma^2 T_{\zeta\zeta} - \alpha\zeta T_\zeta + (-1 + \varepsilon\zeta\cos^2\varphi)T_\varphi + \varepsilon p\zeta\cos\varphi\sin\varphi\, T]P\,\mathrm{d}t + \sigma T_\zeta P\,\mathrm{d}W. \tag{17}$$

For bounded and non-singular transformation $T(\zeta, \varphi)$, both processes P and S are expected to have the same stability behaviour. Therefore, $T(\zeta, \varphi)$ is chosen so that the drift term of the Itô differential equation (17) is independent of the noise process $\zeta(t)$ and the phase process φ, so that

$$\mathrm{d}S = \Lambda S\,\mathrm{d}t + \sigma T_\zeta T^{-1} S\,\mathrm{d}W. \tag{18}$$

Comparing equations (17) and (18), it is seen that such a transformation $T(\zeta, \varphi)$ is given by the equation

$$\tfrac{1}{2}\sigma^2 T_{\zeta\zeta} - \alpha\zeta T_\zeta + (-1 + \varepsilon\zeta\cos^2\varphi)T_\varphi + \varepsilon p\zeta\cos\varphi\sin\varphi\, T = \Lambda T,$$

$$-\infty < \zeta < \infty, \quad -\tfrac{1}{2}\pi \leqslant \varphi \leqslant \tfrac{1}{2}\pi, \tag{19}$$

which defines an eigenvalue problem for a second-order differential operator with Λ being the eigenvalue and $T(\zeta, \varphi)$ the associated eigenfunction. From equation (18), the eigenvalue Λ is seen to be the Lyapunov exponent of the pth moment of system (7), i.e., $\Lambda = \Lambda_{x(t)}(p)$. It is obvious that the differential operator in the eigenvalue problem (19) is the same as the infinitesimal operator $L(p)$ given by equation (15).

In the following section, the method of regular perturbation is applied to the eigenvalue problem (14) to obtain a weak-noise expansion of the moment Lyapunov exponent for system (7).

3. WEAK-NOISE EXPANSION OF THE MOMENT LYAPUNOV EXPONENT

For weak-noise excitation, i.e., small ε, the infinitesimal operator $L(p)$ can be written as

$$L(p) = L_0(p) + \varepsilon L_1(p), \tag{20}$$

where

$$L_0(p) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial\zeta^2} - \alpha\zeta\frac{\partial}{\partial\zeta} - \frac{\partial}{\partial\varphi}, \quad L_1(p) = \zeta\left(\cos^2\varphi\frac{\partial}{\partial\varphi} + p\cos\varphi\sin\varphi\right).$$

Applying the method of regular perturbation, both the eigenvalue $\Lambda_{x(t)}(p)$ and the eigenfunction $T(\zeta, \varphi)$, a periodic function in φ of period π, are expanded in power series of ε as

$$\begin{aligned} \Lambda_{x(t)}(p) &= \Lambda_0(p) + \varepsilon\Lambda_1(p) + \cdots + \varepsilon^n\Lambda_n(p) + \cdots, \\ T(\zeta, \varphi) &= T_0(\zeta,\varphi) + \varepsilon T_1(\zeta,\varphi) + \cdots + \varepsilon^n T_n(\zeta,\varphi) + \cdots, \end{aligned} \tag{21}$$

in which $T_i(\zeta, \varphi)$, $i = 0, 1, \ldots,$ are periodic functions in φ of period π.

Substituting equations (20) and (21) into the eigenvalue problem (14) and equating terms of equal power of ε yields the following equations:

$$\begin{aligned} &\text{0th order:} && L_0T_0 = \Lambda_0T_0,\\ &\text{1st order:} && L_0T_1 + L_1T_0 = \Lambda_0T_1 + \Lambda_1T_0,\\ &\text{2nd order:} && L_0T_2 + L_1T_1 = \Lambda_0T_2 + \Lambda_1T_1 + \Lambda_2T_0,\\ &\quad\vdots && \quad\vdots\\ &n\text{th order:} && L_0T_n + L_1T_{n-1} = \Lambda_0T_n + \Lambda_1T_{n-1} + \cdots + \Lambda_nT_0. \end{aligned} \tag{22}$$

3.1. ZEROTH ORDER PERTURBATION

The equation for the zeroth order perturbation is

$$L_0T_0 = \Lambda_0T_0 \tag{23}$$

or

$$\frac{\sigma^2}{2}\frac{\partial^2T_0}{\partial\zeta^2} - \alpha\zeta\frac{\partial T_0}{\partial\zeta} - \frac{\partial T_0}{\partial\varphi} - \Lambda_0T_0 = 0.$$

Applying the method of separation of variables and letting $T_0(\zeta, \varphi) = Z_0(\zeta)\Phi_0(\varphi)$ results in

$$\frac{\sigma^2}{2}\frac{\ddot{Z}_0}{Z_0} - \alpha\zeta\frac{\dot{Z}_0}{Z_0} - \Lambda_0 = \frac{\Phi_0'}{\Phi_0} = k.$$

Solving the equation for Φ_0 yields $\Phi_0(\varphi) = Ae^{k\varphi}$. For $\Phi_0(\varphi)$ to be a period function, it is required that $k = 0$ and hence $\Phi_0(\varphi)$ can be chosen as 1.

The equation for $Z_0(\zeta)$ becomes

$$\tfrac{1}{2}\sigma^2\ddot{Z}_0 - \alpha\zeta\dot{Z}_0 - \Lambda_0Z_0 = 0. \tag{24}$$

From the property of moment Lyapunov exponent, it is known that

$$\Lambda_{x(t)}(0) = \Lambda_0(0) + \varepsilon\Lambda_1(0) + \cdots + \varepsilon^n\Lambda_n(0) + \cdots = 0,$$

which results in $\Lambda_0(0) = 0$. Since the eigenvalue problem (24) does not contain p, the eigenvalue $\Lambda_0(p)$ is independent of p. Hence, $\Lambda_0(0) = 0$ leads to $\Lambda_0(p) = 0$.

Equation (24) can be easily solved to yield

$$Z_0(\zeta) = C_1\int\exp\left(\frac{\alpha}{\sigma^2}\zeta^2\right)\mathrm{d}\zeta + C_2, \quad -\infty < \zeta < \infty.$$

For $Z_0(\zeta)$ to be bounded, it is required that $C_1 = 0$ and hence $Z_0(\zeta)$ can be taken as 1. Therefore,

$$\Lambda_0(p) = 0, \quad T_0(\zeta, \varphi) = Z_0(\zeta)\Phi_0(\varphi) = 1. \tag{25}$$

Since $\Lambda_0(p) = 0$, the associated adjoint differential equation of (23) is

$$L_0^*T_0^* = \frac{\sigma^2}{2}\frac{\partial^2T_0^*}{\partial\zeta^2} + \alpha\zeta\frac{\partial T_0^*}{\partial\zeta} + \alpha T_0^* + \frac{\partial T_0^*}{\partial\varphi} = 0. \tag{26}$$

Applying the method of separation of variables and letting $T_0^*(\zeta,\varphi)=Z_0^*(\zeta)\Phi_0^*(\varphi)$ leads to

$$\frac{\sigma^2}{2}\frac{\ddot{Z}_0^*}{Z_0^*}+\alpha\zeta\frac{\dot{Z}_0^*}{Z_0^*}+\alpha=-\frac{\Phi_0^{*\prime}}{\Phi_0^*}=\kappa.$$

The equation for Φ_0^* yields $\Phi_0^*(\varphi)=B\,e^{-\kappa\varphi}$. For $\Phi_0^*(\varphi)$ to be a period function, $\kappa=0$ and $\Phi_0^*(\varphi)$ can be taken as

$$\Phi_0^*(\varphi)=\frac{1}{\pi},\quad -\tfrac{1}{2}\pi\leqslant\varphi\leqslant\tfrac{1}{2}\pi, \tag{27}$$

which is the probability density function of a uniformly distributed random variable φ between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$.

The equation for Z_0^* becomes

$$\tfrac{1}{2}\sigma^2\ddot{Z}_0^*+\alpha\zeta\dot{Z}_0^*+\alpha Z_0^*=0. \tag{28}$$

Equation (28) is the Fokker–Planck equation for the stationary transition probability density of the Ornstein–Uhlenbeck process $\zeta(t)$ as defined in equation (8) [1]. Equation (28) may be written as

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{\mathrm{d}Z_0^*}{\mathrm{d}\zeta}+\frac{2\alpha}{\sigma^2}\zeta Z_0^*\right)=0$$

or

$$\frac{\mathrm{d}Z_0^*}{\mathrm{d}\zeta}+\frac{2\alpha}{\sigma^2}\zeta Z_0^*=C_3=\text{probability current}. \tag{29}$$

Since the stationary probability density $Z_0^*(\zeta)$ and the probability current vanishes when $\zeta\to\pm\infty$, the constant of integration $C_3=0$. Equation (29) can be easily solved to give

$$Z_0^*(\zeta)=C_4\exp\left(-\frac{\alpha}{\sigma^2}\zeta^2\right).$$

Since $Z_0^*(\zeta)$ is the stationary probability density, normalizing it yields

$$Z_0^*(\zeta)=\frac{1}{\sqrt{2\pi}\sigma_\zeta}\exp\left(-\frac{\zeta^2}{2\sigma_\zeta^2}\right), \tag{30}$$

i.e., the Ornstein–Uhlenbeck process $\zeta(t)$ is a normally distributed random variable with mean $\mu_\zeta=0$ and standard deviation $\sigma_\zeta=\sigma/\sqrt{2\alpha}$.

Hence, $T_0^*(\zeta,\varphi)=Z_0^*(\zeta)\Phi_0^*(\varphi)$ represents the joint stationary probability density function of the independent random variables ζ and φ, in which ζ is normally distributed with mean $\mu_\zeta=0$ and standard deviation $\sigma_\zeta=\sigma/\sqrt{2\alpha}$ and φ is uniformly distributed between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$.

3.2. FIRST ORDER PERTURBATION

The first order perturbation equation is

$$L_0T_1=\Lambda_1T_0-L_1T_0. \tag{31}$$

Since the homogeneous equation $L_0 T_0 = 0$ has a non-trivial solution as given by equation (25), for equation (31) to have a solution it is required that, from the Fredholm alternative,

$$(\Lambda_1 T_0 - L_1 T_0, T_0^*) = 0, \tag{32}$$

where $T_0^*(\zeta, \varphi)$ is the solution of the adjoint equation (26) as obtained in section 3.1, and (S_1, S_2) denotes the inner product of functions $S_1(\zeta, \varphi)$ and $S_2(\zeta, \varphi)$ defined by

$$(S_1, S_2) = \int_{-\infty}^{\infty} \int_{-1/2\pi}^{1/2\pi} S_1(\zeta, \varphi) S_2(\zeta, \varphi) \, d\varphi \, d\zeta.$$

From equation (32), the first order perturbation of the moment Lyapunov exponent is

$$\Lambda_1 = (L_1 T_0, T_0^*), \tag{33}$$

because $(T_0, T_0^*) = 1$.

It is easy to show that

$$L_1 T_0 = \zeta\left(\cos^2\varphi \frac{\partial T_0}{\partial\varphi} + p\cos\varphi\sin\varphi\, T_0\right) = f_1^{(1)}(\varphi)\,\zeta,$$

where $f_1^{(1)}(\varphi) = p\cos\varphi\sin\varphi$. Hence using equations (27) and (30),

$$(L_1 T_0, T_0^*) = \overline{f_1^{(1)}(\varphi)} E[\zeta],$$

which leads to

$$\Lambda_1 = 0, \tag{34}$$

where

$$\overline{a(\varphi)} = \frac{1}{\pi}\int_{-1/2\pi}^{1/2\pi} a(\varphi)\, d\varphi,$$

denotes the expected value of the random variable $a(\varphi)$, in which φ is the uniformly distributed random variable between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ as defined in equation (27), and $E[b(\zeta)]$ denotes the expected value of the random variable $b(\zeta)$, in which ζ is the normally distributed random variable defined in equation (30).

Hence, equation (31) becomes

$$L_0 T_1 = g_1^{(1)}(\varphi)\zeta, \tag{35}$$

where $g_1^{(1)}(\varphi) = -f_1^{(1)}(\varphi) = -p\cos\varphi\sin\varphi$. Equation (35) is of the form (57) and the solution is given in Appendix A by equation (A.12).

Since, from Appendix A, $E[z(r)] = \mu_{z(r)} = \zeta e^{-\alpha(r-s)}$, the solution of equation (35) is obtained as

$$T_1(\zeta, \psi, s) = \int_0^s g_1^{(1)}(\psi - r)\, e^{-\alpha(r-s)}\, dr \cdot \zeta,$$

or

$$T_1(\zeta, \varphi) = G_1^{(1)}(\varphi)\zeta, \tag{36}$$

where

$$G_1^{(1)}(\varphi) = \frac{p}{2(\alpha^2+4)}(-2\cos 2\varphi + \alpha \sin 2\varphi),$$

in which, as shown in Appendix A, $\psi - s = \varphi$ and $s \to -\infty$ have been employed.

3.3. SECOND ORDER PERTURBATION

The equation for the second order perturbation is

$$L_0 T_2 = \Lambda_2 T_0 - L_1 T_1. \tag{37}$$

From the Fredholm alternative, for equation (37) to have a solution, it is required that

$$(\Lambda_2 T_0 - L_1 T_1, T_0^*) = 0. \tag{38}$$

Since

$$L_1 T_1 = \zeta\left(\cos^2\varphi \frac{\partial T_1}{\partial \varphi} + p\cos\varphi \sin\varphi T_1\right) = f_2^{(2)}(\varphi)\zeta^2,$$

where

$$f_2^{(2)}(\varphi) = \cos^2\varphi G_1^{(1)\prime}(\varphi) + p\cos\varphi\sin\varphi G_1^{(1)}(\varphi)$$

$$= \frac{p}{2(\alpha^2+4)}[2\cos^2\varphi(2\sin 2\varphi + \alpha\cos 2\varphi) + p\cos\varphi\sin\varphi(-2\cos 2\varphi + \alpha\sin 2\varphi)],$$

one obtains from equation (38),

$$\Lambda_2 = (L_1 T_1, T_0^*) = \overline{f_2^{(2)}(\varphi)} E[\zeta^2] = \frac{p(p+2)\sigma^2}{16(\alpha^2+4)}, \tag{39}$$

in which $E[\zeta^2] = \sigma_\zeta^2 = \sigma^2/(2\alpha)$.

Equation (37) becomes

$$L_0 T_2 = g_0^{(2)} + g_2^{(2)}(\varphi)\zeta^2, \tag{40}$$

where

$$g_0^{(2)} = \Lambda_2, \quad g_2^{(2)}(\varphi) = -f_2^{(2)}(\varphi).$$

Since, from Appendix A,

$$E[z^2(r)] = \sigma_{z(r)}^2 + \mu_{z(r)}^2 = \frac{\sigma^2[1 - e^{-2\alpha(r-s)}]}{2\alpha} + \zeta^2 e^{-2\alpha(r-s)},$$

the solution of equation (40) given by equation (68) is

$$T_2(\zeta, \varphi) = G_0^{(2)}(\varphi) + G_2^{(2)}(\varphi)\zeta^2, \tag{41}$$

where

$$G_0^{(2)}(\varphi) = \int_0^s \left\{ g_0^{(2)}(\psi - r) + \frac{\sigma^2}{2\alpha} g_2^{(2)}(\psi - r)[1 - e^{-2\alpha(r-s)}] \right\} dr,$$

$$G_2^{(2)}(\varphi) = \int_0^s g_2^{(2)}(\psi - r) e^{-2\alpha(r-s)} dr,$$

in which $\psi - s = \varphi$ and $s \to -\infty$ are taken after integration.

3.4. HIGHER ORDER PERTURBATION

The procedure presented in sections 3.1–3.3 is very algorithmic. The algebraic manipulation can be performed using a symbolic computation software such as *maple* so that the higher order approximations can be easily obtained. The procedure can be summarized as follows.

For the $(2n)$th order perturbation, $n = 1, 2, \ldots$, the perturbation equation is

$$L_0 T_{2n} = \Lambda_2 T_{2n-2} + \Lambda_4 T_{2n-4} + \cdots + \Lambda_{2n} T_0 - L_1 T_{2n-1}, \tag{42}$$

because $\Lambda_0 = \Lambda_1 = \Lambda_3 = \cdots = \Lambda_{2n-1} = 0$. From the Fredholm alternative, for equation (42) to have a solution, it is required that

$$(\Lambda_2 T_{2n-2} + \Lambda_4 T_{2n-4} + \cdots + \Lambda_{2n} T_0 - L_1 T_{2n-1}, T_0^*) = 0. \tag{43}$$

Since $L_1 T_{2n-1}$ is of the form

$$L_1 T_{2n-1} = f_2^{(2n)}(\varphi)\zeta^2 + f_4^{(2n)}(\varphi)\zeta^4 + \cdots + f_{2n}^{(2n)}(\varphi)\zeta^{2n},$$

equation (43) yields

$$\begin{aligned} \Lambda_{2n} &= (L_1 T_{2n-1}, T_0^*) - \Lambda_2 (T_{2n-2}, T_0^*) - \cdots - \Lambda_{2n-2}(T_2, T_0^*) \\ &= \overline{f_2^{(2n)}(\varphi)} E\,[\zeta^2] + \overline{f_4^{(2n)}(\varphi)} E[\zeta^4] + \cdots + \overline{f_{2n}^{(2n)}(\varphi)} E[\zeta^{2n}] \\ &\quad - \Lambda_2 \{ \overline{G_0^{(2n-2)}(\varphi)} + \overline{G_2^{(2n-2)}(\varphi)} E[\zeta^2] + \cdots + \overline{G_{2n-2}^{(2n-2)}(\varphi)} E[\zeta^{2n-2}] \} \\ &\quad - \cdots - \Lambda_{2n-2} \{ \overline{G_0^{(2)}(\varphi)} + \overline{G_2^{(2)}(\varphi)} E[\zeta^2] \}. \end{aligned} \tag{44}$$

Equation (42) is then of the form

$$L_0 T_{2n} = g_0^{(2n)}(\varphi) + g_2^{(2n)}(\varphi)\zeta^2 + \cdots + g_{2n}^{(2n)}(\varphi)\zeta^{2n}, \tag{45}$$

which can be solved using equation (A.12) to yield $T_{2n}(\zeta, \varphi)$ of the form

$$T_{2n}(\zeta, \varphi) = G_0^{(2n)}(\varphi) + G_2^{(2n)}(\varphi)\zeta^2 + \cdots + G_{2n}^{(2n)}(\varphi)\zeta^{2n}. \tag{46}$$

For the $(2n+1)$th order perturbation, $n = 0, 1, \ldots$, the perturbation equation is

$$L_0 T_{2n+1} = \Lambda_2 T_{2n-1} + \Lambda_4 T_{2n-3} + \cdots + \Lambda_{2n} T_1 + \Lambda_{2n+1} T_0 - L_1 T_{2n}. \tag{47}$$

From the Fredholm alternative, for equation (47) to have a solution, it is required that

$$(\Lambda_2 T_{2n-1} + \Lambda_4 T_{2n-3} + \cdots + \Lambda_{2n} T_1 + \Lambda_{2n+1} T_0 - L_1 T_{2n}, T_0^*) = 0, \tag{48}$$

which yields

$$\Lambda_{2n+1} = (L_1 T_{2n}, T_0^*) - \Lambda_2(T_{2n-1}, T_0^*) - \cdots - \Lambda_{2n}(T_1, T_0^*).$$

Since $L_1 T_{2n}$ is of the form

$$L_1 T_{2n} = f_1^{(2n+1)}(\varphi)\zeta + f_3^{(2n+1)}(\varphi)\zeta^3 + \cdots + f_{2n+1}^{(2n+1)}(\varphi)\zeta^{2n+1},$$

it can be easily shown that $\Lambda_{2n+1} = 0$ because $E[\zeta^{2n+1}] = 0$ for $n = 0, 1, \ldots$.

Equation (47) can be written in the form

$$L_0 T_{2n+1} = g_1^{(2n+1)}(\varphi)\zeta + g_3^{(2n+1)}(\varphi)\zeta^3 + \cdots + g_{2n+1}^{(2n+1)}(\varphi)\zeta^{2n+1}, \tag{49}$$

which can be solved using equation (A.12) to yield T_{2n+1} of the form

$$T_{2n+1}(\zeta, \varphi) = G_1^{(2n+1)}(\varphi)\zeta + G_3^{(2n+1)}(\varphi)\zeta^3 + \cdots + G_{2n+1}^{(2n+1)}(\varphi)\zeta^{2n+1}. \tag{50}$$

Following this procedure, the weak-noise expansion of the moment Lyapunov exponent is obtained as

$$\Lambda_{x(t)}(p) = \varepsilon^2 \Lambda_2 + \varepsilon^4 \Lambda_4 + \varepsilon^6 \Lambda_6 + O(\varepsilon^8), \tag{51}$$

where

$$\Lambda_2 = \frac{p(p+2)\sigma^2}{16(\alpha^2+4)}, \quad \Lambda_4 = \frac{p(p+2)\sigma^4(\alpha^4 + 22\alpha^2 + 48)}{32\alpha(\alpha^2+1)(\alpha^2+4)^3},$$

$$\begin{aligned}\Lambda_6 = p(p+2)\sigma^6[&p^2(99\alpha^{14} + 4274\alpha^{12} + 70379\alpha^{10} + 499596\alpha^8 + 1547568\alpha^6 \\ &+ 2119232\alpha^4 + 1267200\alpha^2 + 262144) + p(198\alpha^{14} + 8548\alpha^{12} + 140758\alpha^{10} \\ &+ 999192\alpha^8 + 3095136\alpha^6 + 4238464\alpha^4 + 2534400\alpha^2 + 524288) \\ &+ (-1080\alpha^{14} - 42960\alpha^{12} - 650680\alpha^{10} - 3903840\alpha^8 - 2981760\alpha^6 + 27553280\alpha^4 \\ &+ 60641280\alpha^2 + 31457280)]/[8192\alpha^2(\alpha^2+16)(9\alpha^2+4)(\alpha^2+1)^2(\alpha^2+4)^5].\end{aligned}$$

The Lyapunov exponent for system (7) can be obtained from equation (51) by using the property of the moment Lyapunov exponent,

$$\lambda_{x(t)} = \left.\frac{\mathrm{d}\Lambda_{x(t)}(p)}{\mathrm{d}p}\right|_{p=0} = \varepsilon^2 \lambda_2 + \varepsilon^4 \lambda_4 + \varepsilon^6 \lambda_6 + \mathrm{O}(\varepsilon^8), \tag{52}$$

where

$$\lambda_2 = \frac{\sigma^2}{8(\alpha^2+4)}, \quad \lambda_4 = \frac{\sigma^4(\alpha^4 + 22\alpha^2 + 48)}{16\alpha(\alpha^2+1)(\alpha^2+4)^3},$$

$$\begin{aligned}\lambda_6 = -5\sigma^6(&27\alpha^{14} + 1074\alpha^{12} + 16267\alpha^{10} + 97596\alpha^8 + 74544\alpha^6 - 688832\alpha^4 \\ &- 1516032\alpha^2 - 786432)/[512\alpha^2(\alpha^2+16)(9\alpha^2+4)(\alpha^2+1)^2(\alpha^2+4)^5].\end{aligned}$$

3.5. STABILITY INDEX

As mentioned in section 1, the stability index is the non-trivial zero of the moment Lyapunov exponent. For system (7), the moment Lyapunov exponent is given by equation

(51). It is seen that $p=0$ and $p=2$ are the two values that lead to $\Lambda_{x(t)}(p)=0$, and hence the stability index $\delta_{x(t)}=-2$.

For system (1), the moment Lyapunov exponent is

$$\Lambda_{q(\tau)}(p)=-p\beta+\omega\Lambda_{x(t)}(p),$$

and the stability index $\delta_{q(\tau)}$ is given by

$$\Lambda_{q(\tau)}(\delta_{q(\tau)})=-\beta\delta_{q(\tau)}+\omega\Lambda_{x(t)}(\delta_{q(\tau)})=0,$$

or

$$\Lambda_{x(t)}(\delta_{q(\tau)})-\varepsilon^2\tilde{\beta}\delta_{q(\tau)}=0, \tag{53}$$

where $\varepsilon^2\tilde{\beta}=\beta/\omega$.

Expanding the stability index $\delta_{q(\tau)}$ in power series of ε as

$$\delta_{q(\tau)}=\delta_0+\varepsilon\delta_1+\cdots+\varepsilon^n\delta_n+\cdots, \tag{54}$$

and substituting equations (51) and (54) into equation (53), expanding and equating terms of equal power of ε yields the equations

$$\begin{aligned}
&\text{2nd order} \quad \delta_0\left[-\tilde{\beta}+\frac{(\delta_0+2)\sigma^2}{16(\alpha^2+4)}\right]=0,\\
&\text{3rd order} \quad \delta_1\left[-\tilde{\beta}+\frac{(3\delta_0+2)\sigma^2}{16(\alpha^2+4)}\right]=0,\\
&\text{4th order} \quad -\tilde{\beta}\delta_2+\frac{[\delta_1^2+2\delta_2(\delta_0+1)]\sigma^2}{16(\alpha^2+4)}+\frac{\delta_0(\delta_0+2)(\alpha^4+22\alpha^2+48)\sigma^4}{32\alpha(\alpha^2+1)(\alpha^2+4)^3}=0,\\
&\qquad\vdots \qquad\qquad\qquad \vdots
\end{aligned} \tag{55}$$

Using a symbolic computation software such as *maple*, these equations can be easily manipulated and solved for δ_i, $i=0,1,\ldots$, to result in

$$\delta_0=-2+\frac{16\tilde{\beta}}{\sigma^2}(\alpha^2+4), \quad \delta_1=0,$$

$$\delta_2=-\frac{8\tilde{\beta}(\alpha^4+22\alpha^2+48)}{\alpha(\alpha^2+1)(\alpha^2+4)}, \quad \delta_3=0,$$

$$\begin{aligned}
\delta_4=\tilde{\beta}[&-\tilde{\beta}^2(3168\alpha^{18}+162112\alpha^{16}+3396960\alpha^{14}+36192384\alpha^{12}+213452800\alpha^{10}\\
&+719785984\alpha^8+1375428608\alpha^6+1417838592\alpha^4+715915264\alpha^2+134217728)\\
&+\tilde{\beta}\sigma^2(396\alpha^{16}+18680\alpha^{14}+349900\alpha^{12}+3124448\alpha^{10}+14183808\alpha^8\\
&+33238016\alpha^6+38976512\alpha^4+21323776\alpha^2+4194304)\\
&+\sigma^4(135\alpha^{14}+5514\alpha^{12}+90039\alpha^{10}+676716\alpha^8+2095344\alpha^6+2482752\alpha^4\\
&+38400\alpha^2-1572864)]/[4\alpha^2\sigma^2(\alpha^2+16)(9\alpha^2+4)(\alpha^2+1)^2(\alpha^2+4)^3],
\end{aligned} \tag{56}$$

where $\tilde{\beta}=\beta/(\varepsilon^2\omega)$.

4. NUMERICAL RESULTS AND CONCLUSIONS

In this paper, the moment Lyapunov exponents of a two-dimensional system under real-noise excitation, an Ornstein–Uhlenbeck process, are studied. The method of regular perturbation is applied to obtain a weak-noise expansion of the moment Lyapunov exponent in terms of the small fluctuation parameter, from which weak-noise expansions of

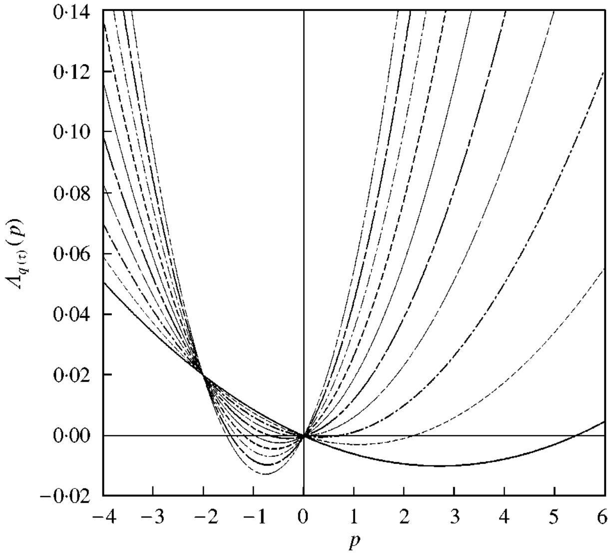

Figure 1. Moment Lyapunov exponents. $\beta = 0{\cdot}01$, $\omega_0 = 1{\cdot}0$, $\alpha_0 = 10{\cdot}0$, $\sigma_0 = 10{\cdot}0$: (——), $\varepsilon_0 = 0{\cdot}15$; (------), $\varepsilon_0 = 0{\cdot}20$; (—·—·-), $\varepsilon_0 = 0{\cdot}25$; (——-—), $\varepsilon_0 = 0{\cdot}30$; (——-·—), $\varepsilon_0 = 0{\cdot}35$; (——), $\varepsilon_0 = 0{\cdot}40$; (------), $\varepsilon_0 = 0{\cdot}45$; (—·—·-), $\varepsilon_0 = 0{\cdot}50$; (-——-), $\varepsilon_0 = 0{\cdot}55$; (--——--), $\varepsilon_0 = 0{\cdot}60$.

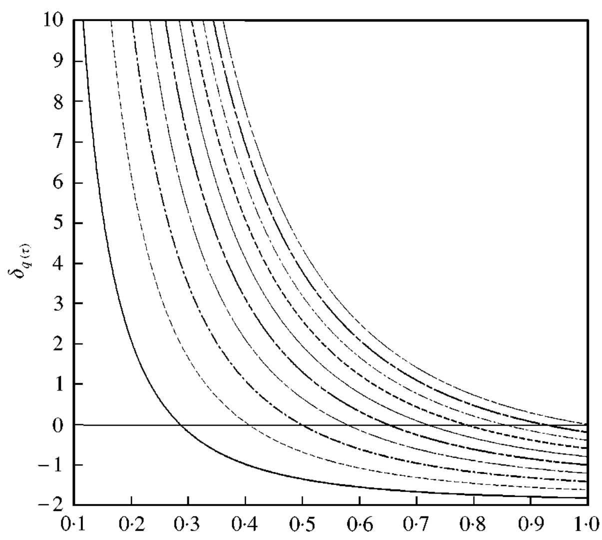

Figure 2. Stability index. $\omega_0 = 1{\cdot}0$, $\alpha_0 = 10{\cdot}0$, $\sigma_0 = 10{\cdot}0$; (——), $\beta = 0{\cdot}01$; (------), $\beta = 0{\cdot}02$; (—·—·-), $\beta = 0{\cdot}03$; (-——-), $\beta = 0{\cdot}04$; (--——--), $\beta = 0{\cdot}05$; (——), $\beta = 0{\cdot}06$; (------), $\beta = 0{\cdot}07$; (—·—·-), $\beta = 0{\cdot}08$; (-——-), $\beta = 0{\cdot}09$; (--——--), $\beta = 0{\cdot}10$.

the Lyapunov exponent and stability index are also obtained. The slope of the moment Lyapunov exponent curve at $p = 0$ is the Lyapunov exponent. When the Lyapunov exponent is negative, i.e., when the slope of the moment Lyapunov exponent at the origin is negative, system (1) is stable w.p.1; otherwise, it is unstable w.p.1. When the system is stable w.p.1, the non-trivial zero of the moment Lyapunov exponent or the stability index $\delta_{q(\tau)}$ is positive. If the system is unstable w.p.1, it is unstable in the pth moment for all $p > 0$. If the system is stable w.p.1, it is unstable in the pth moment for $p > \delta_{q(\tau)}$. The larger the stability index $\delta_{q(\tau)}$, the more stable the system.

Typical results of the moment Lyapunov exponent $\Lambda_{q(\tau)}(p)$ for system (1) are shown in Figure 1 for $\beta = 0{\cdot}01$, $\omega_0 = 1{\cdot}0$, $\alpha_0 = 10{\cdot}0$, $\sigma_0 = 10{\cdot}0$, and various values of fluctuation parameter ε_0. It is seen that, when ε_0 is increased, the slope of the moment Lyapunov exponent curve at the origin decreases from positive to negative. Typical results of the stability index $\delta_{q(\tau)}$ are shown in Figure 2. It is observed that, when ε_0 is increased, the stability index $\delta_{q(\tau)}$ decreases from positive to negative values. It is also seen that the larger the damping coefficient β, the larger the stability index $\delta_{q(\tau)}$.

The procedure presented in this paper is quite algorithmic and the algebraic manipulation can be performed by a symbolic computation software package.

ACKNOWLEDGMENTS

The research for this paper was supported, in part, by the Natural Sciences and Engineering Research Council of Canada through Grant No. OGPO131355. The author is grateful to the referees for the constructive comments which helped to improve the paper.

REFERENCES

1. H. Risken 1989 *The Fokker–Planck Equation: Methods of Solution and Applications*, Berlin: Springer. Second edition.
2. L. Arnold, E. Oeljeklaus and E. Pardoux 1986 *Lyapunov Exponents, Proceedings of a Workshop, Bremen, Germany*, 1984, L. Arnold and V. Wihstutz, editors, Lecture Notes in Mathematics Vol. 1186, 85–125. Berlin: Springer-Verlag, Almost sure and moment stability for linear Itô equations.
3. L. Arnold, W. Kliemann and E. Oeljeklaus 1986 *Lyapunov Exponents, Proceedings of a Workshop, Bremen, Germany*, 1984, L. Arnold and V. Wihstutz, editors, Lecture Notes in Mathematics, Vol. 1186, 129–159. Berlin: Springer-Verlag, Lyapunov exponents of linear stochastic systems.
4. P. H. Baxendale 1985 *Proceedings of the Taniguchi Symposium on Probabilistic Methods in Mathematical Physics, Kinokuniya, Tokyo, Japan*, N. Ikeda, editor, 31–54. Moment stability and large deviations for linear stochastic differential equations.
5. L. Arnold and W. Kliemann 1987 *Stochastic Differential Systems*, H. J. Engelbert and W. Schmidt, editors, Lecture Notes in Control and Information Sciences, Vol. 96, 117–151. Berlin: Springer, Large deviations of linear stochastic differential equations.
6. P. Baxendale and D. Stroock 1988 *Probability Theory and Related Fields* **80**, 169–215. Large deviations and stochastic flows of diffeomorphisms.
7. L. Arnold 1998 *Random Dynamical Systems*. Berlin: Springer-Verlag.
8. L. Arnold, M. M. Doyle and N. Sri Namachchivaya 1997 *Dynamics and Stability of Systems* **12**, 187–211. Small noise expansion of moment Lyapunov exponents for two-dimensional systems.
9. R. Z. Khasminskii and N. Moshchuk 1998 *SIAM Journal of Applied Mathematics* **58**, 245–256. Moment Lyapunov exponent and stability index for linear conservative system with small random perturbation.
10. R. Z. Khasminskii 1967 *Theory of Probability and its Applications* **12**, 144–147. Necessary and sufficient conditions for the asymptotic stability of linear stochastic systems, (English Translation).

11. W. WEDIG 1988 *Stochastic Structural Dynamics–Progress in Theory and Applications*, S. T. Ariaratnam, G. I. Schuëller, and I. Elishakoff, editors, London: Elsevier, 315–327. Lyapunov exponent of stochastic systems and related bifurcation problems.
12. E. ZAUDERER 1989 *Partial Differential Equations of Applied Mathematics*. New York: John Wiley & Sons, Second Edition.
13. L. ARNOLD 1974 *Stochastic Differential Equations: Theory and Applications*. New York: John Wiley & Sons.

APPENDIX A: SOLUTION OF $L_0 T(\zeta, \varphi) = f(\zeta) g(\varphi)$

Consider the partial differential equation

$$L_0 T(\zeta, \varphi) = f(\zeta) g(\varphi)$$

or

$$\left(\frac{\sigma^2}{2}\frac{\partial^2}{\partial \zeta^2} - \alpha\zeta\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \varphi}\right) T(\zeta, \varphi) = f(\zeta) g(\varphi). \tag{A.1}$$

Introducing an auxiliary time t' to equation (A.1) leads to

$$\left(\frac{\partial}{\partial t'} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial \zeta^2} - \alpha\zeta\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \varphi}\right) T(\zeta, \varphi, t') = f(\zeta) g(\varphi). \tag{A.2}$$

Applying the transformation

$$\psi = \tfrac{1}{2}(t' + \varphi), \quad s = \tfrac{1}{2}(t' - \varphi)$$

or

$$t' = \psi + s, \quad \varphi = \psi - s,$$

equation (A.2) becomes

$$\left(\frac{\partial}{\partial s} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial \zeta^2} - \alpha\zeta\frac{\partial}{\partial \zeta}\right) T(\zeta, \psi, s) = f(\zeta) g(\psi - s). \tag{A.3}$$

Applying Duhamel's principle [12], the solution $T(\zeta, \psi, s)$ to equation (A.3) is given by

$$T(\zeta, \psi, s) = \int_0^s V(\zeta, \psi, s; r)\, \mathrm{d}r, \tag{A.4}$$

where $V(\zeta, \psi, s; r)$ is the solution of the homogeneous equation

$$\begin{aligned} &\left(\frac{\partial}{\partial s} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial \zeta^2} - \alpha\zeta\frac{\partial}{\partial \zeta}\right) V(\zeta, \psi, s; r) = 0 \quad &&\text{for } s > r, \\ &V(\zeta, \psi, r; r) = f(\zeta) g(\psi - r) &&\text{for } s = r. \end{aligned} \tag{A.5}$$

To solve equation (A.5), consider the equation

$$\begin{aligned} &\left(\frac{\partial}{\partial s} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial \zeta^2} - \alpha\zeta\frac{\partial}{\partial \zeta}\right) P(s, \zeta; t, z) = 0, \quad s < t, \\ &P(t, \zeta; t, z) = \lim_{s \uparrow t} P(s, \zeta; t, z) = \delta(z - \zeta). \end{aligned} \tag{A.6}$$

Equation (A.6) is the Kolmogorov's backward equation for the transition probability function $P(s, \zeta; t, z)$. It is well known [13] that the transition probability $P(s, \zeta; t, z)$ is also the fundamental solution of the forward or Fokker–Planck equation, i.e., for the initial condition s and ζ fixed,

$$\begin{aligned} &\left[\frac{\partial}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial z}(-\alpha z)\right]P(s, \zeta; t, z) = 0, \quad t > s, \\ &P(s, \zeta; s, z) = \lim_{t \downarrow s} P(s, \zeta; t, z) = \delta(z - \zeta). \end{aligned} \tag{A.7}$$

Applying the Fourier transformation

$$\tilde{P}(s, \zeta; t, k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{ikz} P(s, \zeta; t, z)\,\mathrm{d}z$$

to equation (A.7) leads to

$$\begin{aligned} &\frac{\partial \tilde{P}}{\partial t} + \alpha k \frac{\partial \tilde{P}}{\partial k} = -\frac{\sigma^2 k^2}{2}\tilde{P}, \\ &\tilde{P}(s, \zeta; s, k) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{\mathrm{i}k\zeta}. \end{aligned} \tag{A.8}$$

Equation (A.8) can be solved using the method of characteristics to give

$$\tilde{P}(s, \zeta; t, k) = \frac{1}{\sqrt{2\pi}} \exp\left\{\mathrm{i}k\zeta \mathrm{e}^{-\alpha(t-s)} + \frac{\sigma^2}{4\alpha} k^2 [e^{-2\alpha(t-s)} - 1]\right\}. \tag{A.9}$$

Applying the inverse Fourier transformation

$$P(s, \zeta; t, z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kz} \tilde{P}(s, \zeta; t, k)\mathrm{d}k,$$

to equation (A.9) leads to

$$P(s, \zeta; t, z) = \frac{1}{\sqrt{2\pi}\sigma_{z(t)}} \exp\left[-\frac{(z - \mu_{z(t)})^2}{2\sigma^2_{z(t)}}\right], \tag{A.10}$$

where

$$\mu_{z(t)} = \zeta \mathrm{e}^{-\alpha(t-s)}, \quad \sigma^2_{z(t)} = \frac{\sigma^2[1 - e^{-2\alpha(t-s)}]}{2\alpha}.$$

Hence, for the initial condition $\zeta(s)$ fixed, $z(t)$ is a normally distributed random variable with mean $\mu_{z(t)}$ and standard deviation $\sigma_{z(t)}$.

From equation (A.5) and (A.6), the solution $V(\zeta, \psi, s; r)$ to equation (A.5) is given by

$$V(\zeta, \psi, s; r) = g(\psi - r) \int_{-\infty}^{\infty} f(z) P(s, \zeta; r, z)\,\mathrm{d}z, \tag{A.11}$$

where

$$E[f(z(r))] = \int_{-\infty}^{\infty} f(z) P(s, \zeta; r, z)\,\mathrm{d}z,$$

is the expected value of the random variable $f(z(r))$ with $z(r)$ being the normally distributed random variable as defined in (A.10).

Combining equations (A.4) and (A.11), the solution to equation (A.3) is given by

$$T(\zeta, \psi, s) = \int_0^s g(\psi - r) E[f(z(r))] \, \mathrm{d}r. \tag{A.12}$$

The solution $T(\zeta, \varphi)$ to equation (A.1) is obtained by replacing $\varphi = \psi - s$ and passing the limit $s \to -\infty$.