Journal of Sound and Vibration (2001) **239**(2), 369–378
doi:10.1006/jsvi.2000.3062, available online at http://www.idealibrary.com on IDEAL®

APPROXIMATE RESULTS OF ACOUSTIC IMPEDANCE FOR A COSINE-SHAPED HORN

J. P. Arenas and M. J. Crocker

Department of Mechanical Engineering, Auburn University, AL 36849-5341, *U.S.A.*

(*Received* 15 *October* 1999, *and in final form* 10 *April* 2000)

1. INTRODUCTION

Horns are waveguides that have a varying cross-sectional area. If the horn cross-section expands the wave amplitude is reduced while if the tube becomes narrower an increase in the density of the energy flow occurs. Rayleigh and Webster were the first to derive independently a plane-wave approximation for the acoustic propagation in circular-section ducts of varying section. Putland [1] has given some details about the conditions under which the Webster equation is approximately correct and has shown that every single parameter acoustic field obeys this equation. Salmon [2] used the Webster theory to obtain some famous families of horns. In addition, Weibel [3] used Hamilton's variational principle to obtain the wave equation for horns. This equation can be solved analytically for some shapes [4] such as exponential, conical, parabolic, catenoidal [5] and sinusoidal [6], but in many cases no analytical expression can be found. Moreover, in a horn with a finite length, a reflection from its open end occurs which causes a reflected wave to propagate in the negative direction of the axis.

Hence, the resultant acoustic field in the horn consists of the sum of both solutions of the Webster equation. Mawardi [7] has presented two approaches to solve the Webster equation, which consider an electrical analogue and the singularities of the differential equation. One technique which has been employed to solve this problem involves the partition of the horn into a number of conical-shaped elements, which together coincide approximately with the walls of the horn. In this approach, the solution for the acoustic pressure is a summation of the products of Legendre polynomials and spherical Hankel functions [8]. Another approach consists of using stepped cylindrical elements which give the axial pressure in terms of Bessel functions. Cummings [9] has transformed the problem into matrix form and has used the fourth order Runge–Kutta integration method of solution to study the phenomenon of flow-induced acoustic oscillations in a wine bottle used as a resonating device. Kergomard [10] showed that the continued fraction solution of the Ricatti equation leads to equivalent circuits for acoustical horns of arbitrary shape. Holland *et al.* [11] developed a one-parameter finite-element-type model using exponential elements and compared their results with a numerical solution of a system of differential equations. In a later paper, Holland and Morfey [12] corrected the linearly predicted sound field for non-linear distortion at the end of each exponential element and concluded that the model can predict the degree of non-linear waveform distortion associated with the propagation of large-amplitude waves in loudspeaker horns. Any attempt to calculate the input acoustic impedance of a horn implies a knowledge of the radiation impedance at the mouth. A study of the output impedance of various exponential and catenoidal horns

0022-460X/01/020369 + 10 $35.00/0

has been presented by Fleisher [13] who gives comparisons between experimental results and theoretical values for a piston in plane and spherical baffles. However, a complete description of the inaccuracies introduced by using the impedance of a piston in an infinite baffle is not clear from the literature. For the case of musical horns, the importance of the radiation impedance in determining the input impedance has been investigated by Amir *et al.* [14]. They suggested that a horn function equivalent to a barrier potential, as used in the Schroedinger equation, contains most of the information needed to compute the input impedance. Schuhmacher and Rasmussen [15] have presented a wave-guide-oriented numerical simulation model for horn loudspeakers used for out-door sound reinforcement systems. This model is able to predict the output from horns with square openings under general conditions but the results depend on the number of modes included in the simulation. The directivity, main-axis frequency response and overall efficiency are now of interest as horn design parameters. Finally, modern numerical techniques including the boundary element method (BEM) [16] and the finite element method (FEM) [17] can also be used to design and predict the behavior of horns.

2. DESCRIPTION OF THE COSINE-SHAPED HORN

A cosine-shaped horn is of interest. Most of the typical horn profiles have no null first derivative at the mouth. If we choose a proper cosine-shaped horn it is possible to obtain a profile that has a zero slope at both the throat and the mouth. Webster assumed that the dimensions of the cross-section of the horn are small compared with a wavelength. With this assumption it can be assumed that the acoustic pressure and velocity are independent of the radius r over the whole cross-section at some distance x at any given instant. The Webster equation for a harmonic steady state solution is

$$\frac{\mathrm{d}^2P}{\mathrm{d}x^2} + \frac{1}{S(x)}\frac{\mathrm{d}S}{\mathrm{d}x}\frac{\mathrm{d}P}{\mathrm{d}x} + k^2P(x) = 0, \tag{1}$$

where $P(x)$ is the acoustic amplitude, $S(x)$ is the cross-sectional area of the horn and k is the free-field wave number (ω/c). If the section is circular, the function $S(x) = \pi r^2(x)$, where $r(x)$ is the radial distance from the longitudinal axis of the horn to the profile boundary at some point x.

In order to study the behavior of a cosine-shaped horn we have to find the proper function for $S(x)$, that is to say, $r(x)$.

The most general case can be described using $r(x) = r_0 + A(1 - \cos x/h)$, where r_0 is the radius at $x = 0$ and h is defined as l/π where l is the length of the horn. The geometry of the problem is shown in Figure 1.

We notice that the definition of the non-dimensional parameter $\alpha = A/(r_0 + A)$ implies that

$$r(x) = \frac{A}{\alpha}(1 - \alpha\cos x/h). \tag{2}$$

Then,

$$\frac{\mathrm{d}S}{\mathrm{d}x} = 2\pi r\frac{\mathrm{d}r}{\mathrm{d}x} = 2\pi r\frac{A}{h}\sin x/h. \tag{3}$$

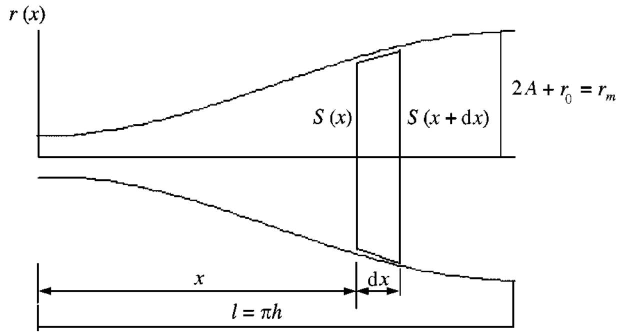

Figure 1. Finite cosine-shaped horn.

Now, substituting for dS/dx in the Webster equation, for steady state conditions we obtain

$$\frac{d^2P}{dx^2} + \frac{2\alpha}{h}\frac{\sin x/h}{(1-\alpha\cos x/h)}\frac{dP}{dx} + k^2P = 0. \tag{4}$$

This is the differential equation to be solved.

3. WKB APPROXIMATION

The WKB method is commonly used in quantum mechanics to solve some problems involving atomic wave functions. The WKB method, named after Wentzel, Kramers, and Brillouin, is a powerful tool for obtaining global approximations for the solution of linear differential equations. In summary, it states that when, in the equation $\Psi'' + K^2\Psi = 0$, the coefficient K does not change by a large fraction over a wavelength, then the WKB approximation yields a complex exponential solution. In other words, the WKB approach is valid whenever K^2 does not change too rapidly with axial position. This is satisfied with most practical horns and by all horns at sufficiently high frequency [18]. Using the transformation

$$P(x) = \frac{f(x)}{(1-\alpha\cos x/h)}, \tag{5}$$

and substituting for dP/dx in equation (4) and after some algebra, we obtain

$$\frac{d^2f}{dx^2} + \left(k^2 - \frac{\alpha}{h^2}\frac{\cos x/h}{(1-\alpha\cos x/h)}\right)f = 0. \tag{6}$$

Equation (6) has no analytic solution, therefore, we must use some approximate method of solution and remember the resulting restrictions. Before using the WKB method we notice that from the theory for differential equations we know that $f'' + Q(x)f = 0$ requires $Q(x)$ to be positive for all values of x in order to find harmonic solutions. This requires the restriction of plane wave propagation. Taking into account that the parameter $\alpha \in [0, 1)$, a value for the cut-off frequency limit f_c for the WKB approximation is obtained from one of the bounds of the function $Q(x)$:

$$f_c = \frac{c}{2\pi h}\sqrt{\frac{A}{r_0}}. \tag{7}$$

We note that below this frequency f_c, the WKB approximation for impedance will be zero.

Thus, using the WKB method, the solution of equation (6) can be approximated by

$$f(x) \sim C_1 K^{-1/4} \exp\left\{-\mathrm{j}\int_0^x \sqrt{K(s)}\,\mathrm{d}s\right\} + C_2 K^{-1/4} \exp\left\{\mathrm{j}\int_0^x \sqrt{K(s)}\,\mathrm{d}s\right\}, \tag{8}$$

where $K(x) = k^2 - (\alpha/h^2)(\cos x/h/(1 - \alpha\cos x/h))$, and C_1 and C_2 are the complex amplitudes of the forward and backward travelling waves [19].

4. ACOUSTIC IMPEDANCE

An estimation of the acoustic impedance presented by a horn to a velocity source placed at the throat is of great importance for evaluating the performance of the horn.

The power output for a given source velocity input is proportional to the real part of this impedance. Hence, sometimes the ratio of the real part of the impedance and the characteristic impedance of the medium is termed as the transmission coefficient of the horn. In the frequency domain, this information gives an indication of the frequency response, efficiency and low-frequency limit of the horn. On the other hand, the imaginary part of the impedance shows the reactive load presented to the source by the horn. An estimate of the acoustic impedance can be obtained using the sound pressure from the WKB approximation and the particle velocity from the Euler equation. Thus, the complex acoustic impedance Z_s for the cosine horn is approximated by

$$Z_s(x) \sim \mathrm{j}\omega\rho g(x)\,\frac{C_1 \mathrm{e}^{-\mathrm{j}\Psi(x)} + C_2 \mathrm{e}^{\mathrm{j}\Psi(x)}}{C_1 \eta(x)\mathrm{e}^{-\mathrm{j}\Psi(x)} + C_2 \eta^* \mathrm{e}^{\mathrm{j}\Psi(x)}}, \tag{9}$$

where

$$g(x) = 1 - \alpha\cos x/h, \tag{10}$$

$$\eta(x) = \frac{\alpha}{h}\sin x/h + g(x)\left(\frac{1}{4K(x)} + \mathrm{j}\sqrt{K(x)}\right), \tag{11}$$

$$\Psi(x) = \int_0^x \sqrt{K(s)}\,\mathrm{d}s, \tag{12}$$

and ω is the circular frequency, ρ is the density of the medium and $*$ indicates complex conjugation. Now, let Z_1 be the impedance at $x = 0$ (throat) and Z_2 be the impedance at the mouth ($x = l$). Then, after some algebra the combination of both boundary conditions gives

$$Z_1(\omega) \sim \mathrm{j}\omega\rho(1 - \alpha)\,\frac{\mathbf{P} - \mathbf{Q}\mathrm{e}^{-\mathrm{j}2\Psi(l)}}{\eta(0)\mathbf{P} - \eta^*(0)\mathbf{Q}\mathrm{e}^{-\mathrm{j}2\Psi(l)}}, \tag{13}$$

where $\mathbf{P} = Z_2\eta^*(l) - \mathrm{j}\omega\rho g(l)$ and $\mathbf{Q} = Z_2\eta(l) - \mathrm{j}\omega\rho g(l)$.

Equation (13) gives a closed-form solution for calculating the approximate impedance at the throat of the cosine horn in terms of the impedance at the mouth Z_2. The final result for the function $\Psi(x)$ can be expressed in terms of a long and complicated result involving elliptic integrals. However, for the purpose of this work, a numerical evaluation of $\Psi(x)$ will be used.

It is common to use the impedance of a piston in an infinite baffle as the impedance at the mouth of the horn, Z_2. Pierce [20] gives this as

$$Z_{piston} = \rho c\left(1 - \frac{2\mathrm{J}_1(2kr_m)}{2kr_m} + \mathrm{j}\,\frac{2Su(2kr_m)}{2kr_m}\right), \tag{14}$$

where r_m is the radius of the mouth, $\mathrm{J}_1(x)$ is the Bessel function of the first order and $\mathrm{Su}(x)$ is the Rayleigh–Struve function of the first order.

In addition, a horn connector can be used to join two pipes of different cross-sectional areas. In this case, the connector acts as a simple discontinuity when its length is short compared with a wavelength and as an acoustic impedance transformer when its length is greater than half the wavelength. A transformer can be used as in electrical circuits to change the acoustic impedances from one value to another without appreciable reflection, for sound reproduction purposes. On the other hand, an appropriately designed connector can produce a certain amount of reflection, which could be useful for noise reduction. In fact, when the horn is used as a connector it transforms acoustical fluctuations with large sound pressures and small volume velocities to those with small sound pressures and large volume velocities and *vice versa*.

If we assume that the second pipe is of infinite extent and ignoring any losses, a plane wave front undergoes no change in cross-sectional area as it propagates, and the normalized acoustic impedance at any point along or across the pipe is purely resistive and equal to unity. Then, this boundary condition can be used instead of the plane piston flush mounted in an infinite baffle in this second horn application.

5. EQUIVALENT SYSTEM OF NON-LINEAR DIFFERENTIAL EQUATIONS FOR THE IMPEDANCE

Another way to calculate the acoustic impedance of a horn begins with the definition of acoustic impedance an the linearized momentum equation for zero mean flow. Substituting the definitions in Webster equation (4) it can be shown that [11]

$$\frac{\mathrm{d}Z}{\mathrm{d}x} = \mathrm{j}k(Z^2 - 1) + F(x)Z, \tag{15}$$

where $F(x) = (2\alpha/h)(\sin x/h/(1 - \alpha\cos x/h))$. That is to say, we can transform the original problem of a linear differential equation for the sound pressure to a new non-linear differential equation for acoustic impedance. To separate the active and reactive parts for the impedance, we denote $Z(x) = \mathscr{X}(x) + \mathrm{j}\mathscr{Y}(x)$, then

$$\frac{\mathrm{d}\mathscr{X}}{\mathrm{d}x} = -\,2k\mathscr{X}\mathscr{Y} + F(x)\mathscr{X}, \qquad \frac{\mathrm{d}\mathscr{Y}}{\mathrm{d}x} = k(\mathscr{X}^2 - \mathscr{Y}^2 - 1) + F(x)\mathscr{Y}, \tag{16}$$

will be the new equations to be solved. Therefore, if we know the boundary condition for the impedance at $x = l$, we can develop a numerical backward integration of the system of non-linear differential equations in order to obtain the value of $\mathscr{X}$ and $\mathscr{Y}$ at $x = 0$. This method has the advantage of using boundary conditions for impedance. This approach is more appropriate for horns, instead of using the boundary conditions given for sound pressure or particle velocity. This latter approach is more useful in resonating cavities problems. Equation (15) can be considered as a general expression for the impedance for a horn with any shape profile, characterized by the function $F(x)$. This function can be expressed in terms of the variable section of the horn as $F(x) = (1/S(x))(\partial S/\partial x)$. This function, for some practical applications of real horns, can be defined or measured. On the other hand, there are several numerical techniques for solving equation (15), which depend on the degree of accuracy required, the computational run-time and the ratio between the step-size and the frequency. Certainly, solving equation (15) numerically, will take a long computer run-time for an acceptable degree of accuracy, particularly at high frequencies where the step-size in the numerical algorithm must be reduced.

6. RESULTS

An example of a cosine-shaped horn was proposed in order to evaluate its throat impedance. The length of the horn was kept constant at $l = 1{\cdot}6256$ m (64 in) and the throat and mouth dimensions were selected to be 0·08 and 0·508 m respectively. The parameters used for the cosine-shaped horn were calculated for use in equation (2).

The resulting expression is $r(x) = 0{\cdot}147[1 - 0{\cdot}7278\ \cos(0{\cdot}615\pi x)]$. The value for the critical frequency obtained from equation (7) is 171 Hz, which limits the validity of the approximate impedance given by equation (13). In the evaluation of the Rayleigh–Struve function its integral representation in terms of the Weber function was used [21].

Figure 2 shows the result for the acoustic impedance for the cosine horn, obtained from a numerical solution of equation (16) compared with the solution obtained using equation (13). The numerical evaluation for the system of differential equations was obtained using a fourth order Runge–Kutta algorithm with a variable step-size. We observe that the horn exhibits resonance characteristics due to the large change in acoustic impedance waves "see" in passing from the mouth to the free atmosphere, which introduces reflections at the mouth, and as a result large variations in the acoustic impedance characteristics with frequency. It is observed that good agreement between both methods can be achieved for frequencies above the second resonance. It is not possible to obtain a WKB solution for frequencies below f_c. It can be observed in Figure 2 that the cosine-shaped horn has its first resonance frequency at about 86 Hz, and the second one very close to f_c. It is noted, that as expected, for very high frequencies the normalized reactance goes to zero and the normalized resistance tends asymptotically to a limiting value of unity. The horn possesses a high value of resistance at its second resonance and this value is approximately 3·5 times greater than the first resonance. This difference is less marked at higher frequencies. This condition can be an undesirable one in sound reproduction but can be useful for other applications. In addition, analysis of equation (16) and from similar numerical results show that shortening the cosine-shaped horn while keeping the throat and mouth dimension fixed, results in a decrease in the amplitude of the resistance at first resonance but that the frequency of this resonance is increased.

In addition, the use of the horn as connector used to join two pipes of different sections was studied (in this case with a ratio $S_2/S_1 = 40{\cdot}3$, were S_2 and S_1 are the areas of the mouth and throat respectively). Assuming that the second pipe is of infinite extent, the results for the impedance at the throat were calculated for the cosine-shaped geometry. Figure 3 shows the results using both methods of solution. The results show good agreement between the

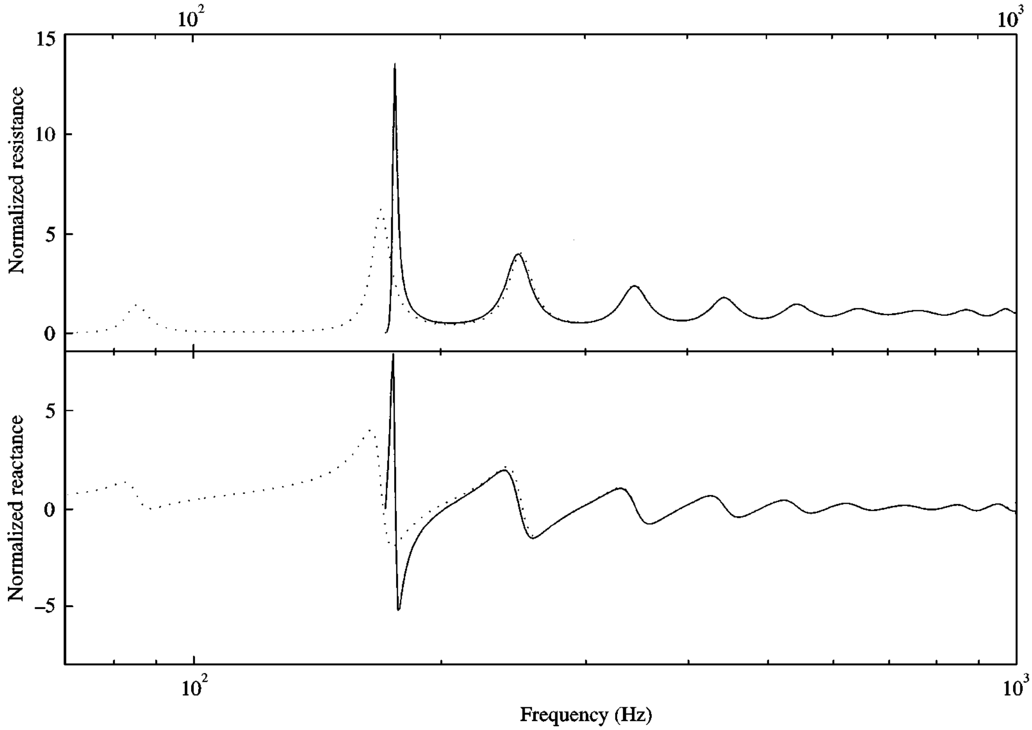

Figure 2. Normalized throat resistance and reactance of a cosine-shaped horn 1·6256 m long with throat and mouth dimensions 8 and 50·8 cm respectively: ——, approximate solution (WKB); · · · · ·, numerical solution of the system of differential equations (SDE).

numerical solution and the WKB approximation for frequencies above the second resonance. The cosine connector has a maximum above 200 Hz and it does not exhibit more significant resonance peaks at higher frequencies. Additional observations can be made for the cosine connector by analyzing equation (13) with a mouth impedance of unity. A reduction in length of the connector produces displacement of the acoustic impedance curve towards high frequency with only a very small change of shape of the curve. This observation has been tested with additional numerical experiments. In addition, it can be observed that the cosine-shaped connector possesses some negative values of normalized reactance.

Finally, the results for the normalized sound pressure level along the axis of the cosine-shaped horn using the boundary condition given by equation (14) were calculated from the WKB approximation. They are presented in Figure 4, for the third, fourth and fifth resonance. Figure 4(b) displays the same results but normalized instead by $\sqrt{S(x)}$. That is to say, the normalization of the sound pressure is performed by multiplying it by the square root of the cross-section. This is important to determine whether the reduction in sound pressure is due to the spreading of the acoustic energy over progressively larger wave fronts or due to the resonance produced by the boundary condition at the mouth. Here, the maximum and minimum of the sound pressure can be seen. The number of minima found between the throat ($x = 0$) and the mouth ($x = l$) gives the order of the resonance. Similar results can be found in the literature for exponential horns (see reference [14]).

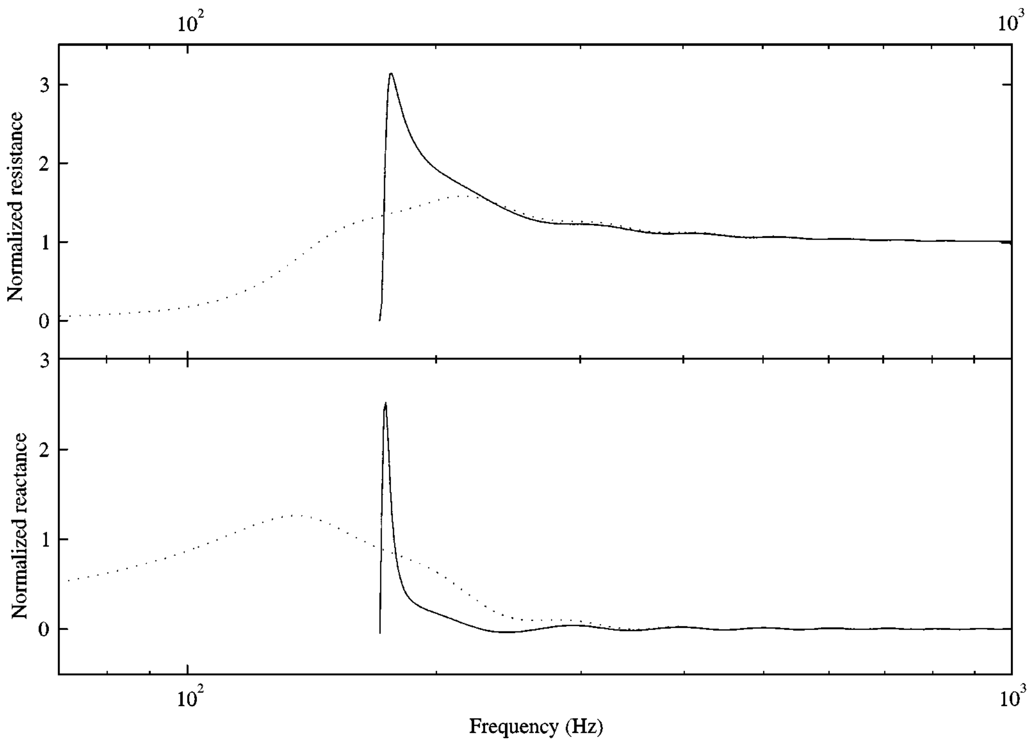

Figure 3. Normalized throat resistance and reactance for a cosine-shaped connector used to joint two pipes of different cross-sectional areas: ——, approximate solution (WKB); ·····, numerical solution of the system of differential equations (SDE).

7. CONCLUSIONS

A WKB method to calculate the approximate value of the impedance of a cosine-shaped horn has been presented. In addition, a comparison with a numerical solution of a system of non-linear differential equations for the acoustic impedance has been presented. Formulae for calculating the cut-off frequency for the approximation have been given in consideration of the differential equation involved. This particular cosine-shaped horn can feature some advantage when it is used as a connector, since the slopes at the ends are zero and with proper design it can avoid turbulence noise when used in a flow duct. In addition, the assumption of a baffled piston radiation impedance at the mouth is obviously more valid for this particular shape. On the other hand, this sort of geometry is commonly found in some acoustical applications such as resonators and some musical instruments. Therefore, it is important to find some harmonic solution for the differential equation. It is clear that the cosine-shaped horn is effectively a high-pass filter and for most designs the ratio between its first and second resonance frequencies is higher than with the most well-known horns. The use of the WKB approximation produces two advantages: (1) it can be used together with a numerical solution at higher frequencies, where the numerical algorithms require a long calculation time and (2) this approximation permits the calculation of the sound pressure distribution along the axis of the horn, so that it can be used as a design tool. It is also noticed that the values of the reactance and resistance at high frequency are the same as those of an infinite tube with the same throat diameter. Thus, it can act as a sound absorber. Certainly, the behavior of the cosine-shaped horn should be studied further using

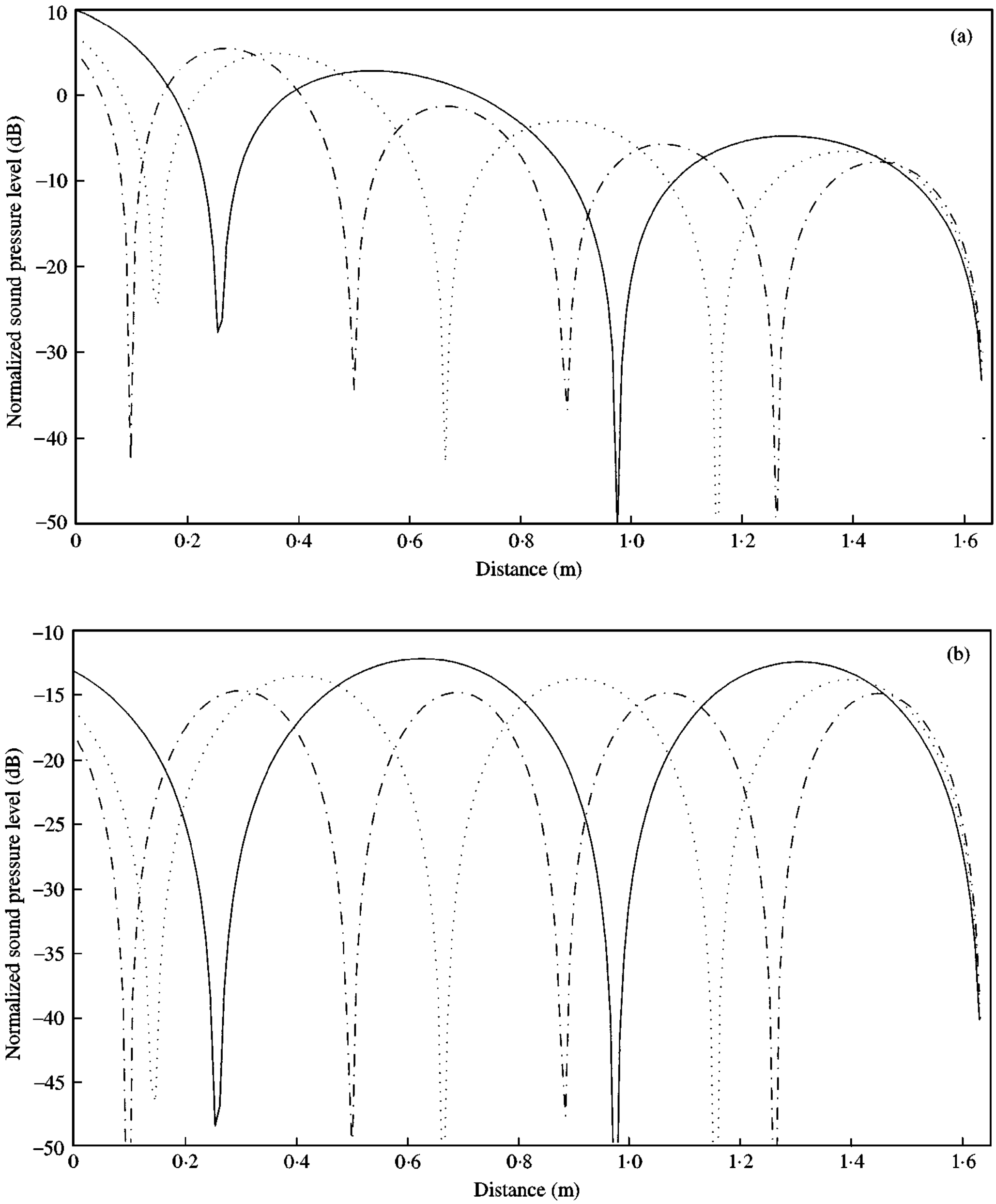

Figure 4. Results for (a) normalized sound pressure level along the axis for the cosine-shaped horn of Figure 2, and (b) sound pressure level along the axis normalized by $\sqrt{S(x)}$: ——, 248·5 Hz; ·····, 344·5 Hz; – · –, 443·0 Hz.

experimental techniques in order to completely characterize this kind of horn. Finally, it is observed that the WKB approximation can be utilized in this particular case, because the variable coefficient K in the linear differential equation does not change too rapidly with axial position. Thus, the WKB approximation should not be used for short connectors or those with high ratios of mouth to throat area.

REFERENCES

1. G. R. PUTLAND 1993 *Journal of the Audio Engineering Society* **41**, 435–450. Every one-parameter acoustic field obeys Webster's horn equation.
2. V. SALMON 1946 *Journal of the Acoustical Society of America* **17**, 212–218. A new family of horns.
3. E. S. WEIBEL 1955 *Journal of the Acoustical Society of America* **27**, 726–727. On Webster's horn equation.
4. H. F. OLSON 1940 *Elements of Acoustical Engineering*. Princeton, NJ: Van Nostrand.
5. G. J. THIESSEN 1950 *Journal of the Acoustical Society of America* **22**, 558–562. Resonance characteristics of a finite catenoidal horn.
6. B. N. NAGARKAR and R. FINCH 1971 *Journal of the Acoustical Society of America* **50**, 23–31. Sinusoidal horns.
7. O. K. MAWARDI 1949 *Journal of the Acoustical Society of America* **21**, 323–330. Generalized solutions of Webster's horn theory.
8. R. J. ALFREDSON 1972 *Journal of Sound and Vibration* **23**, 433–442. The propagation of sound in a circular duct of continuously varying cross-sectional area.
9. A. CUMMINGS 1973 *Journal of Sound and Vibration* **31**, 331–343. Acoustics of a wine bottle.
10. J. KERGOMARD 1988 *Journal of the Audio Engineering Society* **36**, 948–954. General equivalent electric circuits for acoustic horns.
11. K. R. HOLLAND, F. FAHY and C. MORFEY 1991 *Journal of the Audio Engineering Society* **39**, 315–336. Prediction and measurement of the one-parameter behavior of horns.
12. K. R. HOLLAND and C. MORFEY 1996 *Journal of the Audio Engineering Society* **44**, 569–580. A model of nonlinear wave propagation in horns.
13. H. FLEISHER 1995 *Acustica* **81**, 96–106. Über die ausgangsimpedanz von hörnern.
14. N. AMIR, G. ROSENHOUSE and U. SHIMONY 1993 *Applied Acoustics* **38**, 15–35. Input impedance of musical horns and the 'horn function'.
15. A. SCHUHMACHER and K. B. RASMUSSEN 1999 *Applied Acoustics* **56**, 25–37. Modelling of horn-type loudspeakers for outdoor sound reinforcement systems.
16. D. J. HENWOOD 1993 *Journal of the Audio Engineering Society* **41**, 485–496. The boundary-element method and horn design.
17. S. MORITA, N. KYONO, S. SAKAI, T. TAMABUSHI and Y. KAGAWA 1980 *Journal of the Audio Engineering Society* **28**, 482–488. Acoustic radiation of a horn loudspeaker by the finite element method — a consideration of the acoustic characteristic of horns.
18. V. SALMON 1946 *Journal of the Acoustical Society of America* **17**, 199–211. Generalized plane wave horn theory.
19. D. G. CRIGHTON, A. P. DOWLING, J. E. FFOWCS WILLIAMS, M. HECKL and F. G. LEPPINGTON 1992 *Modern Methods in Analytical Acoustics*. London: Springer-Verlag (see p. 219).
20. A. D. PIERCE 1989 *Acoustics, and Introduction to its Physical Principles and Applications*. New York: Acoustical Society of America (see p. 220).
21. M. ABRAMOWITZ and I. STEGUN (editors) 1965 *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. New York: Dover.