Journal of Sound and Vibration (2001) **239**(2), 363–368
doi:10.1006/jsvi.2000.3061, available online at http://www.idealibrary.com on IDEAL®

FUNDAMENTAL FREQUENCIES OF A CIRCULAR MEMBRANE WITH A CENTERED STRIP

L. H. YU AND C. Y. WANG

Department of Mathematics, Michigan State University, East Lansing, MI 48824, *U.S.A.*

(*Received* 30 *March* 2000)

1. INTRODUCTION

Consider the vibrations of a circular membrance with a circular core, Wang [1] showed that when the core diameter shrinks to zero, the frequency decreases to that of a circular membrane without a central core. This means pinpoint constraints, while affecting vibration mode, do not affect the frequency.

The present note studies whether this phenomenon would extend to membrane with an internal line constraint. Unlike a circular core, the line constraint does not change the membrane area. We ask, how does the frequency change when the length of the line shrinks to zero?

Related literature on internal line constraints are few. Gruner [2] studied the equivalent of a rectangular membrance with a rectangular core, the latter can be shrunk to a line. Veselor and Gaydar [3] considered a circular membrane with a central, cross-shaped line constraint. Rozzi *et al.* [4] found frequencies for the elliptic membrane with an internal confocal strip. None of these authors considered asymptotic case when the constraint is very small.

2. ELLIPTIC MEMBRANE WITH AN INTERNAL CONFOCAL STRIP

First, consider the elliptic membrane with a line constraint which connects the foci (Figure 1(a)). As the focal distance approaches zero, the outer elliptical boundary approaches a circle. Thus, its frequency behavior mimics that of a circular membrane with a short centered strip.

The governing Helmholtz equation is

$$\Delta W + k^2 W = 0, \tag{1}$$

where W is the vertical displacement and k is the frequency normalized by $L\sqrt{\text{density/tension per length}}$. L is a characteristic length defined by $\sqrt{\text{area}/\pi}$. The elliptic co-ordinates, (ξ, η) are related to the Cartesian co-ordinates (x, y) by

$$x = c \cosh \xi \cos \eta, \qquad y = c \sinh \xi \sin \eta, \tag{2}$$

0022-460X/01/020363 + 06 \$35.00/0

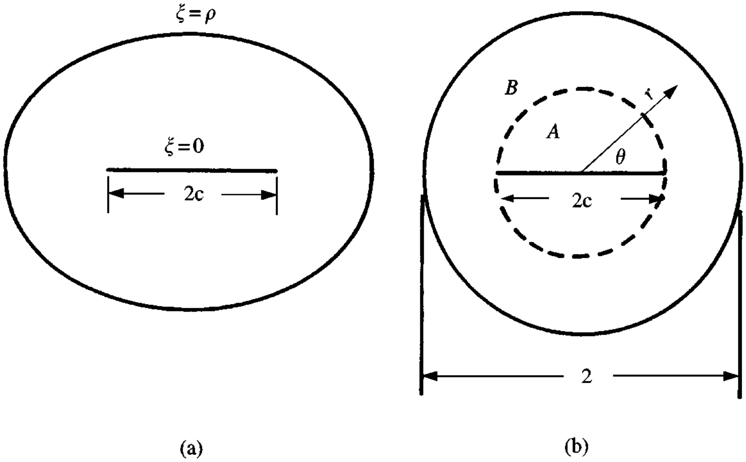

Figure 1. (a) Elliptic membrane with a centered strip, (b) circular membrane with a centered strip.

where $2c$ is the distance between the foci. Equation (1) can be separated by $W = \Psi(\xi)\Phi(\eta)$ resulting in Mathieu equations [4, 5]

$$\frac{d^2\Psi}{d\xi^2} + [h^2\cosh^2\xi - b]\,\Psi = 0, \qquad \frac{d^2\Phi}{d\eta^2} + [b - h^2\cosh^2\eta]\,\Phi = 0, \tag{3, 4}$$

where $h = kc$ and b is a separation constant. For the fundamental frequency, Ψ is the even radial Mathieu function of zeroth order while the solution of Ψ, with the boundary conditions $\Psi(0) = 0$, $\Psi(\rho) = 0$, gives the characteristic equation

$$N_0(h, 0)M_0(h, \rho) - M_0(h, 0)N_0(h, \rho) = 0, \tag{5}$$

where M_0 and N_0 are the even zeroth order radial Mathieu functions of the first and second kind, respectively, related to Bessel functions J_n, Y_n by

$$M_0(h, \xi) = \frac{\sqrt{\pi/2}}{B_0}\sum_{n=0}^{\infty}(-1)^n B_{2n} J_n\left(\frac{h}{2}e^{\xi}\right)J_n\left(\frac{h}{2}e^{-\xi}\right), \tag{6}$$

$$N_0(h, \xi) = \frac{\sqrt{\pi/2}}{B_0}\sum_{n=0}^{\infty}(-1)^n B_{2n} Y_n\left(\frac{h}{2}e^{\xi}\right)J_n\left(\frac{h}{2}e^{-\xi}\right), \tag{7}$$

and the B_n are coefficients depending on h and b.

Since lengths are normalized by L, the family of ellipses has the same area as the circle of radius L, thus $c = (\cosh\rho\sinh\rho)^{-1/2}$. We shall investigate the asymptotic properties of k as $c \to 0$ and $\rho \to \infty$. Since c is small, we expand

$$k = k_0 + k_1\delta_1(c) + k_2\delta_2(c) + o(\delta_2(c)), \tag{8}$$

where $\delta_n(c)$ is an asymptotic sequence to be determined. Then

$$\frac{h}{2}e^{\rho} = \frac{h}{2}\left(\frac{2}{c} + \frac{c^3}{16} + \cdots\right), \qquad B_0 = 1 + \frac{h^2}{8} + \frac{7h^4}{514} + \cdots, \tag{9, 10}$$

$$B_2 = -\frac{h^2}{8} - \frac{h^2}{64} + \cdots, \qquad B_4 = \frac{h^4}{512} + \cdots, \tag{11, 12}$$

where $h = c(k_0 + k_1\delta_1(c) + k_2\delta_2(c) + o(\delta_2(c)))$. Equation (5) becomes

$$\left\{J_0(k_0) - J_1(k_0)k_1\delta_1(c) - J_1(k_0)k_2\delta_2(c) + \frac{1}{4}(J_2(k_0) - J_0(k_0))k_1^2\delta_1^2(c) + \cdots\right\}$$

$$\times(\ln c + \ln k_0 - \ln 4 + \gamma + \cdots)$$

$$-\left\{Y_0(k_0) - Y_1(k_0)k_1\delta_1(c) - Y_1(k_0)k_2\delta_2(c) + \frac{1}{4}(Y_2(k_0) - Y_0(k_0))k_1^2\delta_1^2(c) + \cdots\right\}$$

$$\times\left(\frac{\pi}{2}\right)\left(1 + \frac{3k_0^4}{256}c^4 + \cdots\right) = 0, \tag{13}$$

where $\gamma \approx 0{\cdot}5772$. Comparing the leading orders gives

$$J_0(k_0) = 0. \tag{14}$$

The first root, $k_0 = 2{\cdot}4048$, is the fundamental frequency of the circular membrane. The next orders yield

$$k_1 = \frac{\pi Y_0(k_0)}{2J_1(k_0)} = 1{\cdot}5429, \qquad \delta_1(c) = \frac{1}{|\ln c|}, \tag{15, 16}$$

$$k_2 = \frac{-(\pi/2)k_1Y_1(k_0) + (k_1^2J_2(k_0)/4) + J_1(k_0)k_1(\ln k_0 - \ln 4 + \gamma)}{J_1(k_0)} = 0{\cdot}1208, \tag{17}$$

$$\delta_2(c) = \frac{1}{|\ln c|^2}. \tag{18}$$

We see that the fundamental frequency, for small c, is a quadratic in $|\ln c|^{-1}$. The property would be reflected for a circular membrane with a short centered strip.

3. CIRCULAR MEMBRANE WITH A CENTERED STRIP

Consider the circular membrance with an interior line constraint of length $2c$ (Figure 1(b)). Since an exact formula for the characteristic equation does not exist, the frequency will be found numerically by eigenfunction expansions and matching.

Decompose the membrane into two regions; for region A ($r \leqslant c$) the general solution to equation (1) which is even in θ and satisfies $W = 0$ on the strip is

$$W_A(r, \theta) = \sum_{n=1}^{\infty} (2n)!\, A_n J_{2n-1}(kr)\cos[(2n-1)\theta]. \tag{19}$$

Here, A_n are coefficients to be determined. The factor $(2n)!$ is to ensure that A_n would not be too large. The general solution for region B ($r \geqslant c$) which is even in θ and satisfies $W = 0$ on $r = 1$ is

$$W_B(r, \theta) = \sum_{n=1}^{\infty} C_n H_n(r)\cos(2n\theta), \tag{20}$$

where

$$H_n(r) \equiv \mathrm{J}_{2n}(k)\mathrm{Y}_{2n}(kr) - \mathrm{Y}_{2n}(k)\mathrm{J}_{2n}(kr). \tag{21}$$

Now W_A and W_B are continuous on $r = c$:

$$W_A(c, \theta) = W_B(c, \theta), \qquad \frac{\partial W_A}{\partial r}(c, \theta) = \frac{\partial W_B}{\partial r}(c, \theta). \tag{22, 23}$$

Truncate A_n and C_n to $N + 1$ terms. Multiplying equation (22) by $\cos(2m\theta)$ and integrating from 0 to $\pi/2$ give

$$\sum_{n=1}^{N+1} \frac{(-1)^{n+1}}{2n-1}\,(2n)!\,A_n \mathrm{J}_{2n-1}(kc) = \frac{\pi}{2} C_0 H_0(c), \tag{24}$$

$$\sum_{n=1}^{N+1} \left\{\frac{(-1)^{n+m+1}}{(n-\frac{1}{2}+m)} + \frac{(-1)^{n-m+1}}{(n-\frac{1}{2}-m)}\right\}(2n)!\,A_n \mathrm{J}_{2n-1}(kc) = \pi C_m H_m(c), \quad m = 1, 2, \ldots, N. \tag{25}$$

Similarly, equation (23) gives

$$\sum_{n=1}^{N+1} \frac{(-1)^{n+1}}{2n-1}\,(2n)!\,A_n \mathrm{J}'_{2n-1}(kc) = \frac{\pi}{2} C_0 H'_0(c), \tag{26}$$

$$\sum_{n=1}^{N+1} \left\{\frac{(-1)^{n+m+1}}{(n-\frac{1}{2}+m)} + \frac{(-1)^{n-m+1}}{(n-\frac{1}{2}-m)}\right\}(2n)!\,A_n \mathrm{J}'_{2n-1}(kc) = \pi C_m H'_m(c), \quad m = 1, 2, \ldots, N. \tag{27}$$

Equations (24)–(27) represent $2N + 2$ homogeneous equations and unknowns. For the non-trivial solution, the determinant of the coefficients is set to zero. This gives the characteristic equation which is solved for the minimum value of k. Accuracy is improved by increasing N. Table 1 shows that the convergence occurs when N is about 35.

The characteristic equation is in closed form when $c = 0$ or 1. For $c = 0$, the geometry is the circle and the fundamental frequency is the first root of $\mathrm{J}_0(k) = 0$, or $k = 2{\cdot}4048$. For $c = 1$, the geometry is the semi-circle and the fundamental frequency is the first root of $\mathrm{J}_1(k) = 0$, or $k = 3{\cdot}8317$. For $0 < c < 1$, the method described above is used. Table 2 shows the result for all values of strip lengths c.

TABLE 1

Convergence of k

N	$c = 0{\cdot}1$	$c = 0{\cdot}3$	$c = 0{\cdot}5$	$c = 0{\cdot}7$	$c = 0{\cdot}9$
5	3·052	3·478	3·740	3·821	3·831
10	3·057	3·489	3·748	3·823	3·831
15	3·058	3·493	3·751	3·824	3·831
20	3·059	3·495	3·752	3·824	3·831
25	3·060	3·496	3·753	3·824	3·831
30	3·061	3·497	3·754	3·824	3·831
35	3·061	3·498	3·754	3·824	3·831
40	3·061	3·498	3·754	3·824	3·831

TABLE 2

Fundamental frequency for circular membrane with line constraint

c	0	0·001	0·01	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
k	2·4048	2·629	2·741	3·061	3·297	3·498	3·655	3·754	3·804	3·824	3·830	3·831	3·8317

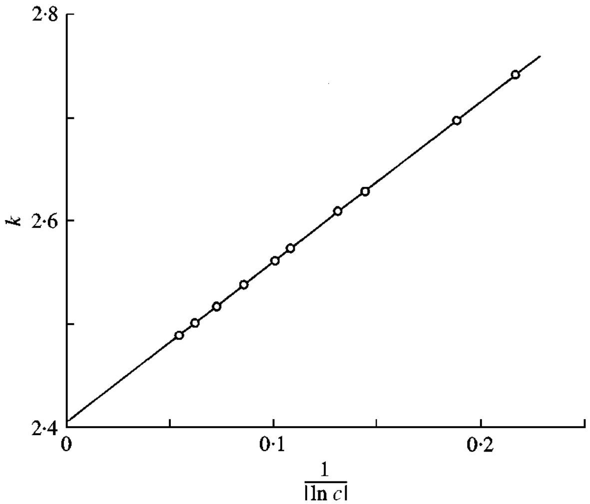

Figure 2. Fundamental frequency of a circular membrane with a centered strip for small c. Circles are computed values. Line is the asymptotic formula from equation (29).

Of interest is the behavior for small c. Guided by our analysis for the elliptic membrane, we propose a similar asymptotic formula

$$k = k_0 + \frac{k_1}{|\ln c|} + \frac{k_2}{|\ln c|^2} + \cdots, \quad c \to 0, \tag{28}$$

The value of k_0 is the basic frequency of the circular membrane as shown from the limit for the elliptic membrane as $c \to 0$. The other coefficients k_1 and k_2 may be different and we used a least-squares fit on our numerical results for the range $c = 10^{-2}$ to $c = 10^{-6}$ (numerical instability occurs for $c < 10^{-6}$). Figure 2 shows the frequency as a function of $|\ln c|^{-1}$ and the curve fit

$$k = 2{\cdot}4048 + \frac{1{\cdot}55}{|\ln c|} - \frac{0{\cdot}012}{|\ln c|^2} + \cdots, \quad c \to 0. \tag{29}$$

Equation (29) describes the rapid rise of fundamental frequency as c is increased from zero. Note the differences in the coefficients k_1 and k_2 as compared to the elliptic membrane case.

4. DISCUSSIONS

A membrane with an internal strip has similarities and differences in comparison to that of an internal circular core. In both cases, for an infinitesimal constraint dimension, the fundamental frequency is the same as that without the constraint, and the increase is proportional to $|\ln c|^{-1}$ which is singular. For large constraint dimensions c, the frequency behavior is quite different. The curvature of $k(c)$ is negative for the line constraint as k approaches a constant value (3·8317) when $c \to 1$ while the curvature is positive for the circular core constraint when $c \to 1$. In fact, if the membrane is circular with a circular core, the frequency approaches infinity as $c \to 1$. This is because the membrane area of the line constraint does not change while the circular core decreases the membrane area by the square of the core radius.

REFERENCES

1. C. Y. WANG 1998 *Journal of Sound and Vibration* **215**, 195–199. On the polygonal membrane with a circular core.
2. L. GRUNER 1967 *IEEE Transactions on Microwave Theory and Techniques* **MTT-15**, 483–485. Higher order modes in rectangular coaxial lines.
3. G. I. VESELOV and V. I. GAYDAR 1970 *Radio Engineering* **25**, 147–149. Analysis of a circular waveguide with an internal cross-shaped conductor.
4. T. ROZZI, L. PIERANTONI and M. RONZITTI 1997 *IEEE Transactions on Microwave Theory and Techniques* **MTT-45**, 1778–1784. Analysis of the suspended strip in elliptical cross section by separation of variables.
5. P. M. MORSE and H. FESHBACH 1953 *Methods of Theoretical Physics*. New York: McGraw-Hill, chapter 6.