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LETTERS TO THE EDITOR

CONCEPT OF DIRECTIONAL NATURAL MODE FOR VIBRATION ANALYSIS OF ROTORS USING COMPLEX VARIABLE DESCRIPTIONS

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1. INTRODUCTION

Rotor dynamics research can be considered as a special sub-discipline of vibration research [1, 2]. In rotor problems, the rotation, whirl and natural mode are all associated with the implication of a circular motion, therefore it becomes important to keep track of their relative directivities. From this viewpoint, it is very convenient to use a time-varying complex variable to describe a planar motion such as shaft whirl, because the direction and amplitude of the motion can be represented by a single complex variable. Crandall [1], Dimentberg [3], and Childs [4] used complex variable descriptions to find the response of single disk rotors in the forward and backward directions. Lee and his colleagues pioneered extensive use of complex variables in the forced and natural response analyses of rotors [5–9]. They developed the concept of directional frequency response functions (dFRF) taking the advantage of the complex description. While their approach has many unique advantages, complicated notation and insufficient definition of natural modes may have prevented it from being more widely accepted. The main purpose of this work is to make a clear definition of the natural mode of the rotor described in complex variables, which will serve as the foundation of the complex-variable-based analysis of general rotor systems.

2. INTERPRETATION OF USING COMPLEX VARIABLES TO REPRESENT A QUANTITY IN A PLANAR MOTION

A complex variable has been used to represent shaft whirl motions [1, 2] and rotating forces [2] in rotors. A planar motion, taking the whirl motion of the shaft center as an example, can be represented by a complex variable, whose real and imaginary parts represent the co-ordinates of the moving points as seen in Figure 1. Besides the obvious advantage of short notation, use of complex variable for this purpose provides one very important advantage of relating the direction of motion directly to the mathematical

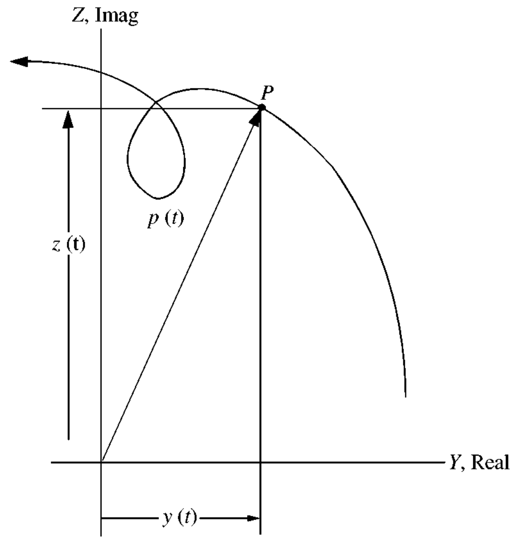

Figure 1. Complex variable representation of planar motion.

expression. If a general planar motion of a point denoted by P in Figure 1 is considered, the position of the point is identified by the complex variable $p(t)$,

$$p(t) = y(t) + \mathrm{j}z(t), \tag{1}$$

where $\mathrm{j} = \sqrt{-1}$. Transformation between real and complex representations can be defined as

$$\begin{Bmatrix} y(t) \\ z(t) \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -\mathrm{j} & \mathrm{j} \end{bmatrix}\begin{Bmatrix} p(t) \\ \bar{p}(t) \end{Bmatrix}, \tag{2}$$

where $\bar{p}(t)$ is the complex conjugate of $p(t)$, and the over bar indicates complex conjugate.

In general, $y(t)$ and $z(t)$ would be arbitrary functions of time. Therefore, if their complex Fourier series descriptions are used, $p(t)$ is

$$\begin{aligned} p(t) = y(t) + \mathrm{j}z(t) &= \sum_{k=0}^{\infty} \{(Y_k \mathrm{e}^{\mathrm{j}\omega_k t} + \bar{Y}_k \mathrm{e}^{-\mathrm{j}\omega_k t}) + \mathrm{j}(Z_k \mathrm{e}^{\mathrm{j}\omega_k t} + \bar{Z}_k \mathrm{e}^{-\mathrm{j}\omega_k t})\} \\ &= \sum_{k=0}^{\infty} \{(Y_k + \mathrm{j}Z_k)\mathrm{e}^{\mathrm{j}\omega_k t} + (\bar{Y}_k + \mathrm{j}\bar{Z}_k)\mathrm{e}^{-\mathrm{j}\omega_k t}\} = \sum_{k=0}^{\infty} \{P_{f_k} \mathrm{e}^{+\mathrm{j}\omega_k t} + P_{b_k} \mathrm{e}^{-\mathrm{j}\omega_k t}\}, \end{aligned} \tag{3}$$

where Y_k and $\bar{Y}_k$ are the kth complex Fourier components, ω_k is $2k\pi/T$ and T is the fundamental period of motion. Because the complex plane is used to define the position in a plane, it is realized that the $\mathrm{e}^{\mathrm{j}\omega_k t}(\mathrm{e}^{-\mathrm{j}\omega_k t})$ terms now represent the vector which is *physically rotating* in the counter-clockwise (clockwise) direction with the angular velocity of ω_k. Therefore, equation (3) tells that an arbitrary planar motion can be considered as sum of many pairs of circular motions rotating at different frequencies. Each of these pairs is composed of two counter-rotating circular motions of different amplitudes, therefore it represents a motion along an elliptic path.

3. EQUATION OF MOTION IN COMPLEX VARIABLES

The equation of motion of a rotor is formulated in

$$[M]\left\{\begin{array}{c}\{\ddot{y}\}\\ \{\ddot{z}\}\end{array}\right\}+[C]\left\{\begin{array}{c}\{\dot{y}\}\\ \{\dot{z}\}\end{array}\right\}+[K]\left\{\begin{array}{c}\{y\}\\ \{z\}\end{array}\right\}=\left\{\begin{array}{c}\{f_y\}\\ \{f_z\}\end{array}\right\}, \tag{4}$$

where vectors $\{y\}$ and $\{z\}$ represent the linear and angular displacements of the shaft at nodal points $\{f_y\}$ and $\{f_z\}$ represent the linear force and moment applied to the shaft at those nodes. Matrices $[M]$, $[K]$ and $[C]$ are the mass, stiffness and damping matrices. $[C]$ is actually the sum of the damping matrices and the gyroscopic matrix. If there is only the viscous damping matrix, $[C]$ becomes

$$[C]=[[C^0]+\Omega[G^0]], \tag{5}$$

where $[C^0]$ is the viscous damping matrix, Ω is the rotational speed of the rotor, and $[G^0]$ is the skew symmetric matrix representing the gyroscopic effect of the rotor.

The relationship between real and complex displacements in equation (2) can be generalized by defining a transformation matrix $[T]$,

$$[T]=\frac{1}{2}\begin{bmatrix} I & I\\ -\mathrm{j}I & jI\end{bmatrix}, \tag{6}$$

where I denotes an identity matrix. Then, the transformation is defined as

$$\left\{\begin{array}{c}\{y\}\\ \{z\}\end{array}\right\}=[T]\left\{\begin{array}{c}\{p\}\\ \{\bar{p}\}\end{array}\right\}. \tag{7}$$

Complex equation of motions result from substituting equation (7) into equation (4), then pre-multiplying by $[T]^{-1}$:

$$[T]^{-1}[M][T]\left\{\begin{array}{c}\{\ddot{p}\}\\ \{\ddot{\bar{p}}\}\end{array}\right\}+[T]^{-1}[C][T]\left\{\begin{array}{c}\{\dot{p}\}\\ \{\dot{\bar{p}}\}\end{array}\right\}+[T]^{-1}[K][T]\left\{\begin{array}{c}\{p\}\\ \{\bar{p}\}\end{array}\right\}=\left\{\begin{array}{c}\{g\}\\ \{\bar{g}\}\end{array}\right\}, \tag{8}$$

where the complex excitation vector is

$$\left\{\begin{array}{c}\{g\}\\ \{\bar{g}\}\end{array}\right\}=[T]^{-1}\left\{\begin{array}{c}\{f_Y\}\\ \{f_Z\}\end{array}\right\}=\left\{\begin{array}{c}\{f_Y\}+j\{f_Z\}\\ \{f_Y\}-j\{f_Z\}\end{array}\right\}. \tag{9}$$

It is noted that pre-multiplication by $[T]^{-1}$ is necessary to convert the excitation, thereby the whole equation, into complex form. Equation (8) can be rewritten as

$$[M_c]\left\{\begin{array}{c}\{\ddot{p}\}\\ \{\ddot{\bar{p}}\}\end{array}\right\}+[C_c]\left\{\begin{array}{c}\{\dot{p}\}\\ \{\dot{\bar{p}}\}\end{array}\right\}+[K_c]\left\{\begin{array}{c}\{p\}\\ \{\bar{p}\}\end{array}\right\}=\left\{\begin{array}{c}\{g\}\\ \{\bar{g}\}\end{array}\right\}. \tag{10}$$

The subscript c is used to emphasize that the matrices and formulation are based upon complex variables.

4. FREE VIBRATION ANALYSIS OF THE COMPLEX ROTOR EQUATION

The free vibration equation is obtained by setting the force vector in equation (10) and the damping matrix $[C^0]$ in equation (5) to zero:

$$[M_c]\begin{Bmatrix}\{\ddot{p}\}\\ \{\ddot{\bar{p}}\}\end{Bmatrix}+[C_c]\begin{Bmatrix}\{\dot{p}\}\\ \{\dot{\bar{p}}\}\end{Bmatrix}+[K_c]\begin{Bmatrix}\{p\}\\ \{\bar{p}\}\end{Bmatrix}=\begin{Bmatrix}\{0\}\\ \{0\}\end{Bmatrix}, \tag{11}$$

where $[C_c]=[T]^{-1}[\Omega[G^0]][T]$. Since the shaft center will go through a planar motion in general, the discussion in section 2 suggests that the motion will have to be described by two components rotating in opposite directions. Therefore, the free vibration response is assumed to be

$$\{p(t)\}=\{P_f\}\,\mathrm{e}^{\mathrm{j}\omega t}+\{P_b\}\,\mathrm{e}^{-\mathrm{j}\omega t}, \tag{12}$$

where the vector $\{P_f\}$ and $\{P_b\}$ define the shape of the components of the natural motion rotating in two opposite directions. Both terms of equation (12) are necessary in general to satisfy equation (11) except for the case of isotropic rotors, which has to be considered a special case. Substituting equation (12) into equation (11) results in

$$[M_c]\begin{Bmatrix}-\omega^2P_f\mathrm{e}^{\mathrm{j}\omega t}-\omega^2P_b\mathrm{e}^{-\mathrm{j}\omega t}\\ -\omega^2\bar{P}_f\mathrm{e}^{-\mathrm{j}\omega t}-\omega^2\bar{P}_b\mathrm{e}^{\mathrm{j}\omega t}\end{Bmatrix}+[C_c]\begin{Bmatrix}\mathrm{j}\omega P_f\mathrm{e}^{\mathrm{j}\omega t}-\mathrm{j}\omega P_b\mathrm{e}^{-\mathrm{j}\omega t}\\ -\mathrm{j}\omega\bar{P}_f\mathrm{e}^{-\mathrm{j}\omega t}-\mathrm{j}\omega\bar{P}_b\mathrm{e}^{\mathrm{j}\omega t}\end{Bmatrix}$$

$$+[K_c]\begin{Bmatrix}P_f\mathrm{e}^{\mathrm{j}\omega t}+P_b\mathrm{e}^{-\mathrm{j}\omega t}\\ \bar{P}_f\mathrm{e}^{-\mathrm{j}\omega t}+\bar{P}_b\mathrm{e}^{\mathrm{j}\omega t}\end{Bmatrix}=\begin{Bmatrix}\{0\}\\ \{0\}\end{Bmatrix}. \tag{13}$$

Equation (13) is satisfied only if coefficients of $\mathrm{e}^{\mathrm{j}\omega t}$ terms and $\mathrm{e}^{-\mathrm{j}\omega t}$ terms are independently zero, therefore it results in two identical equations

$$[-\omega^2[M_c]+\mathrm{j}\omega[C_c]+[K_c]\begin{Bmatrix}P_f\\ \bar{P}_b\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}, \tag{14}$$

$$[-\omega^2[M_c]+\mathrm{j}\omega[C_c]+[K_c]\begin{Bmatrix}\bar{P}_f\\ P_b\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}. \tag{15}$$

5. CONCEPT OF DIRECTIONAL NATURAL MODES

The simplest rotor equation with gyroscopic effect can be represented as [2, 10]

$$\begin{Bmatrix}\ddot{y}\\ \ddot{z}\end{Bmatrix}+\begin{bmatrix}0 & \Omega_p\\ -\Omega_p & 0\end{bmatrix}\begin{Bmatrix}\dot{y}\\ \dot{z}\end{Bmatrix}+\begin{bmatrix}1+\Delta & 0\\ 0 & 1-\Delta\end{bmatrix}\begin{Bmatrix}y\\ z\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}. \tag{16}$$

The equation represents an antisymmetric motion of a geometrically symmetric rigid rotor, which is supported by two identical orthotropic bearings at two ends. In the equation, y and z represent the position of the one end of the shaft, Ω_p represents the normalized rotational speed of the shaft, and Δ is non-dimensional anisotropy. Actual derivation of the equation from the equation of motion of the rigid-body rotor can be found in reference [10].

Applying the transformation defined in equation (7) to equation (16), one obtains

$$\begin{Bmatrix} \ddot{p} \\ \ddot{\bar{p}} \end{Bmatrix} + \begin{bmatrix} -\mathrm{j}\Omega_p & 0 \\ 0 & \mathrm{j}\Omega_p \end{bmatrix} \begin{Bmatrix} \dot{p} \\ \dot{\bar{p}} \end{Bmatrix} + \begin{bmatrix} 1 & \Delta \\ \Delta & 1 \end{bmatrix} \begin{Bmatrix} p \\ \bar{p} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{17}$$

Substituting equation (12) into equation (17), we obtain

$$\begin{bmatrix} -\omega^2 + \omega\Omega_p + 1 & \Delta \\ \Delta & -\omega^2 - \omega\Omega_p + 1 \end{bmatrix} \begin{Bmatrix} P_f \\ \bar{P}_b \end{Bmatrix} \mathrm{e}^{\mathrm{j}\omega t}$$

$$+ \begin{bmatrix} -\omega^2 + \omega\Omega_p + 1 & \Delta \\ \Delta & -\omega^2 - \omega\Omega_p + 1 \end{bmatrix} \begin{Bmatrix} \bar{P}_f \\ P_b \end{Bmatrix} \mathrm{e}^{-\mathrm{j}\omega t} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{18}$$

Two equations obtained from equation (18) results in the following identical form:

$$\begin{bmatrix} -\omega^2 + \omega\Omega_p + 1 & \Delta \\ \Delta & -\omega^2 - \omega\Omega_p + 1 \end{bmatrix} \begin{Bmatrix} P_f \\ \bar{P}_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{19}$$

The characteristic equation from equation (19) is obtained as

$$(1 - \omega^2)^2 - \omega^2\Omega_p^2 - \Delta^2 = 0, \tag{20}$$

and four characteristic roots are obtained as eigenvalues from equation (20),

$$\omega_{1,2,3,4} = \pm\sqrt{\frac{2 + \Omega_p^2}{2} \pm \sqrt{\left(\frac{2 + \Omega_p^2}{2}\right)^2 - 1 + \Delta^2}}. \tag{21}$$

We consider the free vibration solution of an isotropic rotor first, then that of an anisotropic rotor.

5.1. SPECIAL CASE: ISOTROPIC ROTOR

An isotropic rotor can be considered as a special case of equation (19) with $\Delta = 0$. Four eigenvalues in equation (21) become

$$\omega_{1,2} = -\frac{\Omega_p}{2} \pm \sqrt{\left(\frac{\Omega_p}{2}\right)^2 + 1}, \qquad \omega_{3,4} = -\frac{\Omega_p}{2} \pm \sqrt{\left(\frac{\Omega^p}{2}\right)^2 + 1}. \tag{22, 23}$$

Only two of the above four eigenvalues are unique as $\omega_1 = -\omega_4 > 0$, and $\omega_3 = -\omega_2 > 0$. Substituting $\omega_{1,2}$ into equation (19) with $\Delta = 0$, the eigenvector is obtained as

$$\begin{Bmatrix} P_f \\ \bar{P}_b \end{Bmatrix}_{1,2} = \begin{Bmatrix} P_f \\ P_b \end{Bmatrix}_{1,2} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}. \tag{24}$$

Therefore, from equation (12) the natural motions associated with ω_1 and ω_2 are expressed as

$$p_1(t) = 1\mathrm{e}^{-\mathrm{j}\omega_1 t}, \quad \omega_1 > 0, \qquad p_2(t) = 1\mathrm{e}^{-\mathrm{j}\omega_2 t} = 1\mathrm{e}^{\mathrm{j}\omega_3 t}, \quad \omega_3 > 0. \tag{25}$$

Similarly, eigenvectors corresponding to $\omega_{3,4}$ are found:

$$\begin{Bmatrix} P_f \\ \bar{P}_b \end{Bmatrix}_{3,4} = \begin{Bmatrix} P_f \\ P_b \end{Bmatrix}_{3,4} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}. \tag{26}$$

The corresponding two natural motions are expressed as

$$p_3(t) = 1\mathrm{e}^{\mathrm{j}\omega_3 t}, \quad \omega_3 > 0, \qquad p_4(t) = 1\mathrm{e}^{\mathrm{j}\omega_4 t} = 1\mathrm{e}^{-\mathrm{j}\omega_1 t}, \quad \omega_1 > 0. \tag{27}$$

Comparing equations (25) and (27), it is known that the motions represented by $p_1(t)$ and $p_4(t)$ and the motions by $p_2(t)$ and $p_3(t)$ are the same. Therefore, one can choose any two distinct motions, i.e., $p_1(t)$ and $p_2(t)$ or $p_3(t)$ and $p_4(t)$, to describe the natural responses of the rotor. This pair carries the whole information of the natural responses, and is defined as the *directional natural modes*. The mode $p_2(t)$ represents a forward-rotating conical mode and $p_1(t)$ represents a backward-rotating conical mode as shown in Figures 2(a) and 2(b). (Note that the equation of motion (16) is obtained by applying the antisymmetry to the rigid rotor; y and z represent the position of the one end of the shaft that is undergoing a conical motion shown in Figure 2(a).)

If the directional mode as defined in equation (25) or equation (27) had not been considered, natural frequencies in equations (22) and (23) would have been plotted as the four curves corresponding to $\omega_{1,2,3,4}$ as shown in Figure 3. The directional modes described in equation (25) or equation (27) indicate that only two of these curves are physically unique, as the actual rotating directions are depicted in the figure. Therefore, the natural frequency of the rotor has to be plotted as in Figure 4, if the information conveyed by the directional modes is used. In the figure, the frequency curve in the negative (positive) side indicates the rotor is rotating in the backward (forward) direction.

As an example of the benefit of the directional mode description, the response of the rotor to a rotating mass imbalance is considered. Since it is an excitation purely in the forward direction at the rotational speed, the intersection of the frequency curve and the line corresponding to $\omega = \Omega_p$ represents the synchronous critical speed. Figure 4 shows that there will be no intersection, therefore no critical speed due to such an excitation would exist. If the directional information were not used, a false critical speed would be identified from Figure 3 as the point "a" in the figure.

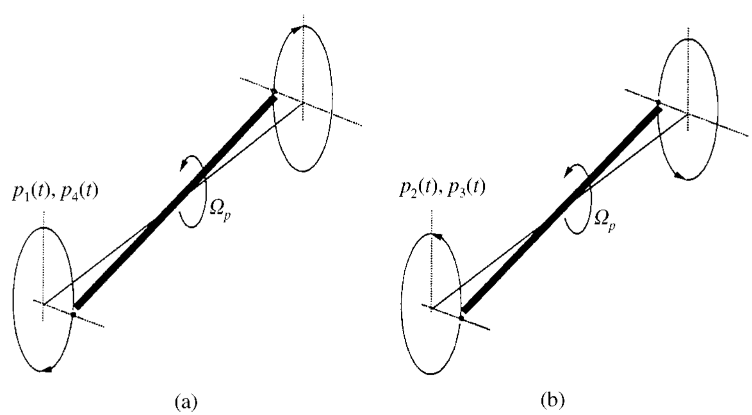

Figure 2. Natural modes of isotropic rigid rotor in antisymmetric natural motion: (a) backward mode; (b) forward mode.

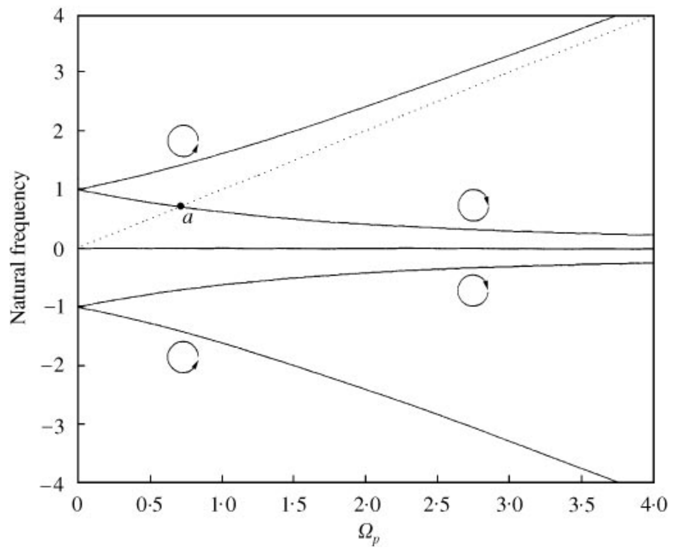

Figure 3. Natural frequencies of isotropic rotor without considering directional information.

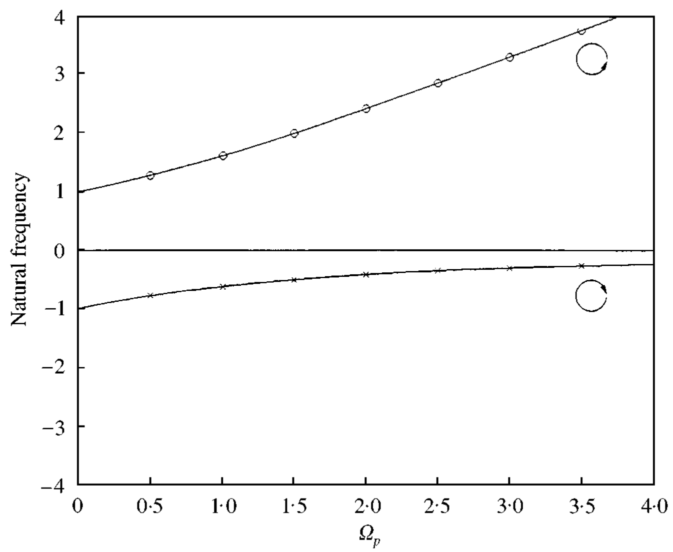

Figure 4. Natural frequencies of isotropic rotor based on the directional natural mode description.

5.2. GENERAL CASE: ANISOTROPIC ROTOR

If there is anisotropy in the rotor ($\Delta \neq 0$ in equation (21)), four eigenvalues are obtained:

$$\omega_{1,3} = \sqrt{\frac{2+\Omega_p^2}{2} \pm \sqrt{\left(\frac{2+\Omega_p^2}{2}\right)^2 - 1 + \Delta^2}},$$
$$\omega_{2,4} = -\sqrt{\frac{2+\Omega_p^2}{2} \pm \sqrt{\left(\frac{2+\Omega_p^2}{2}\right)^2 - 1 + \Delta^2}}. \tag{28}$$

Again, it is known that $\omega_1 = -\omega_4 > 0$ and $\omega_3 = -\omega_2 > 0$. Four eigenvectors are obtained with substitution of the eigenvalues into equation (19):

$$\begin{Bmatrix} P_f \\ P_b \end{Bmatrix}_i = \begin{Bmatrix} 1 \\ \dfrac{\Delta}{\omega_i^2 + \omega_i \Omega_p - 1} \end{Bmatrix}, \quad i = 1, 2, 3, 4. \tag{29}$$

Let us consider a numerical example of $\Delta = 0{\cdot}1$, $\Omega_p = 0{\cdot}9$ for an easier explanation. The eigenvalues become

$$\omega_1 = 1{\cdot}5482 = -\omega_2, \qquad \omega_3 = 0{\cdot}6427 = -\omega_4. \tag{30}$$

The eigenvectors and related natural motions corresponding to ω_1 and ω_2, are

$$\begin{aligned} \begin{Bmatrix} P_f \\ P_b \end{Bmatrix}_1 &= \begin{Bmatrix} 1 \\ 0{\cdot}0354 \end{Bmatrix} \Rightarrow p_1(t) = \mathrm{e}^{1{\cdot}5482\mathrm{j}t} + 0{\cdot}0354\mathrm{e}^{-1{\cdot}5482\mathrm{j}t}, \\ \begin{Bmatrix} P_f \\ P_b \end{Bmatrix}_2 &= \begin{Bmatrix} 0{\cdot}0354 \\ 1 \end{Bmatrix} \Rightarrow p_2(t) = 0{\cdot}0354\mathrm{e}^{-1{\cdot}5482\mathrm{j}t} + 1\mathrm{e}^{1{\cdot}5482\mathrm{j}t}. \end{aligned} \tag{31}$$

The eigenvectors and related natural motions corresponding to ω_3 and ω_4 are

$$\begin{aligned} \begin{Bmatrix} P_f \\ P_b \end{Bmatrix}_3 &= \begin{Bmatrix} -0{\cdot}0858 \\ 1 \end{Bmatrix} \Rightarrow p_3(t) = -0{\cdot}0858\mathrm{e}^{0{\cdot}6427\mathrm{j}t} + 1\mathrm{e}^{-0{\cdot}6427\mathrm{j}t}, \\ \begin{Bmatrix} P_f \\ P_b \end{Bmatrix}_4 &= \begin{Bmatrix} 1 \\ -0{\cdot}0858 \end{Bmatrix} \Rightarrow p_4(t) = 1\mathrm{e}^{-0{\cdot}6427\mathrm{j}t} - 0{\cdot}0858\mathrm{e}^{0{\cdot}6427\mathrm{j}t}. \end{aligned} \tag{32}$$

As in the isotropic case, only two of the natural motions are unique. Therefore, any two unique motions, for example $p_1(t)$ and $p_3(t)$, can be chosen as *the directional natural modes* of the rotor, which carry the full information of the natural modes of the rotor: the directions, shapes and frequency, without any redundancy.

From equation (31) or equation (32), it is seen that each directional natural mode is composed of two circular motions of different strengths rotating in the opposite directions, which are defined as the *forward and backward sub-modes*. In the above example, the directional mode $p_1(t)$ ($p_3(t)$) has a much stronger forward (backward) sub-mode, representing the shaft motion moving along an elliptic path in the forward (backward) direction. Hence, the entire motion described by $p_1(t)$ ($p_3(t)$) can be considered as the *forward* (*backward*) *mode* of the rotor. Forward and backward modes of rotors have been used in rotor dynamics for a long time, however apparently without a rigorous definition, especially for anisotropic rotors.

Figure 5 is the natural frequency chart, which would be obtained if the directional mode information were not used. Four frequencies calculated from equation (28) are plotted as functions of the rotor speed. The figure does not accurately depict the physical situation, because the directions and strengths of the sub-modes are not reflected at all. Figure 6 is the frequency chart constructed based on the information that the directional natural modes in equation (31) carry. In the figure, positive (negative) natural frequencies represent the forward (backward) motion, and solid (dash) lines represent strong (weak) sub-modes. It is noted that the ratio of the strength of the weak sub-mode to the strong sub-mode is the function of the rotational speed as seen from equation (29). In all practical rotor systems, the anisotropy is very small, therefore the ratio will be of very small value.

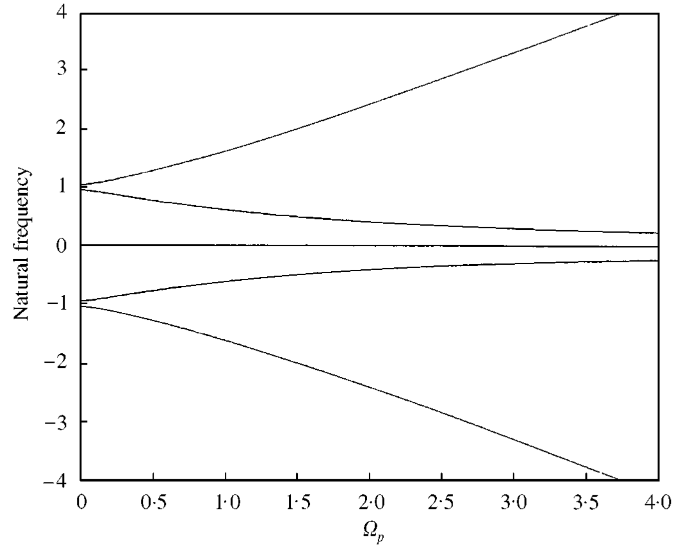

Figure 5. Natural frequencies of anisotropic rotor without considering directional information.

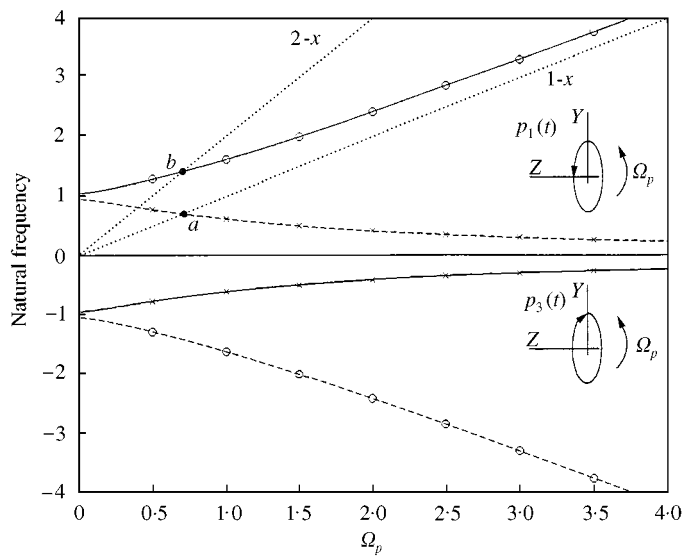

Figure 6. Natural frequencies of anisotropic rotor based on the directional natural mode description.

An important advantage of the directional natural mode concept is seen from Figure 6. The point a indicates the intersection of $1 - x$ line and the frequency curve corresponding to the weak forward sub-mode of the backward directional mode. Therefore, the response at this speed is expected to be much smaller than the typical response at the critical speed. Point b, being an intersection between the strong forward sub-mode of the forward directional mode and the $2 - x$ line, a much more severe response would be expected at this frequency if $2 - x$ excitation existed. Figure 5 would not have provided such information.

6. FURTHER REMARKS ON DIRECTIONAL NATURAL MODE

The directional natural mode introduced in this work defines the frequency, and shape and directions of the natural mode of rotating systems in one complex variable expression. The general expression for the ith directional natural mode can be written as

$$\{p(t)\}_i = \{P_f\}_i \mathrm{e}^{\mathrm{j}\omega_i t} + \{P_b\}_i \mathrm{e}^{-\mathrm{j}\omega_i t}. \tag{33}$$

Elements of $\{P_f\}_i$ and $\{P_b\}_i$ define the mode shapes of the forward and backward submodes of the free, natural response of the rotor. Except for the isotropic rotor, both terms in equation (33) are necessary to satisfy the homogeneous equation of motion formulated in complex variables (equation (17)). Since the mathematical definition of the natural mode is a solution that satisfies the homogeneous equation of motion, the natural mode of an anisotropic rotor has to be defined in the form of equation (33). As discussed, only half of the eigenvalues and eigenvectors are physically unique motions. Because of the way that the complex mode is defined as in equation (33), only positive (or negative) eigenvalues can be chosen to describe all natural modes. These eigenvalues and their corresponding eigenvectors represented in the form of equation (33) are defined as the directional natural modes.

7. CONCLUSIONS

The free vibration analysis of a general rotating system using complex variable descriptions of the equations of motion has been discussed. An important advantage of using complex variables to describe planar motions is recognized as relating physical rotations to mathematical expressions directly. Directional natural modes are obtained as the solution that satisfies the free-vibration equation formulated in complex variable descriptions. The mode description represents all necessary information: frequency, direction and shape, to describe the natural mode of a general rotating system without any redundancy. Especially, it is very difficult to represent the directional information of the mode when the real description is used.

A very simple rotor motion with gyroscopic coupling effect is used as an example to explain the concept of the directional natural mode and the advantage of using it. It is shown that the directional natural mode of a general rotor with small anisotropy is composed of two sub-modes, one rotating in the forward direction and the other in the backward direction. The isotropic rotor can be understood as a special case, in which one of the directional components is zero. Based on the directional natural mode expression, a generalized definition of the forward and backward modes of the rotating system can be made: the forward (backward) mode is defined as the mode whose forward (backward) sub-mode is stronger. Practical advantages of using the directional mode concept are explained, such as finding the critical speed and keeping track of the direction of motion relative to the direction of the excitation force.

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