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ON THE VIBRATIONS OF AN AXIALLY TRAVELLING BEAM ON FIXED SUPPORTS WITH VARIABLE VELOCITY

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1. INTRODUCTION

Axially moving materials are of technological importance, since there are widespread applications in the form of thread lines, high-speed magnetic tapes and paper sheets, strings, power transmission chains and belts, band-saws, fibers, beams, aerial cable tramways and pipes conveying fluid. The work done has been reviewed by Ulsoy *et al.* [1] and Wickert and Mote [2] up to 1978 and 1988 respectively. Wickert and Mote [3] investigated the response of axially travelling strings and beams subjected to arbitrary excitation and initial conditions. They used travelling string eigenfunctions and introduced a convenient orthogonal basis suitable for discretization. They presented a modal analysis using complex state eigenfunctions and their conjugates [4]. Ulsoy [5] dealt with a model for the transverse vibration of an axially moving beam, including elastic coupling between the two adjacent spans. The system was then analyzed using a classical approximate solution method. The simulation showed that a beating phenomenon exists at zero transport velocity, but was destroyed at higher velocities and/or when there were differences in the axial tension in adjacent spans. Al-Jawi *et al.* [6–8] investigated the effects of tension disorder, inter-span coupling and translation speed on the confinement of the natural modes of free vibration. A theoretical basis for the analysis of band vibration and stability was studied by Ulsoy and Mote [9]. The band natural frequencies were found to decrease with increasing axial velocity at a rate dependent on the wheel support system constant, and to increase with increasing axial tension or "strain".

Miranker [10] considered a model using a variational procedure and derived the equations of motion for variable axial velocity. Mote [11] investigated the problem of an axially accelerating string with harmonic excitation at one end and determined stability by Laplace transform techniques. Pakdemirli *et al.* [12] re-derived the equations of motion for an axially accelerating string using Hamilton's principle and made a stability analysis using Floquet theory for sinusoidal velocity function. Pakdemirli and Batan [13] made a stability analysis for periodic constant acceleration–deceleration type of velocity case.

Pakdemirli and Ulsoy [14] investigated the principal parametric resonances and the combination resonances for an axially accelerating string. They obtained instability regions for velocity fluctuation frequencies nearly twice any natural frequency, no instabilities were found for the frequencies close to zero. For combination resonances, instabilities occurred only for those of sum type. No instabilities were detected for difference-type combination resonances in agreement with reference [15]. Öz *et al.* [16] investigated the transition

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behavior from strip to beam for axially moving continua. An outer solution is studied. An approximate analytical expression for the non-linear natural frequency was given for the problem. The stability borders were determined analytically depending on velocity. The beam effects were studied. Öz and Pakdemirli [17] considered a simply supported Euler–Bernoulli beam moving with variable velocity. The natural frequency variation depending on velocity for various flexural stiffness values were determined for the first two modes. The authors arrived at the same results as in references [14, 15] for the axially moving beams. Wickert [18] analyzed free non-linear vibrations of a moving beam over the sub- and superharmonic transport speed ranges. Chakraborty *et al.* [19] investigated the free and forced vibrations of a travelling beam using a model similar to that of reference [18]. Pellicano and Zirilli [20] presented a boundary layer solution for the axially moving beam problem with vanishing flexural stiffness and weak non-linearities. Asokanthan and Ariaratnam [21] investigated flexural instabilities in moving bands under harmonic tension fluctuation. They discussed the effects of damping, mean band speed, and the band compliance on the band stability. Öz *et al.* [22] investigated linear and non-linear vibrations of moving beams with time-dependent velocity. The authors calculated non-linear frequencies depending on mean velocity and flexural stiffness. They obtained amplitude-phase-modulation equations, determined stable and unstable regions for trivial and non-trivial solutions. Öz and Boyaci [23] studied a tensioned pipe conveying fluid with time-dependent velocity. They obtained stability boundaries and reached similar results in agreement with references [14, 16, 17].

In this study, the vibrations of an Euler–Bernoulli-type beam having different flexural stiffness values and moving with harmonically varying velocities are considered. The beam is on fixed–fixed supports. The natural frequency variation depending on transport velocity for various flexural stiffness values are determined for the first two modes. The principal parametric resonances are investigated. For velocity fluctuation frequencies nearly twice any natural frequency, an instability region occurs, whereas for frequencies close to zero, no instabilities are detected. The flexural effects and the effect of support conditions on the natural frequencies and on the stability are discussed.

2. APPROXIMATE ANALYSIS

For the axially moving beam, following a similar derivation as given in reference [18], it can be shown that the linear, time-dependent, dimensionless equation of motion is

$$(\ddot{w} + 2\dot{w}'\upsilon + w'\dot{\upsilon}) + v_f^2 w^{\mathrm{iv}} + (\upsilon^2 - 1)w'' = 0, \tag{1}$$

where w is the transverse displacement, υ is the axial velocity, v_f is the flexural stiffness, $\ddot{w}$, $2\upsilon\dot{w}'$ and $\upsilon^2 w''$ denote local, Coriolis and centripetal acceleration components respectively.

The boundary conditions for a fixed–fixed beam are

$$w(0, t) = w(1, t) = 0, \quad w'(0, t) = w'(1, t) = 0. \tag{2}$$

The dot denotes differentiation with respect to time and the prime denotes differentiation with respect to the spatial variable x.

Assuming that the velocity is harmonically varying about a constant mean value υ_0, one writes

$$\upsilon = \upsilon_0 + \varepsilon\upsilon_1 \sin \Omega t, \tag{3}$$

where ε is a small parameter and $\varepsilon\upsilon_1$ and Ω represent the amplitude of fluctuations and fluctuation frequency respectively. Substituting equation (3) into equation (1) and keeping terms up to the first order of approximation, one has

$$\ddot{w} + 2\upsilon_0\dot{w}' + (\upsilon_0^2 - 1)w'' + v_f^2 w^{\mathrm{iv}} + \varepsilon(\upsilon_1 \Omega \cos\Omega t\, w' + 2\upsilon_1 \sin\Omega t\, \dot{w}' + 2\upsilon_0\upsilon_1 \sin\Omega t\, w'') = 0. \tag{4}$$

Applying the method of multiple scales in a similar way to reference [17], and arranging the equations one obtains

$$v_f^2 Y_n^{\mathrm{iv}} + (\upsilon_0^2 - 1)Y_n'' + 2\mathrm{i}\upsilon_0\omega_n Y_n' - \omega_n^2 Y_n = 0, \tag{5}$$

where ω_n is the natural frequency of oscillations and Y_n is the shape function with the following fixed end conditions:

$$Y_n(0) = 0, \quad Y_n(1) = 0, \quad Y_n'(0) = 0, \quad Y_n'(1) = 0. \tag{6}$$

The solution of equation (5) is

$$Y_n(x) = c_{1n}(\mathrm{e}^{\mathrm{i}\beta_{1n}x} + C_{2n}\mathrm{e}^{\mathrm{i}\beta_{2n}x} + C_{3n}\mathrm{e}^{\mathrm{i}\beta_{3n}x} + C_{4n}\mathrm{e}^{\mathrm{i}\beta_{4n}x}), \tag{7}$$

where β_{in} satisfy the following dispersive relation:

$$v_f^2\beta_{in}^4 + (1 - \upsilon_0^2)\beta_{in}^2 - 2\omega_n\upsilon_0\beta_{in} - \omega_0^2 = 0, \quad i = 1, 2, 3, 4, \quad n = 1, 2 \ldots. \tag{8}$$

Applying the boundary conditions to the solution, one obtains the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \beta_{1n} & \beta_{2n} & \beta_{3n} & \beta_{4n} \\ \mathrm{e}^{\mathrm{i}\beta_{1n}} & \mathrm{e}^{\mathrm{i}\beta_{2n}} & \mathrm{e}^{\mathrm{i}\beta_{3n}} & \mathrm{e}^{\mathrm{i}\beta_{4n}} \\ \beta_{1n}\mathrm{e}^{\mathrm{i}\beta_{1n}} & \beta_{2n}\mathrm{e}^{\mathrm{i}\beta_{2n}} & \beta_{3n}\mathrm{e}^{\mathrm{i}\beta_{3n}} & \beta_{4n}\mathrm{e}^{\mathrm{i}\beta_{4n}} \end{bmatrix} \begin{Bmatrix} 1 \\ C_{2n} \\ C_{3n} \\ C_{4n} \end{Bmatrix} c_{1n} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{9}$$

For non-trivial solutions, the determinant of the coefficient matrix must be zero, which yields the support condition

$$\begin{aligned} &[\mathrm{e}^{\mathrm{i}(\beta_{1n}+\beta_{2n})} + \mathrm{e}^{\mathrm{i}(\beta_{3n}+\beta_{4n})}](\beta_{2n} - \beta_{1n})(\beta_{3n} - \beta_{4n}) \\ &+ [\mathrm{e}^{\mathrm{i}(\beta_{1n}+\beta_{3n})} + \mathrm{e}^{\mathrm{i}(\beta_{2n}+\beta_{4n})}](\beta_{2n} - \beta_{4n})(\beta_{1n} - \beta_{3n}) \\ &+ [\mathrm{e}^{\mathrm{i}(\beta_{2n}+\beta_{3n})} + \mathrm{e}^{\mathrm{i}(\beta_{1n}+\beta_{4n})}](\beta_{1n} - \beta_{4n})(\beta_{3n} - \beta_{2n}) = 0. \end{aligned} \tag{10}$$

Numerical values of ω_n and β_{in} can be calculated using equations (8) and (10) and natural frequencies will be presented in section 4. Using the boundary conditions (6), one can find the coefficients C_{2n}, C_{3n} and C_{4n}:

$$C_{2n} = -\frac{(\beta_{4n} - \beta_{1n})(\mathrm{e}^{\mathrm{i}\beta_{3n}} - \mathrm{e}^{\mathrm{i}\beta_{1n}})}{(\beta_{4n} - \beta_{2n})(\mathrm{e}^{\mathrm{i}\beta_{3n}} - \mathrm{e}^{\mathrm{i}\beta_{2n}})}, \tag{11}$$

$$C_{3n} = -\frac{(\beta_{4n} - \beta_{1n})(\mathrm{e}^{\mathrm{i}\beta_{2n}} - \mathrm{e}^{\mathrm{i}\beta_{1n}})}{(\beta_{4n} - \beta_{3n})(\mathrm{e}^{\mathrm{i}\beta_{2n}} - \mathrm{e}^{\mathrm{i}\beta_{3n}})}, \tag{12}$$

$$C_{4n} = -1 - C_{2n} - C_{3n}. \tag{13}$$

Then, $Y_n(x)$ can be written as

$$\begin{aligned} Y_n(x) = c_{1n}\Bigg(&\mathrm{e}^{\mathrm{i}\beta_{1n}x} - \frac{(\beta_{4n} - \beta_{1n})(\mathrm{e}^{\mathrm{i}\beta_{3n}} - \mathrm{e}^{\mathrm{i}\beta_{1n}})}{(\beta_{4n} - \beta_{2n})(\mathrm{e}^{\mathrm{i}\beta_{3n}} - \mathrm{e}^{\mathrm{i}\beta_{2n}})}\mathrm{e}^{\mathrm{i}\beta_{2n}x} - \frac{(\beta_{4n} - \beta_{1n})(\mathrm{e}^{\mathrm{i}\beta_{2n}} - \mathrm{e}^{\mathrm{i}\beta_{1n}})}{(\beta_{4n} - \beta_{3n})(\mathrm{e}^{\mathrm{i}\beta_{2n}} - \mathrm{e}^{\mathrm{i}\beta_{3n}})}\mathrm{e}^{\mathrm{i}\beta_{3n}x} \\ &+ \left(-1 + \frac{(\beta_{4n} - \beta_{1n})(\mathrm{e}^{\mathrm{i}\beta_{3n}} - \mathrm{e}^{\mathrm{i}\beta_{1n}})}{(\beta_{4n} - \beta_{2n})(\mathrm{e}^{\mathrm{i}\beta_{3n}} - \mathrm{e}^{\mathrm{i}\beta_{2n}})} + \frac{(\beta_{4n} - \beta_{1n})(\mathrm{e}^{\mathrm{i}\beta_{2n}} - \mathrm{e}^{\mathrm{i}\beta_{1n}})}{(\beta_{4n} - \beta_{3n})(\mathrm{e}^{\mathrm{i}\beta_{2n}} - \mathrm{e}^{\mathrm{i}\beta_{3n}})}\right)\mathrm{e}^{\mathrm{i}\beta_{4n}x}\Bigg). \end{aligned} \tag{14}$$

3. PRINCIPAL PARAMETRIC RESONANCES

In this section, three different cases are investigated depending on the numerical value of the frequency.

3.1. Ω AWAY FROM $2\omega_n$ AND 0

The solvability condition requires (see references [17, 24, 25] for details of calculating solvability conditions)

$$D_1 A_n = 0, \tag{15}$$

where A_n is the complex amplitude. This means a constant amplitude solution up to the first order of approximation

$$A_n = A_0. \tag{16}$$

Hence, solutions are bounded for this case up to $O(\varepsilon)$.

3.2. Ω CLOSE TO 0

The nearness of Ω to zero is expressed as

$$\Omega = \varepsilon\sigma. \tag{17}$$

Following the calculations in references [17, 24, 25] for this case one obtains the solvability condition as

$$D_1 A_n + (k_1 \cos\sigma T_1 + k_2 \sin\sigma T_1) A_n = 0, \tag{18}$$

where T_1 denotes slow time scale and

$$k_1 = \frac{\Omega \upsilon_1 \int_0^1 Y_n' \bar{Y}_n \,\mathrm{d}x}{2\left(\mathrm{i}\omega_n \int_0^1 Y_n \bar{Y}_n \,\mathrm{d}x + \upsilon_0 \int_0^1 Y_n' \bar{Y}_n \,\mathrm{d}x\right)}, \quad k_2 = \frac{\upsilon_1 (\mathrm{i}\omega_n \int_0^1 Y_n' \bar{Y}_n \,\mathrm{d}x + \upsilon_0 \int_0^1 Y_n'' \bar{Y}_n \,\mathrm{d}x)}{\left(\mathrm{i}\omega_n \int_0^1 Y_n \bar{Y}_n \,\mathrm{d}x + \upsilon_0 \int_0^1 Y_n' \bar{Y}_n \,\mathrm{d}x\right)}. \tag{19}$$

After solving equation (18), one gets

$$A_n = A_0 \exp\left(-k_1 \frac{\sin\sigma T_1}{\sigma} + k_2 \frac{\cos\sigma T_1}{\sigma}\right). \tag{20}$$

Since $|\sin\sigma T_1| \leqslant 1$ and $|\cos\sigma T_1| \leqslant 1$, the complex amplitudes are bounded in time. Therefore, no instabilities exist up to this order of approximation.

3.3. Ω CLOSE TO $2\omega_n$

In this case, the nearness of velocity-variation frequency to twice that of the natural frequencies can be expressed as

$$\Omega = 2\omega_n + \varepsilon\sigma, \tag{21}$$

where σ is a detuning parameter. The solvability condition is

$$D_1 A_n + k_0 \bar{A}_n \mathrm{e}^{\mathrm{i}\sigma T_1} = 0, \tag{22}$$

where $\bar{A}_n$ is the complex conjugate of amplitude A_n and k_0 *is*

$$k_0 = \frac{\left\{1/2(\Omega - 2\omega_n)\int_0^1 \bar{Y}_n' \bar{Y}_n \,\mathrm{d}x - \mathrm{i}\upsilon_0 \int_0^1 \bar{Y}_n'' \bar{Y}_n \,\mathrm{d}x\right\}}{2\left\{\mathrm{i}\omega_n \int_0^1 \bar{Y}_n Y_n \,\mathrm{d}x + \upsilon_0 \int_0^1 \bar{Y}_n Y_n' \,\mathrm{d}x\right\}} \upsilon_1. \tag{23}$$

Following the solution method in references [17, 24, 25] one obtains

$$\Omega = 2\omega_n \mp 2\varepsilon \sqrt{k_{0R}^2 + k_{0I}^2}, \tag{24}$$

where k_{0R} and k_{0I} are real and imaginary parts of k_0 respectively. The two values of Ω denote the stability boundaries for small ε. When the amplitude of fluctuations υ_1 increases, the stability regions widen.

4. NUMERICAL EXAMPLES

In this section, numerical plots for the natural frequencies and stability borders will be presented.

Natural frequencies are found by solving equations (8) and (10) simultaneously and plotted in Figures 1 and 2 for the first and second modes, respectively, for three different flexural stiffness values. As can be seen the natural frequencies decrease with increasing mean velocity. At the critical velocity divergence instability occurs. Increasing the flexural stiffness values increases natural frequencies in agreement with references [17, 18]. The natural frequencies for the fixed–fixed axially moving beam are higher than those of the simply supported one calculated in reference [17].

In Figures 3 and 4, stable and unstable regions are plotted for principal parametric resonances case for the first mode for two different flexural stiffness values ($v_f = 0{\cdot}6$ and 1)

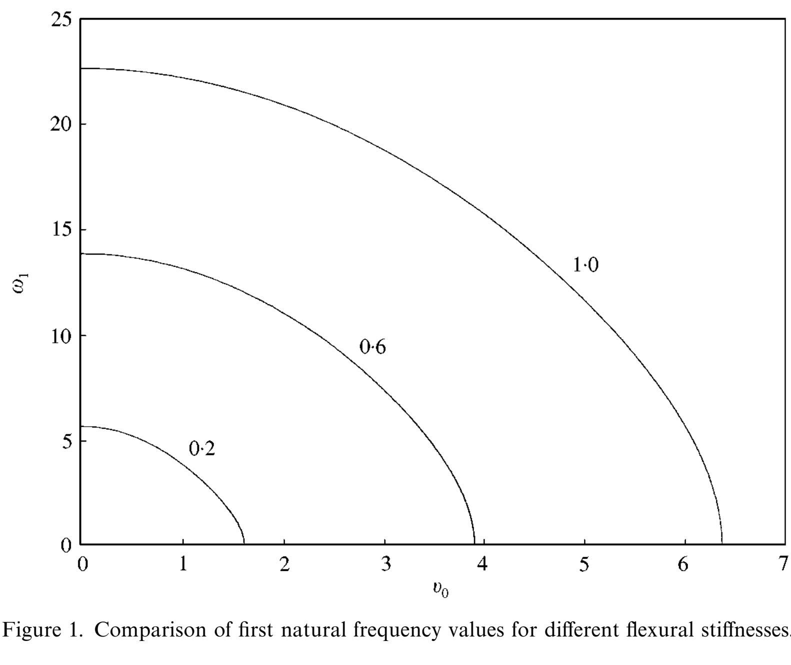

Figure 1. Comparison of first natural frequency values for different flexural stiffnesses.

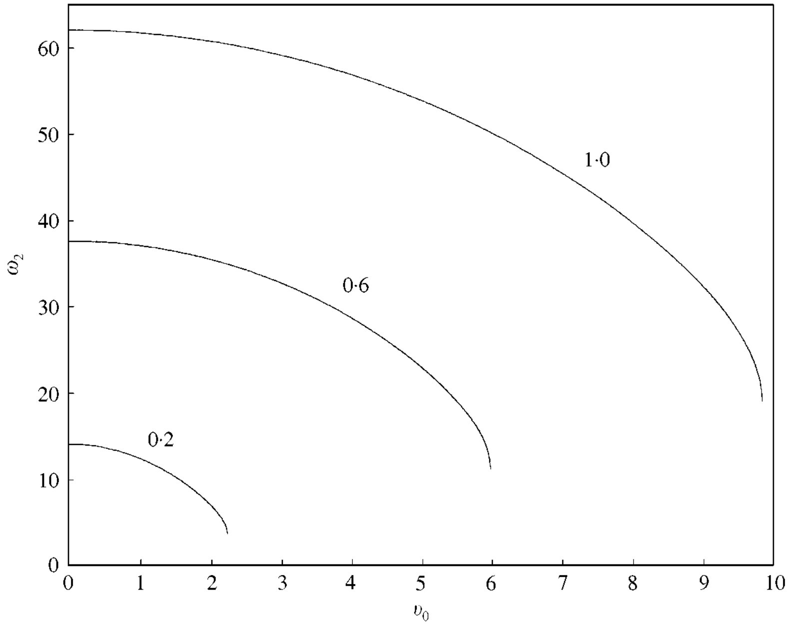

Figure 2. Comparison of second natural frequency values for different flexural stiffnesses.

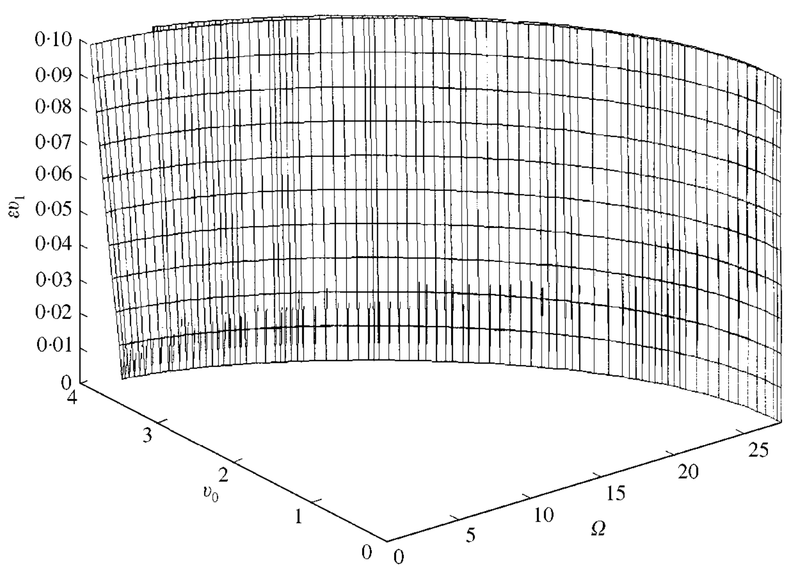

Figure 3. Stable and unstable regions for principal parametric resonances for the first mode ($n = 1$, $v_f = 0{\cdot}6$).

obtained by using equation (24). The regions in between the planar surfaces are unstable whereas the remaining regions are stable. Increasing the flexural stiffness value, the stability regions shift to higher Ω values. Increasing the velocity variation amplitude widen the stability regions in agreement with reference [17]. In Figures 5 and 6, stable and unstable regions are given for the second mode. Similar conclusions can be drawn. The values for the

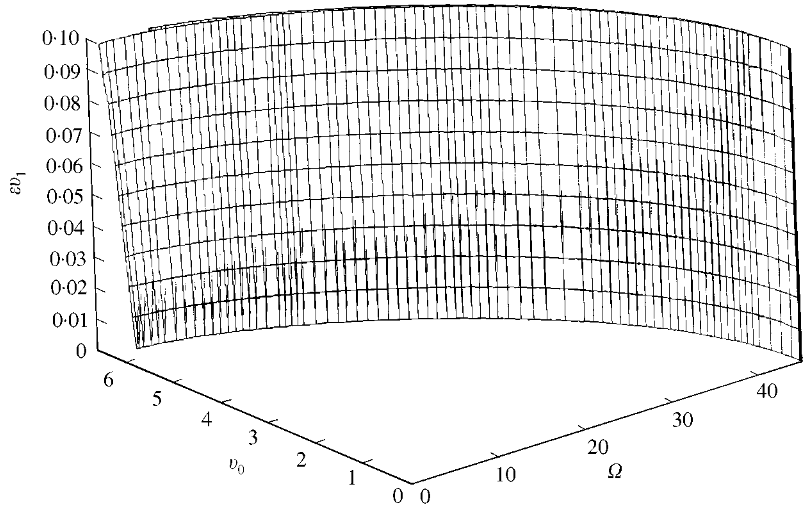

Figure 4. Stable and unstable regions for principal parametric resonances for the first mode ($n = 1$, $v_f = 1{\cdot}0$).

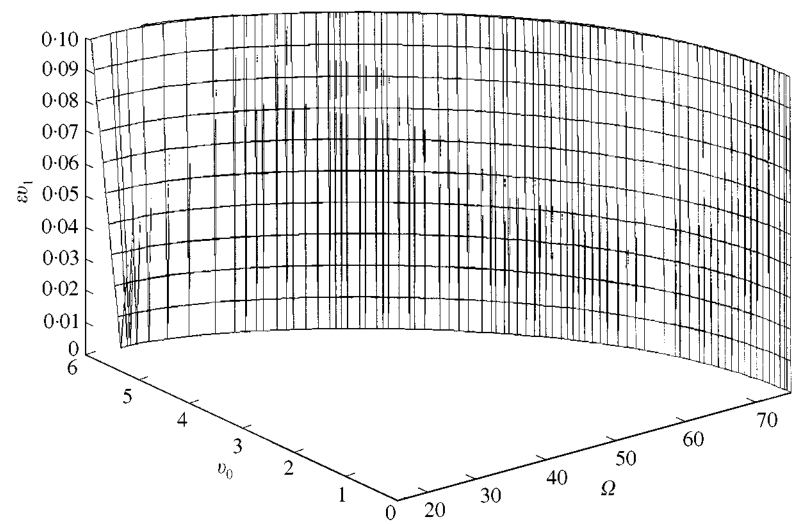

Figure 5. Stable and unstable regions for principal parametric resonances for the second mode ($n = 2$, $v_f = 0{\cdot}6$).

axially moving beam on fixed supports are higher than the values of the simply supported travelling beam in reference [17].

5. CONCLUSIONS

In this study, the vibration of an axially moving Euler–Bernoulli beam with fixed end conditions is investigated. The velocity is assumed to be harmonically varying about a mean value. The velocity-fluctuation amplitude is assumed small. Natural frequencies are found depending on mean velocity by using a standard root-finding algorithm for different

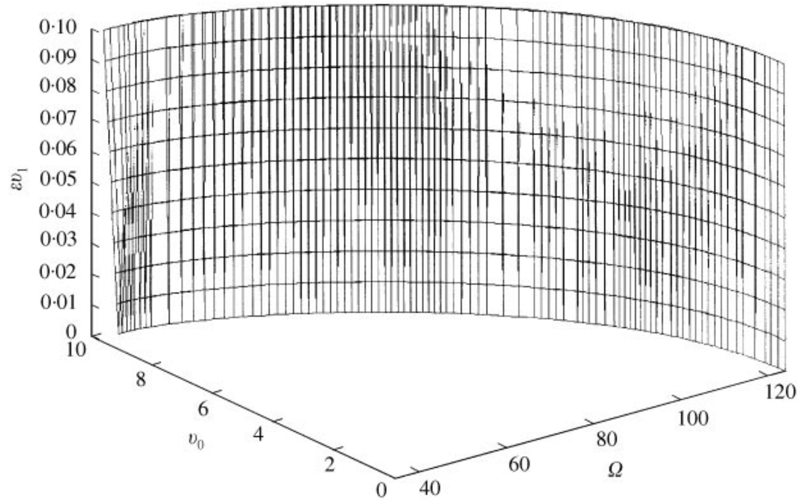

Figure 6. Stable and unstable regions for principal parametric resonances for the second mode ($n = 2$, $v_f = 1{\cdot}0$).

flexural stiffness values for the first two modes. The natural frequencies increase with increasing flexural stiffness and decrease with increasing mean velocity. Divergence instability occurs at the critical velocity. The natural frequencies for the fixed–fixed axially moving beam are higher than those of a simply supported one.

The analysis is valid for the mean value of axial velocity from zero to the critical velocity. Stability boundaries are calculated for the principal parametric resonance cases. For velocity fluctuation frequencies nearly twice any natural frequency, an instability region occurs. For frequencies close to zero, no instabilities are detected up to the first order of approximation. The stability regions are plotted. The beam effects cause the stability boundaries to shift to higher frequency values and increasing the velocity variation amplitude causes the stability regions to be wider. The values for the axially moving beam on fixed supports are higher than the values of a simply supported travelling beam.

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