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WEAK NON-LINEARITY EFFECT ON STOCHASTIC PARAMETRIC RESONANCE

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The question of stability is one of the basic questions in the theory of oscillations. The influence of external forces or parametric perturbation of the system may lead to different resonant phenomena [1, 2]. If the parameters of the system are fluctuating, the corresponding resonance is called a stochastic parametric resonance (SPR) [2–4]. It has to be distinguished from the phenomenon described in reference [5], where external forces cause resonant transitions between the neighboring potential wells. SPR manifests itself in the increase of the moments of higher orders with time, while the mean values of the system remain finite. Then the question of stability of the corresponding oscillations is an important one in the investigation of different technical devices. It is known that relaxation (linear friction) in the system leads to the appearance of the lower boundary for the value of fluctuation intensity necessary for a rise of SPR [3, 4]. Non-linear friction (velocity times the square of the co-ordinate) stabilizes SPR [6, 7], that can be easily understood by simple consideration of energy conservation.

It is of particular interest to consider the effect of weak non-linearity on SPR, because due to the increase of fluctuations in a real system there appears interaction between different harmonics, described by weak non-linearity. Non-linear oscillations under deterministic and stochastic excitations were studied recently [1]. We, in distinction from reference [1], consider the following equation with a fluctuating frequency:

$$\frac{d^2x}{dt^2} + \omega_0^2(1 + z(t))x + \lambda x^3 = 0, \tag{1}$$

where $z(t)$ – is a Gaussian process with zero mean and small intensity σ^2 ($\sigma^2 \ll 1$), $\lambda \ll \omega_0^2 \langle x^2 \rangle^{-1}$ (generalization to non-Gaussian processes will be considered elsewhere).

Non-linearity leads to a shift of the basic frequency of oscillations. The latter causes the change in the order of resonance. Many technical devices have a finite bandwidth. It is therefore interesting to consider a random process $z(t)$ with the following correlation:

$$\langle z(t)z(t) \rangle = \sigma^2 \frac{\sin \omega(t - t)}{\omega(t - t)}, \tag{2}$$

coresponding to a rectangular spectrum with an edge at frequency ω.

If the double frequency of the system $2\omega_0$ is much less than the edge of the device spectrum ω, then its change due to a weak non-linearity cannot lead to stabilization of SPR.

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This case actually corresponds to a white-noise process, when for any system frequency there exists a corresponding harmonic of $z(t)$ excluding the possibility of stabilization. In the other limiting case ($2\omega_0 \gg \omega$) there is no resonance. Therefore, the most essential influence of weak non-linearity on a system takes place, when a shift $\varepsilon = \omega - 2\omega_0$ between the spectrum edge and the double frequency of the system has a small value of the same order as non-linearity $\varepsilon\omega_0 \sim \lambda\langle x^2\rangle$. This case is considered below.

Using equation (1) one obtains equations for the moments of the second order:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x^2\rangle = 2\langle xy\rangle, \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle xy\rangle = \langle y^2\rangle - \omega_0^2\langle x^2\rangle - \lambda\langle x^4\rangle - \omega_0^2\langle z(t)x^2\rangle, \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle y^2\rangle = -2\omega_0^2\langle xy\rangle - 2\lambda\langle x^3y\rangle - 2\omega_0^2\langle z(t)xy\rangle, \tag{5}$$

where $y = \dot{x}$. To close this system of equations, we make use of Furutsu–Novikov formulae [2], $\langle z(t)x^2\rangle$ and present as follows:

$$\langle z(t)x^2\rangle = 2\int_0^t \sigma^2 \frac{\sin[(2\omega_0+\varepsilon)(t-\tau)]}{(2\omega_0+\varepsilon)(t-\tau)} < \frac{\delta x(t)}{\delta z(\tau)}x(t) > \mathrm{d}\tau. \tag{6}$$

The variational derivative in equation (4) is calculated by the standard procedure [2] from equation (1) (neglecting λ and σ terms)

$$\frac{\delta x(t)}{\delta z(\tau)} = \omega_0 x(\tau)\sin[\omega_0(\tau - t)], \quad \frac{\delta y(t)}{\delta z(\tau)} = -\omega_0^2 x(\tau)\cos[\omega_0(\tau-t)]. \tag{7}$$

In the adiabatic approximation $x(\tau)$ can be expressed in terms of $x(t)$ and $y(t)$

$$x(\tau) = x(t)\cos[\omega_0(\tau-t)+\phi] + (1/\omega_0)y(t)\sin[\omega_0(\tau-t)+\phi], \tag{8}$$

where $\phi = \phi(\tau - t)$ is a slow varying phase ($\dot{\phi} \sim \varepsilon$) due to non-linearity, $\phi(0) = 0$.

By the use of equations (7), (8) and of Gaussian approximation $\langle x^4\rangle = 3\langle x^2\rangle$, the system of equations (3)–(5) is replaced by the third order differential equation

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}\langle x^2\rangle + 4\omega_0^2 F_2(t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle x^2\rangle + 4\omega_0^2\left\{1 + \omega_0 F_1(t) + \frac{1}{2}\dot{F}_2(t)\right\}\frac{\mathrm{d}}{\mathrm{d}t}\langle x^2\rangle + 18\lambda\langle x^2\rangle$$
$$+ 4\omega_0^3\langle x^2\rangle\{\dot{F}_1(t) + \omega F_2(t) - \omega_0 F_3(t)\} = 0, \tag{9}$$

where

$$F_1(t) = -\sigma^2\int_0^t \frac{\sin[(2\omega_0+\varepsilon)\tau]\sin\omega_0\tau\cos[\omega_0\tau+\phi]}{(2\omega_0+\varepsilon)\tau}\mathrm{d}\tau, \tag{10}$$

$$F_{21}(t) = \sigma^2\int_0^t \frac{\sin[(2\omega_0+\varepsilon)\tau]\sin\omega_0\tau\sin[\omega_0\tau+\phi]}{(2\omega_0+\varepsilon)\tau}\mathrm{d}\tau, \tag{11}$$

$$F_3(t) = \sigma^2\int_0^t \frac{\sin[(2\omega_0+\varepsilon)\tau]\cos\omega_0\tau\cos[\omega_0\tau+\phi]}{(2\omega_0+\varepsilon)\tau}\mathrm{d}\tau. \tag{12}$$

Assuming that in the absence of non-linearity and fluctuations, the system oscillates harmonically, we are looking for a solution of a perturbed equation (9) in the form

$$\langle x^2 \rangle = C_1 + C_2 \cos(2\omega_0 t + 2\phi) + \sigma^2 x_1(t), \tag{13}$$

where C_1, C_2 and ϕ are slow functions of time ($\dot{C}_1/C_1$, $\dot{C}_2/C_2$, $\dot{\phi}/\phi \sim O(\sigma^2\omega_0)$), and $x_1(t)$ is a correction due to a weak non-linearity and time dependence of frequency. This expression corresponds to a standard Bogolubov–Krylov procedure [8, 9]. The condition of absence of resonant terms (periodic with frequency $2\omega_0$) in the external force in equation for $x_1(t)$ leads to the following equations:

$$2\dot{C}_2 + 3\omega_0^2 F_2 C_2 + \omega_0^2 F_3 C_2 - \omega_0 \dot{F}_1 = 0, \tag{14}$$

$$4\dot{\phi} - 2\omega_0^2 F_1 - \omega_0 \dot{F}_2 - 9\lambda\omega_0^{-1} C_1 = 0, \tag{15}$$

$$\dot{C}_1 + \omega_0 \dot{F}_1 C_1 + \omega_0^2 (F_2 - F_3) C_1 = 0. \tag{16}$$

The equations above describe the solution of equation (9) adequately under conditions $C_{1,2} \ll \omega_0 \dot{C}_{1,2}, \ddot{\phi} \ll \omega_0 \dot{\phi}$, which are satisfied if $\dot{F}_i \sim o(\sigma^2)$. This implies limitations for times at which equations (9)–(11) are justified. The definition of $F_{1,2,3}$ leads to the following condition $(1/2)\omega_0^{-1} \ll t \lesssim (\sigma^2\omega_0)^{-1}$. As it is shown below, this time domain is of interest for the stabilization of SPR.

Using equation (11) one finds

$$\begin{aligned} C_1 = C_{10} \exp\{&(1/4)\omega_0\sigma^2[t\{\mathrm{Si}(4\omega_0 t) - \mathrm{Si}((\dot{\phi} - \varepsilon)t)\} + (1/4\omega_0)(\cos 4\omega_0 t - 1) \\ &- (\dot{\phi} - \varepsilon)^{-1}(\cos[(\dot{\phi} - \varepsilon)t], \end{aligned} \tag{17}$$

where $C_{10} \equiv C_1(t = 0)$, $\mathrm{Si}(x)$ is an integral sinus, and $\dot{\phi}$ is found from equation (15). If $\dot{\phi} > \varepsilon$, then the stabilization of SPR takes place at times $t \sim (\sigma^2\omega_0)^{-1}$. Indeed, it is obvious from equation (17), that the maximal value of the expression in the exponent is achieved (up to $\ddot{\phi}$ terms) when both integral sinuses saturate and become equal. It means that for all allowed times C_1 is less or of the order

$$C_{10} \exp\left\{\frac{\sigma^2\omega_0}{2(\dot{\phi} - \varepsilon)}\right\}. \tag{18}$$

Therefore, we just need to satisfy the condition $\dot{\phi} - \varepsilon \gtrsim \sigma^2\omega_0$ (necessary for the saturation of the second integral sinus in equation (12). As it is easy to show from equation (15), it is satisfied for

$$\lambda \gtrsim (4/9)\omega_0^2 C_{10}^{-1}(\sigma^2 + \varepsilon\omega_0^{-1} + (1/16)\sigma^2 \ln(4\sigma^{-2})). \tag{19}$$

It follows from equation (14) that C_2 decays with time as

$$C_2 = C_{20} \exp\{-(1/2)\omega_0 \int (3F_2(\tau) + F_3(\tau))\,\mathrm{d}\tau\}, \tag{20}$$

$F_{2,3}$ being positive by definition at any t.

We, therefore, conclude that the weak non-linearity could lead to stabilization of SPR, if double eigenfrequency of the system $2\omega_0$ is close to the spectrum edge of the fluctuations ω. The value of non-linearity λ, necessary for stabilization, is proportional to the frequency shift $\varepsilon = \omega - 2\omega_0$ and to the intensity of fluctuations σ^2.

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