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A NOTE ON THE VIBRO-ACOUSTIC RESPONSE OF A PERIODICALLY SUPPORTED BEAM SUBJECTED TO A TRAVELLING, TIME-HARMONIC LOADING

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The effects of periodic spring supports on the acoustic response of an infinite, fluid-loaded beam subjected to a travelling, time-harmonic loading are investigated. To account for the numerous wavenumber components caused by the elastic supports; the wavenumber-harmonic series proposed in previous study (C. C. Cheng *et al.* 2000 *Journal of Vibration and Acoustics ASME*) is used to represent the transverse response. Results show that for a periodically, elastically supported beam subjected to a nearly stationary force, an acoustic radiation peak occurs when the excitation frequency coincides with one of the bounding frequencies of the propagation zones. Furthermore, each enhanced acoustic radiation will split into two peaks due to the Doppler shift when the harmonic force becomes travelling and a formula based upon equations of propagation constant and the Doppler effect is derived to help determine the wavenumber ratios corresponding to these peaks.

1. INTRODUCTION

Flexural wave motion in periodically supported structures has been studied extensively in the past. Motion of the whole structure was usually deduced from the analysis of a single substructure instead of maintaining the whole structure and the wave-propagating constant is used to describe the flexural wave relation between two adjacent substructures [1]. The wave propagating constant method pioneered by Heckel [1], followed by Gupta [2], Mace [3, 4], Mead *et al.* [5–13], Morse [14], Yuan [15] and others have proved to be efficient in analyzing such structures. However, the calculation of the complex propagation constants for a periodic structure becomes very difficult when the fluid-loading effect is taken into account in the formulation.

Instead of using the wave propagation method to reduce the analysis to a substructure, Cheng and Chui [16] formulated the vibration response on the whole structure in wavenumber domain through Fourier transform. For an air-loaded, periodically simply supported beam subjected to a stationary line force, they showed that the radiated sound power exhibited peaks at certain wavenumber ratios. The wavenumber ratios at which the radiation peaks occur nearly coincide with the lower bounding wavenumber ratios of the odd number of propagation zones. However, Cheng's formulation did not include the presence of numerous wavenumber components induced from the elastic supports and is subjected to the restriction that the external force is located on one of the elastic supports.

To account for the influences of numerous wavenumber components caused by the elastic supports on the vibration responses, Cheng *et al.* [17] presented an alternative approach

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named as wavenumber-harmonic method in which a wavenumber-harmonic series was proposed to represent the transverse response of a periodically supported beam. Results showed that the number of acoustic radiation peaks increases, compared with a periodically, elastically supported beam subjected to a stationary force, when the external loading is travelling. Unfortunately, in their studies no formula could precisely determine the wavenumber ratios at which the acoustic radiation peaks occur when the supports are elastic and the external loading is travelling.

This paper may be considered as an extension of the previous study [17] in which the wavenumber-harmonic method is presented. We focus on three topics that remain elusive in the previous research. The first is to find the discrepancy in responses between an infinite beam on an elastic foundation and an infinite beam with periodic, elastic supports. The second is to determine the wavenumber ratios at which the acoustic radiation peaks occur when the periodic supports are elastic and the external loading is nearly stationary. The third is to derive a formula to determine the wavenumber ratios corresponding to these peaks when the external loading becomes travelling.

2. WAVENUMBER-HARMONIC ANALYSIS

As a first step, a general formulation for the vibro-acoustic response of a periodically spring-supported, fluid-loaded Timoshenko beam subjected to a travelling, time-harmonic loading is presented. Details of the derivation using wavenumber harmonic series can be found in reference [17]. However, some of them will be repeated here in order to introduce the notation that will be used as well as to clarify ideas.

Assume a periodically spring-supported beam of height h and of infinite extent lying in the plane $y = 0$. A stationary acoustic fluid of density ρ_0 and sound speed C_0 occupies the half space ($y > 0$) and there is a vacuum under the beam ($y < 0$). The beam is excited by a travelling, time-harmonic loading with subsonic speed V and frequency ω as shown in Figure 1. The equation of motion for a Timoshenko beam is given by Junger and Feit [18] as

$$\bar{E}I\frac{\partial^4 u(x,t)}{\partial x^4}+\rho_v h\frac{\partial^2 u(x,t)}{\partial t^2}-\left(\rho_v I+\frac{\bar{E}I\rho_v}{\kappa^2\bar{G}}\right)\frac{\partial^4 u(x,t)}{\partial x^2\partial t^2}+\rho_v I\frac{\rho_v}{\bar{G}\kappa^2}\frac{\partial^4 u(x,t)}{\partial t^4}$$
$$=\left(1-\frac{\bar{E}I}{\bar{G}\kappa^2 h}\frac{\partial^2}{\partial x^2}+\frac{\rho_v h^2}{12\bar{G}\kappa^2}\frac{\partial^2}{\partial t^2}\right)[f(x,t)-p(x,y=0,t)-p_1(x,t)], \tag{1}$$

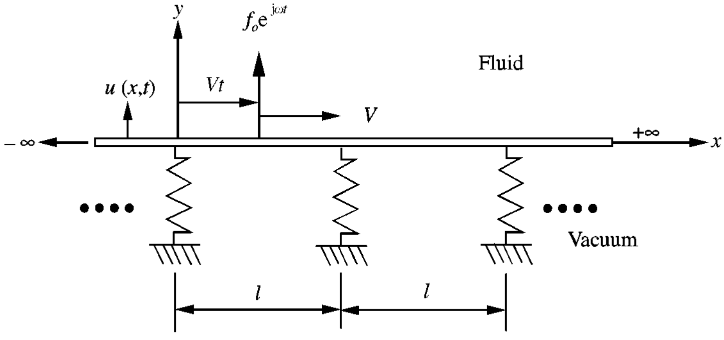

Figure 1. Schematic representation of problem geometry.

where $u(x, t)$ represents the transverse displacement of the beam, I the cross-sectional second moment of area per unit width, ρ_v the density of the beam, κ^2 the cross-sectional shape factor, $f(x, t)$ the external moving force, $p(x, y = 0, t)$ the acoustic pressure acting on the beam surface, $p_1(x, t)$ the force from the spring supports, $\bar{E}(= E(1 + j\eta))$ and $\bar{G}(= \bar{E}/2(1 + v))$ are the complex elastic and shear modulus, respectively, η the structural damping and v the Poisson ratio. By applying the spatial Fourier transformation to equation (1), the transformed equation is written as

$$\left(\bar{E}I\xi^4 + \rho_v h \frac{\partial^2}{\partial t^2} - \left(\rho_v I + \frac{\bar{E}I\rho_v}{\kappa^2 \bar{G}}\right)\frac{\partial^2}{\partial t^2} + I\frac{\rho_v^2}{\bar{G}\kappa^2}\frac{\partial^4}{\partial t^4}\right)\tilde{U}(\xi, t)$$
$$= \left(1 - \frac{\bar{E}I}{\bar{G}\kappa^2 h}\xi^2 + \frac{\rho_v h^2}{12\bar{G}\kappa^2}\frac{\partial^2}{\partial t^2}\right)[\tilde{F}(\xi, t) - \tilde{P}(\xi, y = 0, t) - \tilde{P}_1(\xi, t)], \tag{2}$$

where ξ is the wavenumber variable. The dimensionless wavenumber response of the infinite, periodically supported, fluid-loaded beam is given by [17]

$$\bar{U}(\zeta, t) = \sum_{n=-\infty}^{\infty} \bar{u}_n(\zeta)\exp^{(j[(\zeta + \frac{2n\pi}{(k_0 h)(l/h)})M + 1]\omega t)}. \tag{3}$$

where $k_0(= \omega/C_0)$ is the acoustic wavenumber, $\zeta(= \xi/k_0)$ the dimensionless wavenumber variable, l the spacing between two adjacent supports and $M(= V/C_0)$ the Mach number of the travelling loading. The external moving force $\tilde{F}(\zeta, t)$, the sound pressure $\tilde{P}(\zeta, y, t)$ and the reactions $\tilde{P}_1(\zeta, t)$ from the elastic supports in the wavenumber domain are expressed as follows:

$$\tilde{F}(\zeta, t) = \bar{f}_0(\zeta)e^{j(\zeta M + 1)\omega t}, \tag{4}$$

$$\tilde{P}(\zeta, y = 0, t) = \sum_{n=-\infty}^{\infty} j\frac{(k_0 h)(C_0/C_L)^2(\rho_0/\rho_v)H_n(\zeta)}{\sqrt{H_n(\zeta) - \zeta^2}}\bar{u}_n(\zeta)\exp^{(j[(\zeta + \frac{2n\pi}{(k_0 h)(l/h)})M + 1]\omega t)}, \tag{5}$$

$$\tilde{P}_1(\zeta, t) = \frac{k_s}{l}\sum_{n=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} \bar{u}_{n-r}\left[\zeta + \frac{2r\pi}{(k_0 h)(l/h)}\right]\exp^{(j[(\zeta + \frac{2n\pi}{(k_0 h)(l/h)})M + 1]\omega t)}, \tag{6}$$

where $\bar{f}_0(\zeta)(= f(\xi)/f_0)$ is the dimensionless strength of external force, f_0 the strength of the external force per unit width, ρ_0 the acoustic medium density, k_s the stiffness of the spring support per unit width, and

$$C_L = \sqrt{\frac{E}{\rho_v}}, \tag{7}$$

$$\bar{u}_n(\zeta) = \frac{E}{f_0 h}u_n(\xi), \tag{8}$$

$$H_n(\zeta) = \left\{\left[\zeta + \frac{2n\pi}{(k_0 h)(l/h)}\right]M + 1\right\}^2. \tag{9}$$

Substituting equations (4–6) into the transformed equation (2), then the resulting equation is simplified to two equations as follows:

$$\bar{u}_0(\zeta) = \bar{X}_0(\zeta)\bar{f}_0(\zeta) - \bar{C}\bar{X}_0(\zeta)\sum_{r=-\infty}^{\infty}\bar{u}_{-r}\left[\zeta + \frac{2r\pi}{(k_0 h)(l/h)}\right], \tag{10}$$

$$\bar{u}_p\left(\zeta - \frac{2p\pi}{(k_0 h)(l/h)}\right) = -\bar{C}\bar{X}_p\left(\zeta - \frac{2p\pi}{(k_0 h)(l/h)}\right)\sum_{r=-\infty}^{\infty}\bar{u}_{-r}\left(\zeta + \frac{2r\pi}{(k_0 h)(l/h)}\right), \quad p \neq 0, \tag{11}$$

where

$$\bar{f}_0(\zeta) = \frac{f(\xi)}{f_0}, \tag{12}$$

$$\bar{X}_n(\zeta) = \frac{\bar{B}_n(\zeta)}{\bar{A}_n(\zeta) + \bar{B}_n(\zeta)\bar{D}_n(\zeta)}, \tag{13}$$

$$\bar{A}_n(\zeta) = \frac{A_n(\xi)}{E/h} = \left(\frac{1+j\eta}{12}\right)(k_0h)^4\zeta^4 - H_n(\zeta)\left(\frac{C_0}{C_L}k_0h\right)^2\left[1 + \frac{(k_0h)^2}{12}\left(1 + \frac{2(1+v)}{\kappa^2}\right)\zeta^2\right]$$

$$+ \frac{(1+v)[(C_0/C_L)(k_0h)]^4}{6\kappa^2(1+j\eta)} H_n(\zeta)^2, \tag{14}$$

$$\bar{B}_n(\zeta) = 1 + \left(\frac{1+v}{6\kappa^2}\right)[(k_0h)\cdot\zeta]^2 - \left[\frac{1+v}{6\kappa^2(1+j\eta)}\right]\left[\left(\frac{C_0}{C_L}\right)(k_0h)\right]^2 H_n(\zeta), \tag{15}$$

$$\bar{C} = \frac{k_s/E}{l/h}, \tag{16}$$

$$\bar{D}_n(\zeta) = \frac{D_n(\xi)}{E/h} = j\,\frac{(k_0h)(C_0/C_L)^2(\rho_0/\rho_v)H_n(\zeta)}{\sqrt{H_n(\zeta) - \zeta^2}}. \tag{17}$$

Define elastic support stiffness ratio as

$$S = \frac{k_s}{E}. \tag{18}$$

Equations (10) and (11) can be rearranged into a set of linear algebraic equations as

$$\begin{Bmatrix} \vdots \\ \bar{u}_{-p}\left(\zeta + \frac{2p\pi}{(k_0h)(l/h)}\right) \\ \vdots \\ \vdots \\ \bar{u}_0(\zeta) \\ \vdots \\ \vdots \\ \bar{u}_p\left(\zeta - \frac{2p\pi}{(k_0h)(l/h)}\right) \\ \vdots \end{Bmatrix} = \left[\begin{bmatrix} \vdots & \vdots & \vdots \\ \cdots & \bar{C}\bar{X}_{-p}\left(\zeta + \frac{2p\pi}{(k_0h)(l/h)}\right) & \cdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \cdots & \bar{C}\bar{X}_0(\zeta) & \cdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \cdots & \bar{C}\bar{X}_p\left(\zeta - \frac{2p\pi}{(k_0h)(l/h)}\right) & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix} + [I]\right]^{-1} \begin{Bmatrix} \vdots \\ 0 \\ \vdots \\ 0 \\ \bar{X}_0(\zeta)\bar{f}_0(\zeta) \\ 0 \\ \vdots \\ 0 \\ \vdots \end{Bmatrix}. \tag{19}$$

The unknown solutions of expansion coefficients $\bar{u}_n(\zeta)$ can be obtained by using a Gaussian elimination algorithm after this set of equations is truncated to a set with a finite number of equations. Through integrating the surface acoustic intensity over the entire beam, the dimensionless radiated sound power W, which is the sum of the sound power contributed from each wavenumber component is given by

$$W = (k_0 h)^3 \sum_{n=-\infty}^{\infty} \int_{\zeta_1}^{\zeta_2} H_n(\zeta)^{3/2} |\bar{u}_n(\zeta)|^2 \,\mathrm{Re}\left[\frac{1}{\sqrt{H_n(\zeta) - \zeta^2}}\right] \mathrm{d}\zeta. \tag{20}$$

The limits of integration are given as follows:

$$\zeta_1 = \frac{-\left[\dfrac{2n\pi}{(k_0 h)(l/h)}\right]M - 1}{1+M} \leqslant \zeta \leqslant \frac{\left[\dfrac{2n\pi}{(k_0 h)(l/h)}\right]M + 1}{1-M} = \zeta_2, \quad \text{for}$$

$$\frac{\left[\dfrac{2n\pi}{(k_0 h)(l/h)}\right]M + 1}{1-M} \geqslant \frac{-\left[\dfrac{2n\pi}{(k_0 h)(l/h)}\right]M - 1}{1+M} \tag{21}$$

or

$$\zeta_1 = \frac{\left[\dfrac{2n\pi}{(k_0 h)(l/h)}\right]M + 1}{1-M} \leqslant \zeta \leqslant \frac{-\left[\dfrac{2n\pi}{(k_0 h)(l/h)}\right]M - 1}{1+M} = \zeta_2, \quad \text{for}$$

$$\frac{\left[\dfrac{2n\pi}{(k_0 h)(l/h)}\right]M + 1}{1-M} \leqslant \frac{-\left[\dfrac{2n\pi}{(k_0 h)(l/h)}\right]M - 1}{1+M}. \tag{22}$$

3. NUMERICAL RESULTS AND DISCUSSION

The specific model analyzed consisted of an infinite steel beam (elastic modulus $E = 2 \times 10^{11}$ N/m^2, density $\rho_v = 7800$ kg/m^3, height $h = 2{\cdot}54 \times 10^{-2}$ m, Poisson's ratio $v = 0{\cdot}3$, cross-sectional shape factor $\kappa^2 = 0{\cdot}85$, structural damping $\eta = 0{\cdot}01$) submerged in air ($C_0 = 343$ m/s, $\rho_0 = 1{\cdot}24$ kg/m^3). Two different stiffnesses of the elastic supports were examined, $S = 10^{-4}$ and 10^{-2}, besides the unsupported beams as baseline data. The support spacing l/h is chosen to vary from 1 to 10^5. Results were calculated at three different Mach numbers of a travelling, time-harmonic point force, $M = 10^{-4}$, 0·25 and 0·4 respectively.

Shown in Figure 2 are the wavenumber responses at frequency $k_0 h = 0{\cdot}05$ for a periodically supported, air-loaded beam with the support stiffness ratio $S = 10^{-2}$ and support spacing $l/h = 10$. The amplitude in logarithmic scale is a summation of the root-mean-square of the responses of wavenumber components from $\bar{u}_{-15}(\zeta)$ to $\bar{u}_{15}(\zeta)$:

$$\text{Amplitude} = \log_{10}\left(\sum_{n=-15}^{15} \frac{\bar{u}_n^2(\zeta)}{2}\right)^{1/2}. \tag{23}$$

Note that the numerical solution that requires the infinite set of equations as shown in equation (3) to be truncated has been discussed in the previous work [17]. For $M = 10^{-4}$,

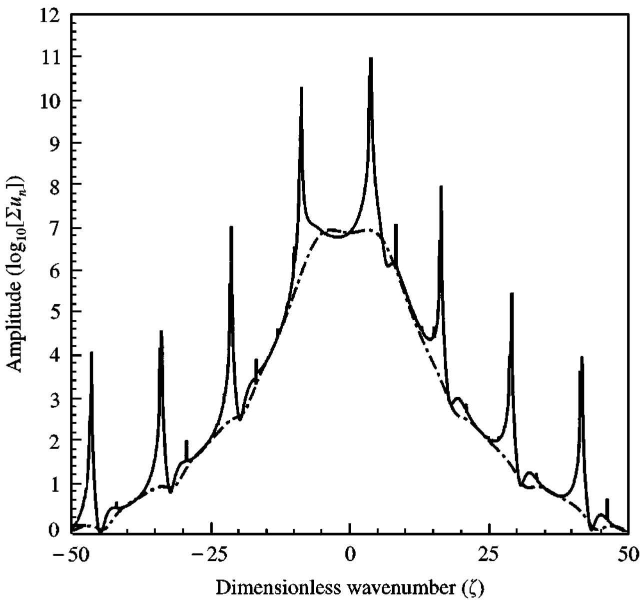

Figure 2. Wavenumber response for an air-loaded, periodically supported beam for a range of M values, $s = 10^{-2}$, $l/h = 10$, $k_0h = 0{\cdot}05$. -----, $M = 10^{-4}$, ——, $M = 0{\cdot}4$.

notice that there is no peak at low moving speed, which indicates that the beam exhibits static deflection. However, when $M = 0{\cdot}4$, it shows that the wavenumber response consists of many peaks; and the spacing, $\Delta\xi$, between two adjacent peaks is found to be

$$\Delta\xi = \frac{2\pi}{l}. \tag{24}$$

Rewrite equation (24) in dimensionless form as

$$\Delta\xi = \frac{2\pi}{k_0h(l/h)}. \tag{25}$$

It shows that the difference of two adjacent dominating wavenumber components induced by the periodic supports is $2\pi/l$.

Figure 3 shows the wavenumber response for a periodically spring-supported, air-loaded beam for $S = 10^{-2}$, $k_0h = 0{\cdot}05$, $M = 0{\cdot}4$ and three different support spacings, $l/h = 1$, 10^2, and 10^5 respectively. For a periodically supported beam with such large support spacing $l/h = 10^5$, the spring supports have almost no effect on the response, hence, the model behaves as an unsupported beam and there appear only two pronounced peaks. However, for support spacing $l/h = 10^2$, the support spacing and support stiffness play an important role in determining the peak values of the propagating waves and the corresponding wavenumbers. From the curve of $l/h = 10^2$, the wavenumber response consists of many peaks caused by the elastic supports and the spacing between two adjacent peaks can be determined by equation (25). If the support spacing is reduced to $l/h = 1$, no dominating peak appears. It implies that the beam exhibits static deflection and in fact the response is similar to that of a beam on an elastic foundation. Naturally, a fundamental issue that needs to be addressed is what is the difference between a periodically, elastically supported beam

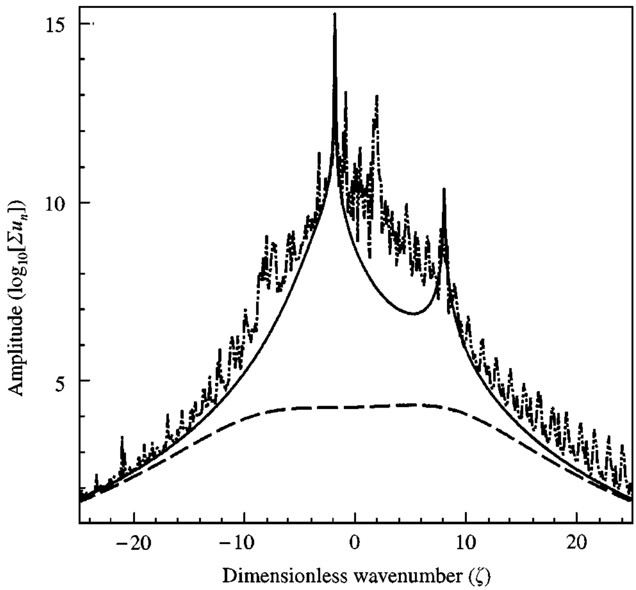

Figure 3. Wavenumber response for an air-loaded, periodically supported beam for a range of l/h values, $s = 10^{-2}$, $k_0 h = 0{\cdot}05$, $M = 0{\cdot}4$. - - -, $l/h = 1$; - - - - - - -, $l/h = 10^2$; ——, $l/h = 10^5$.

with a small support spacing and a beam on an elastic foundation at low frequency. For a Timoshenko beam on an elastic foundation, the equation of motion has the same form as equation (1) except that the force $p_1(x, t)$ from the elastic support is replaced by that from the elastic foundation:

$$P_e(x, t) = k_e u(x, t), \tag{26}$$

where k_e is the stiffness per unit area of the elastic foundation. The displacement $\tilde{U}(\xi, t)$, sound pressure $\tilde{P}(\xi, y, t)$ and the force $\tilde{P}_e(\xi, t)$ from the elastic foundation in a wavenumber domain can be rewritten as follows:

$$\tilde{U}(\xi, t) = U(\xi)\mathrm{e}^{\mathrm{j}(\xi V + \omega)t}, \tag{27}$$

$$\tilde{P}_e(\xi, t) = k_e U(\xi)\mathrm{e}^{\mathrm{j}(\xi V + \omega)t}, \tag{28}$$

$$\tilde{P}(\xi, y = 0, t) = \frac{\mathrm{j}\rho_0(\xi V + \omega)^2}{K_y} U(\xi)\mathrm{e}^{\mathrm{j}(\xi V + \omega)t}, \tag{29}$$

where

$$K_y = \begin{cases} -\mathrm{j}\sqrt{\xi^2 - (\xi M + k_0)^2} & \text{for } \xi^2 > (\xi M + k_0)^2, \\ \sqrt{(\xi M + k_0)^2 - \xi^2} & \text{for } \xi^2 < (\xi M + k_0)^2. \end{cases} \tag{30}$$

Substituting equations (27–29) into equation (2), the latter is expressed as

$$\begin{aligned} A(\xi)U(\xi)\mathrm{e}^{\mathrm{j}(\xi V + \omega)t} = {} & B(\xi)f(\xi)\mathrm{e}^{\mathrm{j}(\xi V + \omega)t} - B(\xi)D(\xi)U(\xi)\mathrm{e}^{\mathrm{j}(\xi V + \omega)t} \\ & - k_e B(\xi)U(\xi)\mathrm{e}^{\mathrm{j}(\xi V + \omega)t}, \end{aligned} \tag{31}$$

where

$$A(\xi) = \bar{E}I\xi^4 - \rho_v A(\xi V + \omega)^2 - \rho_v I\left(1 + \frac{\bar{E}}{\bar{G}k^2}\right)\xi^2(\xi V + \omega)^2 + \frac{\rho_v^2 I}{\bar{G}k^2}(\xi V + \omega)^4, \quad (32)$$

$$B(\xi) = 1 + \frac{\bar{E}I}{\bar{G}k^2 A}\xi^2 - \frac{\rho_v I}{\bar{G}k^2 A}(\xi V + \omega)^2, \quad (33)$$

$$D(\xi) = \frac{\mathrm{j}\rho_0(\xi V + \omega)^2}{K_y}. \quad (34)$$

The displacement simplifies to

$$U(\xi) = \frac{B(\xi)f(\xi)}{A(\xi) + B(\xi)D(\xi) + k_e B(\xi)}. \quad (35)$$

Equation (35) is rewritten in dimensionless form as

$$U(\zeta) = \frac{B(\zeta)\bar{f}_0(\zeta)}{A(\zeta) + B(\zeta)D(\zeta) + k_e hB(\zeta)/E}, \quad (36)$$

where

$$A(\zeta) = \frac{A(\xi)}{E/h} = \left(\frac{1 + \mathrm{j}\eta}{12}\right)(k_0h)^4\zeta^4 - H(\zeta)\left(\frac{C_0}{C_L}k_0h\right)^2\left[1 + \frac{(k_0h)^2}{12}\left(1 + \frac{2(1+v)}{\kappa^2}\right)\zeta^2\right]$$

$$+ \frac{(1+v)[(C_0/C_L)(k_0h)]^4}{6\kappa^2(1 + \mathrm{j}\eta)}H(\zeta)^2, \quad (37)$$

$$B(\zeta) = 1 + \left(\frac{1+v}{6\kappa^2}\right)[(k_0h)\cdot\zeta]^2 - \left[\frac{1+v}{6\kappa^2(1+\mathrm{j}\eta)}\right]\left[\left(\frac{C_0}{C_L}\right)(k_0h)\right]^2 H(\zeta), \quad (38)$$

$$D(\zeta) = \frac{D(\xi)}{E/h} = j\frac{(k_0h)(C_0/C_L)^2(\rho_0/\rho_v)H(\zeta)}{\sqrt{H(\zeta) - \zeta^2}}, \quad (39)$$

$$H(\zeta) = (\zeta M + 1)^2, \quad (40)$$

$$C_L = \sqrt{\frac{E}{\rho_v}}. \quad (41)$$

For a small value of l/h and at low frequency, the summation in equation (10) can be approximated by the term at $r = 0$ and the influences of the spring support on the beam vibration are dominated from supports close to the excitation. Hence, equation (10) can be simplified as

$$\bar{u}_0(\zeta) = \frac{\bar{X}_0(\zeta)\bar{f}_0(\zeta)}{1 + \bar{C}\bar{X}_0(\zeta)}. \quad (42)$$

Substituting equations (13) and (16) into equation (42), the latter is rewritten as

$$\bar{u}_0(\zeta) = \frac{\bar{B}_0(\zeta)\bar{f}_0(\zeta)}{\bar{A}_0(\zeta) + \bar{B}_0(\zeta)\bar{D}_0(\zeta) + k_s h\bar{B}_0(\zeta)/EI}. \tag{43}$$

Comparing equation (43) with equation (36), one can find that the response of a beam on an elastic foundation is equivalent to that of a periodically, elastically supported beam with the following simple relation as expected:

$$k_s = k_e l. \tag{44}$$

However, for the relation to hold would require that the support spacing is much smaller than the flexural wavelength for the periodically, elastically supported beam. Although this result is so obvious and is logically predictable, the intention is to show that the wavenumber-harmonic method can also be used to study the vibration behavior of a beam on a continuous foundation.

Figure 4 illustrates the sound power radiated from the specific model with support stiffness $S = 10^{-4}$, support spacing $l/h = 10$ and at the Mach numbers $M = 10^{-4}$ and 0·25 respectively. The sound power radiated from an unsupported beam at $M = 10^{-4}$ and 0·25 is also plotted for the purpose of comparison. Define wavenumber ratio as

$$\gamma = \frac{k_0}{k_B}, \tag{45}$$

where k_B is the free bending wavenumber

$$k_B = \left[\frac{12\rho_v\omega^2}{Eh^2}\right]^{1/4}. \tag{46}$$

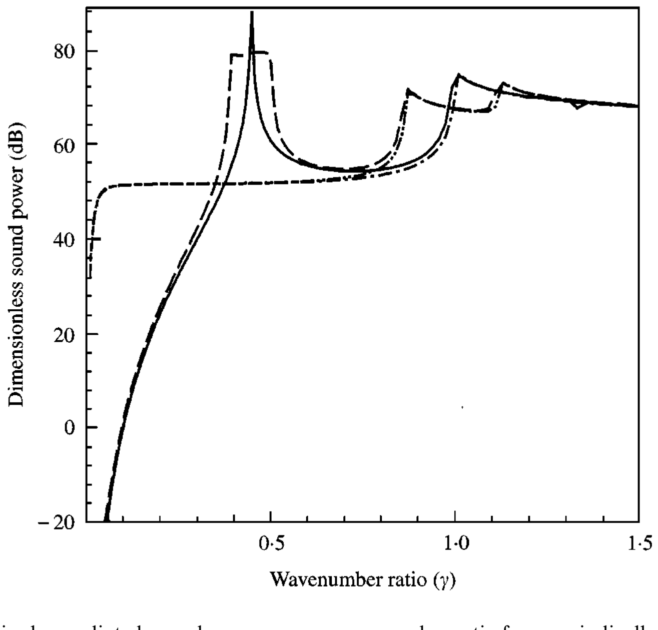

Figure 4. Dimensionless radiated sound power versus wavenumber ratio for a periodically supported beam for a range of M values, $s = 10^{-4}$, $l/h = 10$, in air.——, $M = 10^{-4}$; – – –, $M = 0{\cdot}25$; – – – – –, $M = 10^{-4}$ (Unsupported); – – – – – – –, $M = 0{\cdot}25$ (Unsupported).

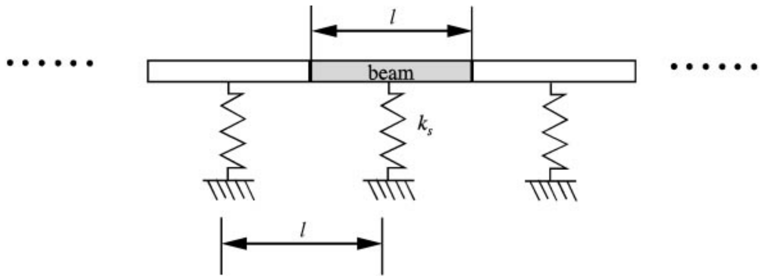

Figure 5. A spring mass model of the spring-supported system.

Notice that the strong unique coincidence radiation peak located at $\gamma = 1$ for the unsupported beam subjected to a nearly stationary force $M = 10^{-4}$, is split into two coincidence peaks when the force is travelling with $M = 0{\cdot}25$. According to Keltie [19], one peak now is located at the wavenumber ratio $\gamma = \sqrt{1 - M}$, and the other at $\gamma = \sqrt{1 + M}$. For the periodic spring-supported beam subjected to a nearly stationary force at $M = 10^{-4}$, it is of interest to find that the radiated sound power shows another pronounced peak at the wavenumber ratio $\gamma = 0{\cdot}45$. The appearance of this special radiation peak could be explained using a simple lumped mass model that represents a single section of the periodically supported beam as shown in Figure 5.

The natural frequency of this model is simply given by

$$\omega = \sqrt{\frac{k_s}{\rho_v A l}}. \tag{47}$$

The wavenumber ratio corresponding to equation (47) is

$$\gamma = \left[\frac{k_s E h}{12 \rho_v^2 C_0^4 l}\right]^{1/4}. \tag{48}$$

According to equation (48), the wavenumber ratio is calculated to be 0·446 and agrees with the location of the observed peak in Figure 4. At this specific wavenumber ratio, all the substructures move almost in phase, and therefore generate a radiation peak.

For the periodic supported beam subjected to a moving load with speed $M = 0{\cdot}25$, notice that the radiation peak at $\gamma = 0{\cdot}446$ becomes flat and the associated wavenumber ratios bandwidth increases. The wavenumber ratios corresponding to the upper and lower ends of this bandwidth can be obtained using the Doppler formula

$$\omega_{1,2} = \frac{\omega}{1 \pm M}. \tag{49}$$

Substituting equation (49) into equation (45), the corresponding wavenumber ratios can be written as

$$\gamma_{1,2} = \frac{\gamma}{\sqrt{1 \pm M}}, \tag{50}$$

where γ_1 and γ_2 are the upper and lower ends of the wavenumber ratio bandwidth respectively. For the periodically supported beam at $M = 0{\cdot}25$, the wavenumber ratios, γ_1

and γ_2, are calculated to be 0·399 and 0·515, respectively, which agree with the ends of the bandwidth in Figure 4.

For a periodically spring-supported beam subjected to a stationary force, the bounding frequencies corresponding to each propagating zone can be calculated using the equation of propagation constant given by Mead [9]:

$$\cosh^2\mu + \cosh\mu\left[\frac{G}{\sqrt{\Omega^3}}(\sinh\sqrt{\Omega} - \sin\sqrt{\Omega}) - (\cosh\sqrt{\Omega} + \cos\sqrt{\Omega})\right] + \left[\cosh\sqrt{\Omega}\cos\sqrt{\Omega} + \frac{G}{\sqrt{\Omega^3}}(\cosh\sqrt{\Omega}\sin\sqrt{\Omega} - \sinh\sqrt{\Omega}\cos\sqrt{\Omega}.)\right] = 0, \quad (51)$$

where μ is the complex propagation constant, Ω is the non-dimensional frequency and G is the parameter which describes influences of the support stiffness and support spacing

$$\Omega = (k_B l)^2 = \sqrt{\frac{\omega^2\rho_v A l^4}{EI}}, \quad (52)$$

$$G = 3\,\frac{k_s}{E}\left(\frac{l}{h}\right)^3. \quad (53)$$

The relation between the wavenumber ratio γ and the non-dimensional frequency Ω is given by

$$\gamma = \frac{\sqrt{\Omega}}{(l/h)\sqrt{12\rho_v C_0^2/E}}. \quad (54)$$

Note that one can substitute equations (47) and (52) into equation (54), then obtain the same result as equation (48). When the support spacing is increased from $l/h = 10$ to 100, the radiated sound power is showed in Figure 6. The number of radiation peaks is greater than that shown in Figure 4, especially in the region of low wavenumber ratios. However, for $M = 10^{-4}$, the wavenumber ratio corresponding to each radiation peak can be determined by equations (51) and (54). Table 1 lists the lower and upper bounding non-dimensional frequencies and the corresponding wavenumber ratios of the first five propagation zones of a periodically supported beam with the support stiffness $S = 10^{-4}$ and spacing $l/h = 100$. Each bounding wavenumber ratio agrees with that corresponding to each radiation peak in Figure 6 except for the first lower bounding wavenumber ratio $\gamma = 0{\cdot}133$ due to hydrodynamic cancellation. For the same specific model subjected to a travelling, time-harmonic loading at speed $M = 0{\cdot}25$, it is worth noting that the number of radiation peak increases due to the Doppler shift. Although it may not be apparent in Figure 6, the wavenumber ratio corresponding to each radiation peak can be determined approximately using equation (50) and is listed in Table 2.

4. CONCLUSION

The advantage of expressing the response in terms of a wavenumber harmonic series arises from the fact that the periodic boundary conditions and the phase relation between two adjacent substructures will not be used. Furthermore, the fluid-loading effect is easily

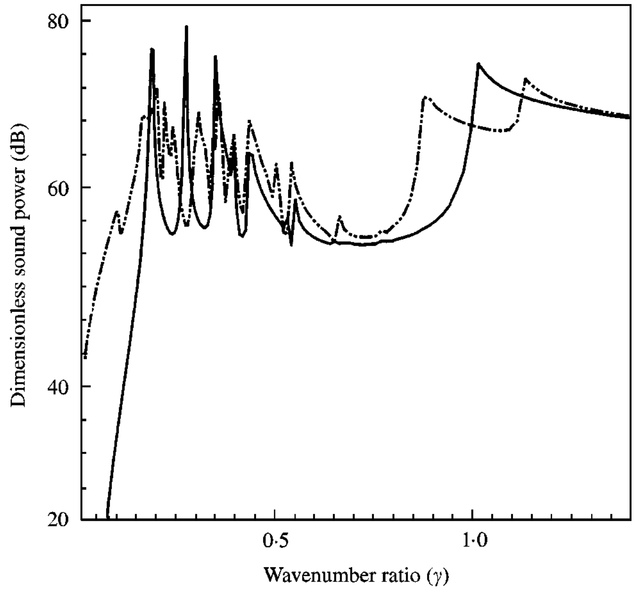

Figure 6. Dimensionless radiated sound power versus wavenumber ratio for a periodically supported beam for a range of M values, $s = 10^{-4}$, $l/h = 100$, in air. ——, $M = 10^{-4}$; -------, $M = 0{\cdot}25$.

TABLE 1

Bounding frequencies for the first five propagation zones of a periodically spring-supported beam, $S = 10^{-4}$, $l/h = 100$, *in air*

Propagation zone no.	Non-dimensional frequency Ω	Wavenumber ratio γ
1	9·8	0·133
	19·5	0·188
2	39·4	0·268
	65·8	0·346
3	88·9	0·402
	103·6	0·434
4	158·0	0·536
	166·1	0·549
5	246·8	0·670
	251·8	0·676

taken into account in the formulation. Results show that the response of a beam on an elastic foundation can be approximated using a periodically, elastically supported beam when the support spacing is small compared with the flexural wavelength. For a periodically, elastically supported beam subjected to a nearly stationary force, an acoustic radiation peak occurs when the excitation frequency coincides with one of the bounding frequencies of the propagating zones. Each enhanced acoustic radiation will split into two peaks due to the Doppler shift when the force becomes travelling. A formula based upon equations of propagation constant and the Doppler effect is derived to determine the wavenumber ratios corresponding to these peaks.

TABLE 2

Wavenumber ratio $\gamma_{1,2}$ *for a periodically spring-supported beam with* $S = 10^{-4}$, $l/h = 100$, $M = 0{\cdot}25$, *in air*

Propagation zone no.	Wavenumber ratio γ ($M = 10^{-4}$)	Wavenumber ratio ($M = 0{\cdot}25$) γ_1	γ_2
1	0·133	0·119	0·154
	0·188	0·168	0·217
2	0·268	0·240	0·309
	0·346	0·309	0·399
3	0·402	0·359	0·464
	0·434	0·388	0·501
4	0·536	0·479	0·619
	0·549	0·491	0·634
5	0·670	0·599	0·774
	0·676	0·605	0·781

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APPENDIX A: NOMENCLATURE

C_L	longitudinal wave speed
C_0	sound speed in the acoustic medium
E	elastic modulus
$\bar{E}$	complex elastic modulus
f_0	external force strength per unit width
$f(x, t)$	moving force
$\tilde{F}(\xi, t)$	moving force in wavenumber domain
$\bar{G}$	complex shear modulus
h	height of the beam
I	cross-sectional second moment of inertia per unit width
j	$\sqrt{-1}$
κ^2	cross-sectional shape factor
k_B	free bending wavenumber
k_e	stiffness of the elastic foundation per unit area
k_0	acoustic wavenumber
ks	stiffness of the spring support per unit width
l	spacing between two adjacent supports
M	Mach number
$p(x, y=0, t)$	acoustic pressure acting on the beam surface
$\tilde{P}(\xi, y, t)$	acoustic pressure in wavenumber domain
$p_1(x, t)$	spring support force
$p_e(x, t)$	elastic foundation force
$\tilde{P}_1(\xi, t)$	spring support force in wavenumber domain
$\tilde{P}_e(\xi, t)$	elastic foundation force in wavenumber domain
S	stiffness ratio
$u(x, t)$	transverse displacement of the beam
$\bar{U}(\zeta, t)$	dimensionless transverse displacement of the beam in wavenumber domain
V	moving force speed
W	dimensionless radiated acoustic power per unit width
γ	wavenumber ratio
ζ	dimensionless wavenumber variable
ξ	wavenumber variable
η	structural damping
v	Poisson ratio
ρ_0	acoustic medium density
ρ_v	beam density
ω	moving force frequency
Ω	non-dimensional frequency