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PROPER ORTHOGONAL DECOMPOSITION OF TURBULENT FLOWS FOR AEROACOUSTIC AND HYDROACOUSTIC APPLICATIONS

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This paper describes the use of proper orthogonal decomposition (POD) for problems in which sound radiation is caused by turbulent flow interacting with rigid surfaces. For a complete solution the statistics of the turbulent velocity fluctuations must be defined, and it is customary to base these on spectral analysis. This paper will describe how proper orthogonal modes provide a better approach than spectral analysis for inhomogeneous turbulent flows. One of the additional advantages of POD is that sound radiation problems may be analyzed in the time domain and this may prove useful in rotor noise applications where time-domain calculations are frequently used. For flows that are stationary in time or homogeneous in at least one direction, it will be shown that there are advantages in using a combination of POD and a linear stochastic estimator to describe the flow. This provides a decomposition of the two-point cross-correlation function of the turbulence that requires fewer modes than POD.

1. INTRODUCTION

It is customary to specify turbulent flows in terms of their time-averaged statistics such as the time-averaged turbulent stresses, the cross-correlation function or the cross-spectrum. An alternative approach is to use proper orthogonal decomposition (POD) [1] in which the flow is defined using a set of modes with time-varying amplitudes. It was demonstrated by Lumley [1] that the optimum modes were orthogonal and had amplitudes which were uncorrelated with each other. POD requires the specification of the two-point cross-correlation function throughout the flow, which is very time consuming to measure with devices such as hot-wire probes. Most of the interest in POD has therefore been from the theoretical viewpoint [2]. However, the development of direct numerical simulations (DNS) and laser light sheet measurements (PIV) has provided capabilities to specify turbulent flow fields in the detail required for POD and this has lead to renewed interest in the method [3–5]. The purpose of this paper is to consider how POD may be applied to typical problems found in aero- or hydroacoustics, given the data set required to define the modes.

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Proper orthogonal decomposition is optimal because the modal expansion requires the minimum number of modes to describe the flow. For homogeneous flows the optimum modes are harmonic, and so the problem is reduced to classical Fourier analysis. For inhomogeneous flows POD provides a rigorous method for reducing the data set to its simplest description in the spatial domain. Applying this approach to problems in aeroacoustics, such as the noise radiated from a strut in a turbulent flow (see Figure 1), enables the radiated noise from the flow structure interaction to be calculated directly. The contribution of each mode to the sound field can then be identified. For homogeneous flows this offers little new insight over and above the existing, well-known, Fourier analysis techniques. However, time-domain calculations for stochastic flows can be defined in a rigorous fashion using POD, as will be discussed in section 3.

Homogeneous turbulent flows require a large number of wavenumber components to describe completely all the appropriate scales, and so for flows which are homogeneous in at least one direction, the number of proper orthogonal modes (POMs) required to describe the flow may be very large. An alternative to POD is linear stochastic estimation (LSE) [6], which seeks to minimize the error between the turbulent fluctuations and the terms in a modal expansion. The modes obtained from LSE are time-averaged representations of the flow, and differ from POMs because they cannot be used to reconstruct the time history of the flow. However, as will be discussed in section 4, they can be used in combination with POD to decompose the cross-correlation function into a set of uncorrelated modes. This new decomposition approach leads to the definition of compact eddy structures (CES) which are obtained from applying POD in the inhomogeneous directions of the flow, and LSE in the homogeneous directions. The CES can be used to calculate the response of a structure to the incoming flow, in the same way as POMs, and the response can be associated directly with particular structures that have clearly identifiable characteristics. This will be illustrated in section 5 by considering the noise radiated by a turbulent wake of finite width incident on a stationary airfoil. It will be shown how each mode couples with the sound field, and interestingly, how only a very limited number of modes are responsible for the sound radiation.

2. PROPER ORTHOGONAL DECOMPOSITION

We will start by considering the problem of calculating the surface pressure fluctuations on an airfoil induced by an unsteady inflow. We will assume that the linearized equations of motion may be used, which is a valid assumption for thin airfoils at high Reynolds numbers [7]. We will also assume that the surface pressure at the location $\mathbf{x}$ may be calculated provided that the flow is specified upstream of the computational domain shown in Figure 1.

Since the problem is assumed linear, a solution can be obtained by considering an inflow disturbance that is harmonic in space and time, and by superimposing the results for each wavenumber and frequency to give the complete solution. Consequently, if a vortical inflow disturbance $w_j \exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{y} - \mathrm{i}\omega t)$ is defined far upstream of the airfoil, the surface pressure can be specified in the form

$$P(\mathbf{x}, t) = w_j g_j(\mathbf{x}, \mathbf{k}, \omega) \exp(-\mathrm{i}\omega t), \tag{1}$$

where g_j is the response of the system to the jth component of the incoming gust. For a general inflow disturbances $u_j(\mathbf{y}, t)$, we define w_j using a space–time Fourier

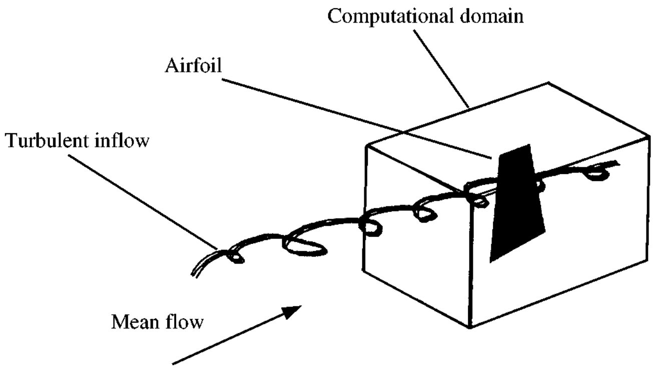

Figure 1. A typical problem in aeroacoustics: an airfoil excited by a turbulent inflow.

transform so that

$$w_j(\mathbf{k}, \omega) = \frac{1}{(2\pi)^4} \int_{-T}^{T} \int_V u_j(\mathbf{y}, t) \mathrm{e}^{\mathrm{i}\omega t + \mathrm{i}\mathbf{k}\cdot\mathbf{y}} \,\mathrm{d}t\,\mathrm{d}V = \frac{1}{(2\pi)^3} \int_V u_j(\mathbf{y}, \omega) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{y}} \,\mathrm{d}V$$

and

$$P(\mathbf{x}, t) = \int_{-\infty}^{\infty} \int_{\mathbf{k}} w_j(\mathbf{k}, \omega) g_j(\mathbf{x}, \mathbf{k}, \omega) \exp(-\mathrm{i}\omega t) \,\mathrm{d}\mathbf{k}\,\mathrm{d}\omega, \tag{2}$$

where V is defined as a large volume upstream unaffected by the presence of the airfoil. This is a modelling requirement that is necessary to define the response of the system to the flow disturbance. This requirement could also be met by defining the flow through the volume of interest, but without the airfoil being present. If the inflow is turbulent we can only define its mean or average statistics and so we are primarily interested in the cross-power spectrum of the surface pressure fluctuations at $\mathbf{x}$ and $\mathbf{x}'$ in the form

$$\begin{aligned} C_{PP}(\mathbf{x}, \mathbf{x}', \omega) &= (\pi/T) Ex[P(\mathbf{x}, \omega) P^*(\mathbf{x}', \omega)] \\ &= \int_{\mathbf{k}'} \int_{\mathbf{k}} \frac{\pi}{T} Ex[w_i(\mathbf{k}, \omega) w_j^*(\mathbf{k}', \omega)] g_i(\mathbf{x}, \mathbf{k}, \omega) g_j^*(\mathbf{x}', \mathbf{k}', \omega) \,\mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{k}'. \end{aligned} \tag{3}$$

This basic result shows that the surface pressure spectrum is defined in terms of the wavenumber transforms of the inflow disturbance. This can be related to the cross-spectrum of the velocity fluctuations by

$$\frac{\pi}{T} Ex[w_i(\mathbf{k}, \omega) w_j^*(\mathbf{k}', \omega)] = \frac{1}{(2\pi)^6} \int_V \int_{V'} C_{ij}(\mathbf{y}, \mathbf{y}', \omega) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{y}'} \mathrm{d}V\,\mathrm{d}V', \tag{4}$$

where

$$C_{ij}(\mathbf{y}, \mathbf{y}', \omega) = (\pi/T) Ex[u_i(\mathbf{y}, \omega) u_j^*(\mathbf{y}', \omega)].$$

A double-volume integral is therefore required to specify the random inflow to the computational domain. Furthermore, the interpretation of equation (4) is far from obvious and it is not clear how the features of the flow are coupled to the surface pressure fluctuations. Inevitably, simplifying assumptions have been used to help with the interpretation of this problem and the most useful of these is to assume a homogeneous turbulent flow so that the cross spectrum is a function of $\mathbf{y}-\mathbf{y}'$ only. Then equation (4) yields

$$\frac{\pi}{T} Ex[w_i(\mathbf{k}, \omega) w_j^*(\mathbf{k}', \omega)] = \frac{1}{(2\pi)^3} \int_V C_{ij}(\mathbf{y} - \mathbf{y}', \omega) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{y}-\mathbf{y}')} \, \mathrm{d}V \, \delta(\mathbf{k} - \mathbf{k}')$$

$$= \mathbf{\Phi}_{ij}(\mathbf{k}, \omega) \delta(\mathbf{k} - \mathbf{k}') \tag{5}$$

so that

$$C_{PP}(\mathbf{x}, \mathbf{x}', \omega) = \int_{\mathbf{k}} \mathbf{\Phi}_{ij}(\mathbf{k}, \omega) g_j(\mathbf{x}, \mathbf{k}, \omega) g_j^*(\mathbf{x}', \mathbf{k}, \omega) \, \mathrm{d}\mathbf{k}. \tag{6}$$

In this representation the surface pressure is defined in terms of the wavenumber energy spectrum $\mathbf{\Phi}(\mathbf{k}, \omega)$ of the turbulent flow. Various models are available for this function which allow it to be specified in terms of a turbulence intensity and integral length scale (or for anisotropic turbulence, lengthscales in each orthogonal direction). This simplifies flow measurements by an order of magnitude. The wavenumber spectrum model also allows the integral in equation (6) to be carried out analytically and so a closed-form solution can be obtained [6]. While this leads to a relatively attractive result, it imposes a very severe condition on the description of the inflow, namely that it should be homogeneous. Unfortunately, this assumption is unrealistic for almost all flows of interest, and, unless conditions such as "local homogeneity" can be applied, the full wavenumber integral defined in equation (4) must be used to evaluate equation (3).

An alternative approach is to use a modal expansion of the inflow of the type originally proposed by Lumley [1]. The concept is to expand the unsteady inflow velocity as a set of orthogonal modes in the form

$$u_i(\mathbf{y}, \omega) = \sum_n a_n(\omega) \phi_i^{(n)}(\mathbf{y}, \omega). \tag{7}$$

It is shown by Lumley (see Appendix A) that the optimal set of modes is obtained by optimizing the projection of $u_i(\mathbf{y}, \omega)$ onto the mode function $\phi_i^{(n)}$. This leads to the definition of the modes in terms of the eigenvalue problem

$$\frac{1}{V'} \int_{V'} C_{ij}(\mathbf{x}, \mathbf{x}', \omega) \phi_j^{(n)}(\mathbf{x}', \omega) \, \mathrm{d}V' = \frac{\pi}{T} \lambda_\omega^{(n)} \phi_i^{(n)}(\mathbf{x}, \omega), \tag{8}$$

where $\lambda_\omega^{(n)}$ are the eigenvalues of the nth mode at the frequency ω. Because the cross-spectrum function is symmetric we can use spectral theory to show that the eigenvalues are real, the modes are orthogonal (see equation (A.7)), and each mode is uncorrelated (see

Appendix A) so that

$$Ex[a_n(\omega)a_m^*(\omega)] = \lambda_\omega^{(n)}\delta_{mn} \tag{9}$$

$$\mathrm{C}_{ij}(\mathbf{y}, \mathbf{y}', \omega) = \frac{\pi}{T}\sum_n \lambda_\omega^{(n)}\phi_j^{(n)}(\mathbf{y}, \omega)(\phi_j^{(n)}(\mathbf{y}', \omega))^*. \tag{10}$$

These results summarize the theory of proper orthogonal decomposition. They show a number of simple relationships for functions that would require multiple Fourier integrals to be evaluated if spectral analysis was used.

We will apply these results to the analysis of the gust/airfoil interaction problem. First, by taking Fourier transforms of equation (7) we find

$$Ex[w_i(\mathbf{k}, \omega)w_j^*(\mathbf{k}', \omega)] = \sum_n \lambda_\omega^{(n)}\{\phi_i^{(n)}(\mathbf{k}, \omega)\}\{\phi_j^{(n)}(\mathbf{k}', \omega)\}^*, \tag{11}$$

where functions in { } are Fourier transforms as defined in equation (2). Using the result in equation (3) we obtain

$$C_{PP}(\mathbf{x}, \mathbf{x}', \omega) = \sum_n \frac{\pi}{T}\lambda_\omega^{(n)}\left[\int_{\mathbf{k}}\{\phi_i^{(n)}(\mathbf{k}, \omega)\}g_i(\mathbf{x}, \mathbf{k}, \omega)\,\mathrm{d}\mathbf{k}\right]\left[\int_{\mathbf{k}}\{\phi_j^{(n)}(\mathbf{k}, \omega)\}g_j(\mathbf{x}', \mathbf{k}, \omega)\,\mathrm{d}\mathbf{k}\right]^*. \tag{12}$$

The advantage of this approach is that computations of the surface pressure can be carried out for each mode individually. The acoustic power spectrum will then be the independent sum of the mean square output from each modal calculation. Using the modal expansion (7) the flow has not been restricted in any sense and we can allow for inhomogeneous turbulence without difficulty. Furthermore, we can identify dominant modes and their coupling efficiency, and this may lead to a better understanding of the features of the inflow which affect the blade response.

The computation time for the evaluation of either equation (3) or equation (12) will typically be dominated by the calculation of the response functions $g_i(\mathbf{x}, \mathbf{k}, \omega)$. Consequently, the orthogonal modes do not necessarily offer major computational advantages if the calculation of the response function is carried out in the wavenumber domain. However, if the computations are carried out for each mode velocity vector $\phi_i^{(n)}(\mathbf{y}, \omega)$ and the response to this mode is defined as $P^{(n)}(\mathbf{x}, \omega)$ then, since the contribution from each mode is uncorrelated, the surface pressure spectrum is simply the linear sum of the modal response functions in the form

$$C_{PP}(\mathbf{x}, \mathbf{x}', \omega) = \sum_n \frac{\pi}{T}\lambda_\omega^{(n)}(\omega)P^{(n)}(\mathbf{x}, \omega)(P^{(n)}(\mathbf{x}', \omega))^*. \tag{13}$$

This clearly provides a major reduction in the computational effort required to solve this problem and, since the modal description is optimal, the number of terms required to be evaluated in equation (13) will be minimized, providing the most efficient computational approach.

The optimal set of modes is obtained from the solution of the eigenvalue problem defined in equation (8). This requires that the cross-correlation function be specified everywhere in the inflow volume. It would appear therefore that the modal decomposition approach has not provided any reduction in detail required for the description (or measurement) of the flow. However, the modes provide a more rigorous basis on which to interpret the flow. Furthermore, the number of modes required to describe the flow [4] can be significantly less than the number of wavenumber components required to define the same flow and so computational savings in evaluating equations (12) or (13) rather than equation (3) may be significant.

3. TIME-DOMAIN APPLICATION

The analysis in the previous section applies to linear problems that can be considered in the frequency domain. For non-linear problems, or problems which are more amenable to time-domain formulations, this approach does not apply. Also, the formulation in the frequency domain requires a modal decomposition to be carried out for each frequency which results in a very non-compact description of the unsteady flow. However, there is no reason why proper orthogonal decomposition cannot be applied in four dimensions, giving both time- and space-dependent modes. The time-domain representation of the unsteady velocity is given by the expansion

$$u_i(\mathbf{y}, t) = \sum_n b_n(t_R)\psi_i^{(n)}(\mathbf{y}, t), \tag{14}$$

where $b_n(t_R)$ are a set of uncorrelated stochastic random variables which describe the amplitude of the modes and are evaluated for a given realization specified by the reference time t_R. The time-domain modes can be obtained from the solution to the eigenvalue problem (see Appendix A)

$$\frac{1}{2TV'}\int_{-T}^{T}\int_{V'} R_{ij}(\mathbf{y}, \mathbf{y}', t, t')\psi_j^{(n)}(\mathbf{y}', t')\,\mathrm{d}V'\,\mathrm{d}t' = \lambda_t^{(n)}\psi_i^{(n)}(\mathbf{y}, t). \tag{15}$$

This requires the complete space–time correlation function for the entire flow over all time. The detail required to invert equation (15) in four dimensions is large, but for non-stationary problems this is the only approach which is available since time histories in the input volume are an important feature of the results.

Another application for time-domain modes is the calculation rotor noise. Very accurate time-domain methods [8] have been developed for this by using a direct evaluation of the Ffowcs Williams and Hawkings equation. The loading noise component of the radiated noise is obtained from formulae of the type

$$p(\mathbf{x}, t) = -\frac{\partial}{\partial x_i}\int_S \left[\frac{P(\mathbf{y}, \tau)n_i}{4\pi r(1 - M_r)}\right]_{\tau = t - r/c_0} \mathrm{d}S, \tag{16}$$

where p is the radiated acoustic pressure, P is the blade surface pressure, $r = |\mathbf{x} - \mathbf{y}|$, n_i is the blade surface normal and M_r the convection Mach number. Accurate calculations of the blade surface pressure are obtained from direct computations and the acoustic field is then calculated from equation (16) or its variations. This approach has been used successfully for sound radiation from steady loadings and blade vortex interactions for which the inflow is specified precisely. However, it is hard to apply in the time domain for stochastic variations of the surface pressure. This problem is overcome by using proper orthogonal decomposition. Formally, the surface pressure is defined using a set of proper orthogonal modes $P^{(n)}$ which are the solutions to the eigenvalue problem

$$\frac{1}{2TS'}\int_{-T}^{T}\int_{S'} R_{PP}(\mathbf{y}, \mathbf{y}', t, t')P^{(n)}(\mathbf{y}', t')\,\mathrm{d}S'\,\mathrm{d}t' = \lambda_P^{(n)}P^{(n)}(\mathbf{y}, t). \tag{17}$$

Then, by using $P^{(n)}$ in equation (16), we can calculate the contribution of each mode to the farfield sound giving an acoustic field $p^{(n)}(\mathbf{x}, t)$. It is customary to calculate the farfield sound pressure spectrum for broadband rotor-noise sources and this can be obtained by taking the Fourier transform of $p^{(n)}(\mathbf{x}, t)$ with respect to time and summing the modal contributions using

$$S_{pp}(\mathbf{x}, \omega) = \frac{\pi}{T}\sum_n \lambda_P^{(n)}|p^{(n)}(\mathbf{x}, \omega)|^2. \tag{18}$$

The advantage of this approach is that broadband noise calculations can be carried out using the very accurate time-domain methods which have been developed for rotor noise calculations [9]. There are obvious limitations, for example the modal decomposition is not trivial and calculations have to be carried out for each mode. However, the advantage is that proper orthogonal decomposition provides a formal approach for the time-domain stochastic calculations.

4. COMPACT EDDY STRUCTURES

For time stationary flows and/or flows which are homogeneous in at least one direction, proper orthogonal modes are not necessarily the best description to use. The modes in the homogeneous directions are Fourier modes, and so, if the flow is only locally correlated, large numbers of modes will result. A more compact description can be obtained by using compact eddy structures (CES) as described in Appendix B. These give the best estimate of the flow as a function of displacement in time and/or space in the homogeneous directions, given the proper orthogonal modes in the inhomogeneous directions. For a flow that is stationary in time, these modes are defined using the cross-correlation of the velocity fluctuations as

$$\kappa_i^{(n)}(\mathbf{y}, \tau) = \frac{1}{V'\lambda_0^{(n)}}\int_{V'} R_{ij}(\mathbf{y}, \mathbf{y}', \tau)\phi_j^{(n)}(\mathbf{y}')\,\mathrm{d}V', \tag{19}$$

where $\phi_i^{(n)}$ are the modes obtained at a fixed time by solving the eigenvalue problem

$$\frac{1}{V'}\int_{V'} R_{ij}(\mathbf{y}, \mathbf{y}', 0)\phi_j^{(n)}(\mathbf{y}')\,\mathrm{d}V' = \lambda_0^{(n)}\phi_i^{(n)}(\mathbf{y}). \tag{20}$$

The CES cannot be used to reconstruct the flow in the same way as the proper orthogonal modes, but they can be used to reconstruct the ensemble averages of the flow. For example, the cross-spectrum of the velocity fluctuations can be obtained as

$$C_{ij}(\mathbf{y}, \mathbf{y}', \omega) = \sum_n \lambda_0^{(n)}\kappa_j^{(n)}(\mathbf{y}', \omega)\phi_i^{(n)}(\mathbf{y}). \tag{21}$$

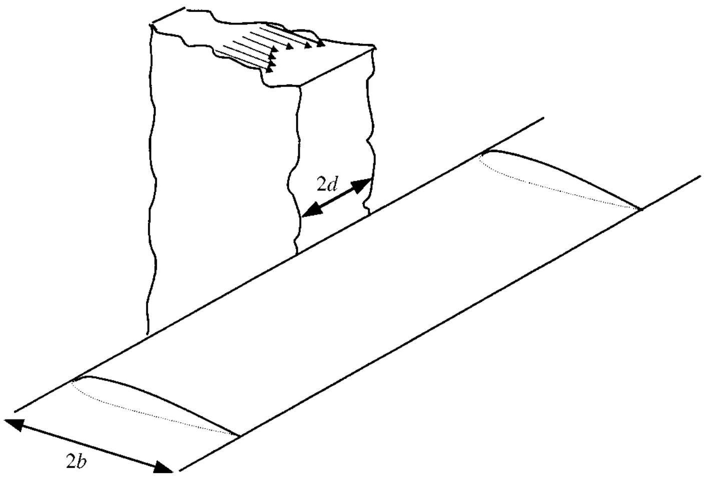

Figure 2. Sound radiation from a wake of width $2d$ incident on an airfoil of infinite span and chord $2b$.

Using this in equations (3) and (4) we obtain a result which is equivalent to equation (12) which is

$$C_{PP}(\mathbf{x}, \mathbf{x}', \omega) = \sum_n \lambda_0^{(n)} \left[\int_{\mathbf{k}} \{\phi_i^{(n)}(\mathbf{k})\} g_i(\mathbf{x}, \mathbf{k}, \omega)\, \mathrm{d}\mathbf{k} \right] \left[\int_{\mathbf{k}} \{\kappa_j^{(n)}(\mathbf{k}, \omega)\} g_j(\mathbf{x}', \mathbf{k}, \omega)\, \mathrm{d}\mathbf{k} \right]^*. \quad (22)$$

Note that the CES are related to the ensemble average of the output. The eddies themselves do not necessarily occur in the flow as specific features, but they do represent average flow structures. The response of the blade to the averaged flow structures then gives the spectral average of the response.

5. WAKE–AIRFOIL INTERACTION

As an example of how these methods may be applied, we will consider the well-known problem of a turbulent wake interacting with an airfoil, modelled as a flat plate (see Figure 2). The wake is assumed to be at 90° to the span of the airfoil. We will also assume that the plate is of infinite span and chord $2b$, but that the gust has a spanwise width of $2d$. Since the flow is bounded it will be inhomogeneous in the spanwise direction, but may be considered homogeneous in the direction of the flow.

The radiated acoustic field is given by Amiet [7] as

$$p(\mathbf{x}, \omega) = \int_S \left[\frac{-\,\mathrm{i}\omega x_3 F(y_1, y_2, \omega)}{4\pi c_0 \sigma^2} \right] \mathrm{e}^{-\mathrm{i}\omega(M(x_1 - y_1) - \sigma)/c_0\beta^2 - \mathrm{i}\omega(x_1 y_1 + x_2 y_2 \beta^2)/c_0\beta^2\sigma}\, \mathrm{d}y_1\, \mathrm{d}y_2, \quad (23)$$

where $\sigma = (x_1^2 + \beta^2(x_2^2 + x_3^2))^{1/2}$ and $\beta^2 = 1 - M^2$. The function $F(\,)$ represents the pressure jump across the blade and the observer is located at $\mathbf{x} = (x_1, x_2, x_3)$. The pressure jump across the blade is calculated from the response of the blade to an upwash velocity $w(y_1, y_2)$ in the y_3-direction, and the gust is assumed to be convected with the mean flow. This gives

$$F(y_1, y_2, \omega) = 2\pi\rho_0 b \int w_f(\gamma_0, \upsilon) g(y_1, \gamma_0, \upsilon) \mathrm{e}^{-\mathrm{i}\upsilon y_2}\, \mathrm{d}\upsilon \quad (24)$$

with $\gamma_0 = -\omega/U$, and $w_f(\gamma_0, \upsilon)$ defined as the spatial Fourier transform of $w(y_1, y_2)$,

$$w_f(\gamma_0, \upsilon) = \frac{1}{(2\pi)^2}\int_S w(y_1, y_2)\mathrm{e}^{\mathrm{i}\gamma_0 y_1 + \mathrm{i}\upsilon y_2}\,\mathrm{d}y_1\,\mathrm{d}y_2.$$

Combining these results gives

$$p(\mathbf{x}, \omega) = (2\pi)^2\left[\frac{-\mathrm{i}\omega x_3 \rho_0 b g_f(\gamma_1, \gamma_0, \upsilon_0) w_f(\gamma_0, \upsilon_0)}{2c_0\sigma^2}\right]\mathrm{e}^{-\mathrm{i}\omega(Mx_1-\sigma)/c_0\beta^2}, \tag{25}$$

where $\upsilon_0 = \omega x_2/c_0\sigma$ and $\gamma_1 = -\omega x_1/c_0\beta_2\sigma + \omega M/c_0\beta$, and

$$g_f(\gamma_1, \gamma_0, \upsilon_0) = \frac{1}{2\pi}\int_{-R}^{R} g(y_1, \gamma_0, \upsilon_0)\mathrm{e}^{\mathrm{i}\gamma_1 y_1}\,\mathrm{d}y_1.$$

In this formulation the inflow gust is defined in the y_1-, y_2-plane and Taylor's hypothesis is assumed. Therefore, when a POD is used to describe the flow the decomposition need only be carried out in two dimensions. This is a simplification of the formulation given in Appendix A. The velocity field is defined as

$$u_i(y_1, y_2) = \sum_n a_n \phi_i^{(n)}(y_1, y_2),$$

$$\frac{1}{4RZ}\int_{-R}^{R}\int_{-Z}^{Z} R_{ij}(y_1, y_1', y_2, y_2')\phi_j^{(n)}(y_1', y_2')\,\mathrm{d}y_2' = \lambda_w^{(n)}\phi_i^{(n)}(y_1, y_2). \tag{26}$$

If the upwash is in the $i = 3$ direction then the radiated field has a power spectral density of

$$S_{PP}(\mathbf{x}, \omega) = \frac{\pi}{T}\sum_n \lambda_w^{(n)}\left|\frac{2\pi^2\omega x_3\rho_0 b g_f(\gamma_1, \gamma_0, \upsilon_0)\{\phi_3^{(n)}(\gamma_0, \upsilon_0)\}_f}{c_0\sigma^2}\right|^2, \tag{27}$$

where $\{\ \}_f$ represents the two-dimensional Fourier transform of the modes. Alternatively, since the flow is homogeneous in the x-direction we can use a compact-eddy structure approach to obtain in the same result. First, we calculate the POMs in the y_2-direction so the velocity can be expanded as the series

$$u_i(y_1, y_2) = \sum_n b_n(y_1)\phi_i^{(n)}(y_2), \quad \frac{1}{2Z}\int_{-Z}^{Z} R_{ij}(y_2, y_2')\phi_j^{(n)}(y_2')\,\mathrm{d}y_2' = \lambda_c^{(n)}\phi_i^{(n)}(y_2) \tag{28}$$

with eigenvalues $\lambda_c^{(n)}$ and then we calculate the CES as

$$\kappa_i^{(n)}(y_1 - y_1', y_2) = \frac{1}{2Z\lambda_c^{(n)}}\int_{-Z}^{Z} R_{ij}(y_1 - y_1', y_2, y_2')\phi_j^{(n)}(y_2')\,\mathrm{d}y_2' \tag{29}$$

so that the radiated sound power spectral density is given as

$$S_{PP}(\mathbf{x}, \omega) = \frac{\pi}{T}\sum_n \lambda_c^{(n)}\left|\frac{2\pi^2\omega z\rho_0 b g_f(\gamma_1, \gamma_0, \upsilon_0)}{c_0\sigma^2}\right|^2 \{\kappa_3^{(n)}(\gamma_0, \upsilon_0)\}_f\{\phi_3^{(n)}(\upsilon_0)\}_f^*. \tag{30}$$

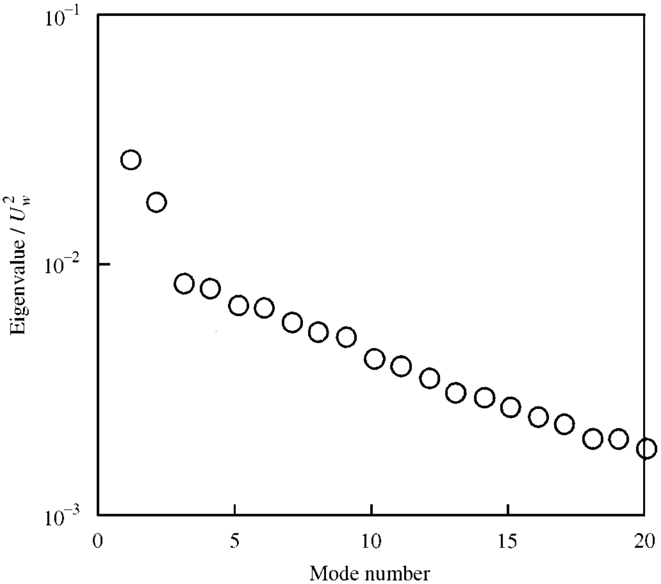

Figure 3. Eigenvalue spectra from the one-dimensional proper orthogonal decomposition in the y_2-direction. U_w is the axial mean velocity deficit at the wake centerline.

This relates the acoustic field directly to the CESs and hence provides some insight into the types of flow structures which radiate sound. To demonstrate this we will consider the results of a recent study [10] in which the CESs of a turbulent wake were measured. The eigenvalue spectrum for the one-dimensional modes given by equation (28) is shown in Figure 3. The contributions from the first two modes are dominant, but the amplitudes of the higher order modes decay quite slowly.

It was found that the first four modes were dominated by the velocity fluctuations in the y_1- and y_2-directions that did not affect the upwash incident on the airfoil. Only modes of orders 5, 6, 9, 10, etc. included upwash components and their CESs are shown in Figure 4. To demonstrate how each compact-eddy contributes to the sound field, Figure 5 shows a plot of $J_n = \lambda_c^{(n)} \{\kappa_3^{(n)}(\gamma_0, 0)\}_f \{\phi_3^{(n)}(0)\}_f^*$ as a function of frequency. This gives the contribution of each term in equation (30) to the radiated field in the direction normal to the airfoil.

Note that those compact eddies that are antisymmetric across the airfoil span do not contribute to the radiated field because the integral of the upwash across the span is zero (e.g., eddies 6 and 10 in Figure 4). This is the result of choosing the observation position to be directly above the intersection between the wake centerline and airfoil. If the observer were at any other angle, there would be a propagation delay across the span and even order modes would contribute. Consequently, Figure 5 only includes the contributions from the symmetric modes.

Figure 5 shows that all the contributing compact eddies produce the same spectral shape. This is an inevitable result of them having the same time dependence; see equation (19). The spectral levels, however, depend both on the eigenvalues, and the radiation efficiency, which vary substantially with mode number. Indeed, spectral levels fall off very rapidly with mode number suggesting that a reasonably accurate sound calculation could be performed using only a few, say 4, eddies. (Note that the function J_n given in Figure 5 does not include the effect of blade response, which will tend to increase the contribution from the higher

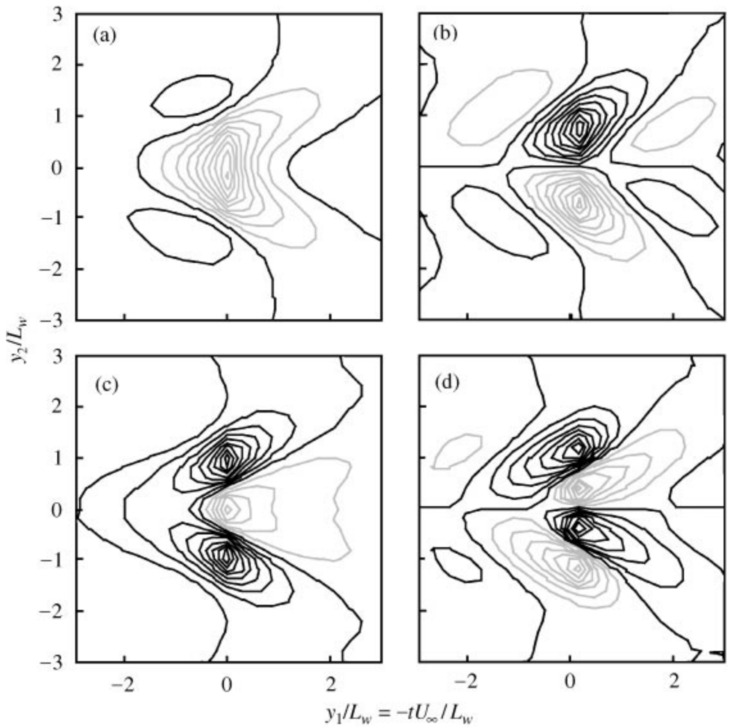

Figure 4. Contours of $\kappa_3^{(n)}(y_1, y_2)$ for the compact eddies corresponding to the first four non-zero W modes. Distances normalized on the half-wake width, L_w. Gray contours—negative values, Black contours—zero and positive values. (a) mode 5, (b) mode 6, (c) mode 9, (d) mode 10.

frequencies.) Furthermore, the dominant eddy (mode 5, Figure 4) has a particularly simple structure.

This example shows how compact eddy structures can provide both a rigorous and efficient basis for performing aeroacoustic calculations. Perhaps more importantly, though, it illustrates how the present approach can provide, at least in a time-average sense, a view of the specific types of fluid motions most responsible for the sound radiation. Such information may be valuable in providing physical understanding of the sound generation process itself and in formulating control strategies.

6. CONCLUSIONS

The objective of this paper has been to show how proper orthogonal decomposition can be used to solve problems in aeroacoustics where the complete description of a turbulent flow is required. It has been shown that the proper orthogonal decomposition of time stationary flows can be given in terms of compact eddy structures, which represent the time-average characteristics of the flow. The response of an airfoil and the subsequent sound radiation has been given in terms of the CESs and hence can be related to specific features of the flow. This is important because it provides a rigorous way in which to interpret turbulent flows and define turbulent flow “structures”. The energy in the structure is defined by the eigenvalue of the associated POM and each structure may be considered as uncorrelated. In previous analyses of the turbulence/airfoil interaction problem [7]

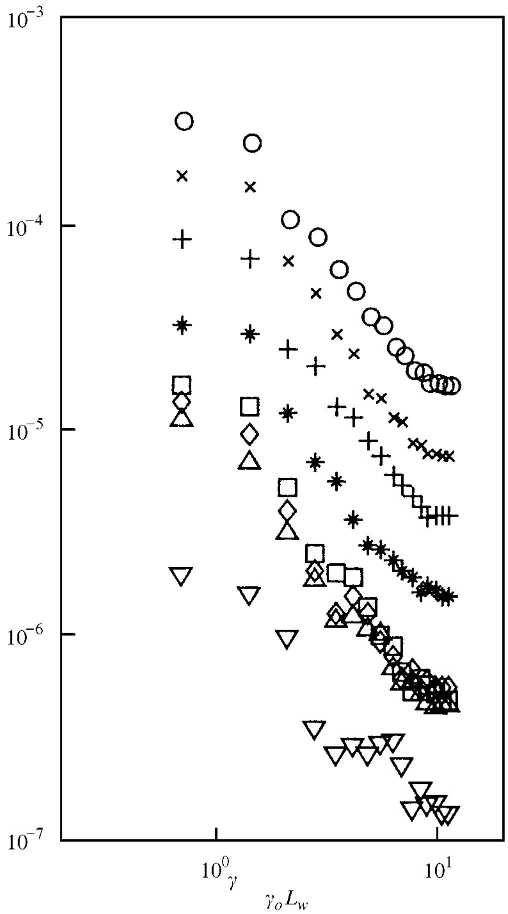

Figure 5. Spectra $|\lambda_c^{(n)}\kappa_3^{(n)}(\gamma_0, 0)\phi_3^{(n)}(0)|$ for compact eddies corresponding to the first eight non-zero upwash modes normalized on the half-wake width L_w and centerline axial mean velocity deficit: ○, mode 5; × mode 9; ∗, mode 13; +, mode 18; □, mode 24; ◇, mode 29; ▽, mode 34; △, mode 37.

restrictive assumptions were required to include the effect of finite span on the radiated sound. In the approach given here these assumptions are not required. To illustrate the application of this theory, results have been given which show the CES in a wake and the sound radiation from it's interaction with a downstream airfoil.

This paper is dedicated to Phil Doak for his 80th birthday. While I was a graduate student at the ISVR I took a course on Advanced Acoustics taught by Phil. In the course he showed what could be learned from attention to detail and sparked my imagination on the application of theoretical concepts to practical applications. His course has had a bigger effect on my subsequent career than any of the others which I took during my time as a student. Subsequently, I have enjoyed working with Phil as both a colleague and a friend, and have always been amazed at how he could correct a manuscript at the same time as holding a conversation and cleaning his pipe! One of the things that Phil always encouraged his students to do was to "dig around in text books for nuggets of gold". This paper, written with William Devenport, is the result of doing just that in reference [2]. Hope you enjoy it Phil, Happy Birthday.

SG

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APPENDIX A: THE THEORY OF PROPER ORTHOGONAL DECOMPOSITION

For completeness this appendix is included to summarize the theory of POD. For a more detailed discussion of this theory the reader is referred to References [1, 2].

The objective of proper orthogonal decomposition (POD) is to decompose an unsteady velocity field into the optimum set of normalized modes. For example, if we consider the Fourier transform with respect to time of an unsteady velocity field, then we can define a modal expansion in the form

$$u_i(\mathbf{y}, \omega) = \sum_n a_n(\omega)\phi_i^{(n)}(\mathbf{y}, \omega). \tag{A.1}$$

The optimal set of modes is obtained by optimizing the projections of $u_i(\mathbf{y}, \omega)$ onto the mode functions $\phi_i^{(n)}$, with a suitable normalization, by maximizing

$$\frac{Ex[|(1/V)\int_V u_i(\mathbf{y}, \omega)\phi_i^*(\mathbf{y}, \omega)\,\mathrm{d}V|^2]}{(1/V)\int_V \phi_i(\mathbf{y}, \omega)\phi_i^*(\mathbf{y}, \omega)\,\mathrm{d}V}, \tag{A.2}$$

where summation is implied over the repeated indices. To achieve this we define the functional

$$J(\alpha) = Ex\left[\left|\frac{1}{V}\int_V u_i(\mathbf{y},\omega)\mu_i^*(\mathbf{y},\omega)\,\mathrm{d}V\right|^2\right] - \lambda\left[\frac{1}{V}\int_V \mu_i(\mathbf{y},\omega)\mu_i^*(\mathbf{y},\omega)\,\mathrm{d}V - 1\right],$$

$$\mu_i(\mathbf{y},\omega) = \phi_i(\mathbf{y},\omega) + \alpha\psi_i(\mathbf{y},\omega). \tag{A.3}$$

The optimum modes are then obtained by maximizing the functional for any mode shape by evaluating $[\partial J/\partial\alpha]_{\alpha=0} = 0$, which gives

$$\frac{\partial J}{\partial\alpha} = 2\mathrm{Re}\left\{Ex\left[\frac{1}{VV'}\int_{V'} u_j^*(\mathbf{y}',\omega)\phi_j(\mathbf{y}',\omega)\,\mathrm{d}V'\int_V u_i(\mathbf{y},\omega)\psi_i^*(\mathbf{y},\omega)\,\mathrm{d}V\right]\right.$$

$$\left. - \frac{\lambda}{V}\int_V \phi_i(\mathbf{y},\omega)\psi_i^*(\mathbf{y},\omega)\,\mathrm{d}V\right\}$$

$$= 2\mathrm{Re}\left\{\frac{1}{V}\int_V \psi_i^*(\mathbf{y},\omega)\left[\frac{1}{V'}\int_{V'} Ex\langle u_i(\mathbf{y},\omega)u_j^*(\mathbf{y}',\omega)\rangle\phi_j(\mathbf{y}',\omega)\,\mathrm{d}V'\right.\right.$$

$$\left.\left. - \lambda\phi_i(\mathbf{y},\omega)\right]\mathrm{d}V\right\} = 0. \tag{A.4}$$

This is satisfied if the terms in the brackets [ ] are zero. Hence, the optimum modes are defined as the solution to the eigenvalue problem

$$\frac{1}{V'}\int_{V'} C_{ij}(\mathbf{y},\mathbf{y}',\omega)\phi_j^{(n)}(\mathbf{y}',\omega)\,\mathrm{d}V' = \frac{\pi}{T}\lambda_\omega^{(n)}\phi_i^{(n)}(\mathbf{y},\omega), \tag{A.5}$$

where $C_{ij}(\mathbf{y},\mathbf{y}',\omega) = (\pi/T)Ex[u_i(\mathbf{y},\omega)u_j^*(\mathbf{y}',\omega)]$ is the cross spectrum of the velocity fluctuations and $\lambda_\omega^{(n)}$ are the eigenvalues of the nth mode at the frequency ω. Because the cross-spectrum function is symmetric we can use spectral theory to show that the eigenvalues are real, and the modes are orthogonal. To verify this consider

$$\frac{\pi}{TV}\lambda_\omega^{(n)}\int_V \phi_i^{(n)}(\mathbf{y},\omega)(\phi_i^{(m)}(\mathbf{y},\omega))^*\,\mathrm{d}V = \frac{1}{VV'}\int_{V'}\int_V C_{ij}(\mathbf{y},\mathbf{y}',\omega)\phi_j^{(n)}(\mathbf{y}',\omega)(\phi_i^{(m)}(\mathbf{y},\omega))^*\,\mathrm{d}V\,\mathrm{d}V'$$

$$= \frac{1}{VV'}\int_{V'}\int_V C_{ji}^*(\mathbf{y}',\mathbf{y},\omega)\phi_j^{(n)}(\mathbf{y}',\omega)(\phi_i^{(m)}(\mathbf{y},\omega))^*\,\mathrm{d}V\,\mathrm{d}V'$$

$$= \frac{\pi}{TV'}(\lambda_\omega^{(m)})^*\int_{V'} \phi_j^{(n)}(\mathbf{y}',\omega)(\phi_j^{(m)}(\mathbf{y}',\omega))^*\,\mathrm{d}V'. \tag{A.6}$$

If the eigenvalues $\lambda_\omega^{(m)}$ are unique then the only possible solutions to this equation occur if the eigenvalues are real and

$$\frac{1}{V}\int_V \phi_i^{(n)}(\mathbf{y},\omega)(\phi_i^{(m)}(\mathbf{y},\omega))^*\,\mathrm{d}V = \delta_{mn}. \tag{A.7}$$

Hence, the modes are orthogonal when the eigenvalues are distinct. Spectral theory also shows that this is also true for eigenvalues that are duplicated. For example, if two or three eigenvalues are the same the associated eigenvectors can be arbitrarily rearranged into two or three orthogonal vectors.

Next, we will show that the modes are uncorrelated. If we combine equations (A.7) and (A.6) with equation (A.1) we find

$$\frac{1}{VV'}\int_{V'}\int_{V} C_{ij}(\mathbf{y},\mathbf{y}'\ \omega)\phi_j^{(n)}(\mathbf{y}',\omega)(\phi_i^{(m)}(\mathbf{y},\omega))^*\,\mathrm{d}V\,\mathrm{d}V' = \frac{\pi}{T}(\lambda_\omega^{(m)})\delta_{mn}$$

so

$$\frac{1}{VV'}\int_{V'}\int_{V}\frac{\pi}{T}Ex\left[\sum_k\sum_l a_k(\omega)a_l^*(\omega)\phi_i^{(k)}(\mathbf{y},\omega)(\phi_j^{(l)}(\mathbf{y}',\omega))^*\right]\phi_j^{(n)}(\mathbf{y}',\omega)(\phi_i^{(m)}(\mathbf{y},\omega))^*\,\mathrm{d}V\,\mathrm{d}V'$$
$$=\frac{\pi}{T}(\lambda_\omega^{(m)})\delta_{mn}.$$

Then, using the orthogonality condition, we find that

$$Ex[a_m(\omega)a_n^*(\omega)] = \lambda_\omega^{(m)}\delta_{mn}, \tag{A.8}$$

which proves that each mode is uncorrelated. Then it follows that

$$C_{ij}(\mathbf{y},\mathbf{y}',\omega) = \frac{\pi}{T}\sum_n \lambda_\omega^{(n)}\phi_i^{(n)}(\mathbf{y},\omega)(\phi_j^{(n)}(\mathbf{y}',\omega))^*. \tag{A.9}$$

For homogeneous turbulence the cross-correlation function is only dependent on the displacement vector $(\mathbf{y}-\mathbf{y}')$, in which case the proper orthogonal modes are Fourier modes [1] given by $A_i\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{y})$ where $A_i^2=1$. Hence, for homogeneous turbulence proper orthogonal decomposition is exactly the same as conventional wavenumber decomposition. The advantage of POD is that it is rigorous for both homogeneous and inhomogeneous flows.

In some problems we need to define the time history of the inflow and this needs a slightly different formulation. First, we will consider the spatial decomposition of the flow at a fixed time, say t_R. In this case, we choose the expansion

$$u_i(\mathbf{y},t_R) = \sum_n a_n(t_R)\phi_i^{(n)}(\mathbf{y}), \tag{A.10}$$

where the modes are now defined only as a function of position and $a_n(t_R)$ specifies a set of random stochastic coefficients which are associated with the time of evaluation, but are not necessarily related to the time history of the flow (since this would imply that the separation of time and space variables was valid for the flow). Using the same analysis as before, keeping t_R fixed, leads to the symmetric eigenvalue problem for the optimum modes in the form

$$\frac{1}{V'}\int_{V'} R_{ij}(\mathbf{y},\mathbf{y}',0)\phi_j^{(n)}(\mathbf{y}')\,\mathrm{d}V' = \lambda_0^{(n)}\phi_i^{(n)}(\mathbf{y}), \tag{A.11}$$

where $R_{ij}(\mathbf{y}, \mathbf{y}', 0)$ is the cross-correlation tensor at zero time delay. Here again we can use spectral theory to show that the modes are uncorrelated, and, because of the optimization procedure, the number of modes required to define the flow is a minimum.

To obtain a complete description of the flow, we also require the time history of the modes and this can be achieved by carrying out proper orthogonal decomposition in four dimensions. To illustrate this we use the modal expansion

$$u_i(\mathbf{y}, t) = \sum_n b_n(t_R)\psi_i^{(n)}(\mathbf{y}, t). \tag{A.12}$$

In this approach, the coefficient b_n are determined by a reference time, but should also be considered as a set of stochastic random variables. We then optimize the modes by minimizing their projection onto the velocity field using

$$\frac{Ex[|(1/2TV)\int_{-T}^{T}\int_V u_i(\mathbf{y}, t)\psi_i(\mathbf{y}, t)\,\mathrm{d}V\,\mathrm{d}t|^2]}{(1/2TV)\int_{-T}^{T}\int_V u_i(\mathbf{y}, t)\psi_i(\mathbf{y}, t)\,\mathrm{d}V\,\mathrm{d}t}. \tag{A.13}$$

Following the same procedure as before the optimal modes are found as the solution to the eigenvalue problem

$$\frac{1}{2TV'}\int_{-T}^{T}\int_V R_{ij}(\mathbf{x}, \mathbf{x}', t, t')\psi_j^{(n)}(\mathbf{x}', t')\,\mathrm{d}V'\,\mathrm{d}t' = \lambda_t^{(n)}\psi_i^{(n)}(\mathbf{x}, t). \tag{A.14}$$

To show that the time-varying modes are orthogonal consider

$$\begin{aligned}
&\lambda_t^{(n)}\frac{1}{2TV}\int_{-T}^{T}\int_V \psi_i^{(n)}(\mathbf{y}, t)\psi_i^{(m)}(\mathbf{y}, t)\,\mathrm{d}V\,\mathrm{d}t\\
&= \left(\frac{1}{2T}\right)^2\frac{1}{VV'}\int_{-T}^{T}\int_{-T}^{T}\int_{V'}\int_V R_{ij}(\mathbf{y}, \mathbf{y}', t, t')\psi_j^{(n)}(\mathbf{y}', t')\psi_i^{(m)}(\mathbf{y}, t)\,\mathrm{d}V\,\mathrm{d}t\,\mathrm{d}V'\,\mathrm{d}t'\\
&= \left(\frac{1}{2T}\right)^2\frac{1}{VV'}\int_{-T}^{T}\int_{-T}^{T}\int_{V'}\int_V R_{ji}(\bar{\mathbf{y}}', \bar{\mathbf{y}}, t', t)\psi_j^{(n)}(\mathbf{y}', t')\psi_i^{(m)}(\mathbf{y}, t)\,\mathrm{d}V\,\mathrm{d}t\,\mathrm{d}V'\,\mathrm{d}t'\\
&= \lambda_t^{(m)}\frac{1}{2TV'}\int_{-T}^{T}\int_V \psi_j^{(n)}(\mathbf{y}', t')\psi_j^{(m)}(\mathbf{y}, t')\,\mathrm{d}V'\,\mathrm{d}t'
\end{aligned} \tag{A.15}$$

and so, if the eigenvalues are unique, we obtain the orthogonality condition

$$\frac{1}{2TV}\int_{-T}^{T}\int_V \psi_i^{(n)}(\mathbf{y}, t)\psi_i^{(m)}(\mathbf{y}, t)\,\mathrm{d}V\,\mathrm{d}t = \delta_{mn}. \tag{A.16}$$

It then follows that the modes are uncorrelated and that

$$R_{ij}(\mathbf{y}, \mathbf{y}', t, t') = \sum_n \lambda_t^{(n)}\psi_i^{(n)}(\mathbf{y}, t)\psi_j^{(n)}(\mathbf{y}', t'). \tag{A.17}$$

For a stationary time series we expect Fourier modes [1]. To illustrate this consider the expansion

$$u_i(\mathbf{y}, t) = \sum_m \sum_{k=0}^{\infty} \psi_i^{(m)}(\mathbf{y})[b_{m,k}(t_R)\cos(k\pi t/T + \theta^{(m,k)}(\mathbf{y})) + c_{m,k}(t_R)\sin(k\pi t/T + \theta^{(m,k)}(\mathbf{y}))], \tag{A.18}$$

where $\psi_i^{(m)}(\mathbf{y})$ represent an orthogonal set of functions, so that each term in equation (A.18) satisfies the orthogonality condition given by equation (A.16). The expansion represents two modes with uncorrelated random amplitudes and, if these have the properties that

$$Ex[b_{m,k}(t_R)b_{m',k'}(t_R)] = Ex[c_{m,k}(t_R)c_{m',k'}(t_R)] = \lambda_t^{(m,k)}\delta_{mm'}\delta_{kk'},$$
$$Ex[b_{m,k}(t_R)c_{m',k'}(t_R)] = 0, \tag{A.19}$$

the correlation function becomes

$$R_{ij}(\mathbf{y}, \mathbf{y}', t, t') = \sum_m \sum_{k=0}^{\infty} \lambda_t^{(m,k)}\, \psi_i^{(m)}(\mathbf{y})\psi_j^{(m)}(\mathbf{y}')\cos(k\pi(t-t')/T + \theta^{(m,k)}(\mathbf{y}) - \theta^{(m,k)}(\mathbf{y}')), \tag{A.20}$$

which is a function of the time delay $t-t'$ only. This shows that for a stationary time series two modes with the same eigenvalues are required to represent the correlation function correctly. It follows that the same approach may be used in any direction where the correlation function is only dependent on the separation of the measurement points. Fourier modes are therefore the optimum modes to describe the flow in the homogeneous directions.

APPENDIX B: COMPACT EDDY STRUCTURES

One of the problems with proper orthogonal decompositions is that a large number of Fourier modes are required to describe flows which are homogeneous (or stationary in time) and compactly correlated (i.e., the correlation function decays rapidly as a function of the separation in time or space). This is not necessarily so for flows with inhomogeneous directions for which proper orthogonal decomposition provides the minimal number of modes needed to describe the flow. To investigate this we will consider expansions that use POD in the inhomogeneous directions, and an alternative formulation in the homogeneous directions. For simplicity, we will consider a time stationary flow so that by using equation (A.10), the flow can be described as

$$u_i(\mathbf{y}, t_R) = \sum_n a_n(t_R)\phi_i^{(n)}(\mathbf{y}). \tag{B.1}$$

This only describes the flow at the reference time t_R. To obtain the best estimate of the time history we choose a set of "compact eddies" defined by $\kappa_i(\mathbf{y}, \tau)$ which minimize the mean-square error

$$E = Ex\left[\left(u_i(\mathbf{y}, t) - \sum_n a_n(t_R)\kappa_i^{(n)}(\mathbf{y}, t - t_R)\right)^2\right]. \tag{B.2}$$

The optimum eddy functions are obtained by considering the functional

$$\mathrm{J}(\alpha) = \mathrm{Ex}\left[\left(u_i(\mathbf{y}, t) - \sum_n a_n(t_R)(\kappa_i^{(n)}(\mathbf{y}, t - t_R) + \alpha\xi_i^{(n)}(\mathbf{y}, t - t_R))\right)^2\right]. \quad \text{(B.3)}$$

The optimum modes are obtained from the minimum of the functional when $\alpha = 0$, so

$$\frac{\partial J}{\partial \alpha} = 2\sum_n \xi_i^{(n)}(\mathbf{y}, t - t_R)\{Ex[a_n(t_R)u_i(\mathbf{y}, t)] - \lambda_0^{(n)}\kappa_i^{(n)}(\mathbf{y}, t - t_R)\} = 0.$$

If all terms in this series are required to be zero then

$$\lambda_0^{(n)}\kappa_i^{(n)}(\mathbf{y}, t - t_R) = Ex[a_n(t_R)u_i(\mathbf{y}, t)]. \quad \text{(B.4)}$$

The solution to this equation is obtained by substituting for $a_n(t_R)$. By using equation (B.1) and the orthogonality of the modes we find that

$$a_n(t_R) = \frac{1}{V'}\int_{V'} u_j(\mathbf{y}', t_R)\phi_j^{(n)}(\mathbf{y}')\,\mathrm{d}V', \quad \text{(B.5)}$$

then, by combining equations (B.4) and (B.5) we obtain

$$\kappa_i^{(n)}(\mathbf{y}, t - t_R) = \frac{1}{\lambda_0^{(n)}V'}\int_{V'} R_{ij}(\mathbf{y}, \mathbf{y}', t - t_R)\phi_j^{(n)}(\bar{\mathbf{y}}')\,\mathrm{d}V'. \quad \text{(B.6)}$$

The functions $\kappa^{(n)}$ provide a more compact definition of the time varying part of the flow, and given the coefficients $a_n(t_R)$, are the best estimate for the time history of the modes. They also provide an expansion of the correlation function since by multiplying equation (B.4) by $\phi_j^{(n)}(\mathbf{y})$ and summing over all modes we obtain

$$\sum_n \lambda_0^{(n)}\kappa_j^{(n)}(\bar{\mathbf{y}}', t - t_R)\phi_i^{(n)}(\mathbf{y}) = Ex\left[\sum_n a_n(t_R)\phi_i^{(n)}(\mathbf{y})u_j(\mathbf{y}', t)\right] = R_{ij}(\mathbf{y}, \mathbf{y}', t - t_R). \quad \text{(B.7)}$$

This suggests that the compact eddies may be used to decompose the statistical averages of the flow space–time history and hence represent averaged flow structures. By taking the Fourier transform with respect to the time delay $\tau = t - t_R$ we obtain the cross-spectral density

$$C_{ij}(\mathbf{y}, \mathbf{y}', \omega) = \sum_n \lambda_0^{(n)}\kappa_j^{(n)}(\mathbf{y}', \omega)\phi_i^{(n)}(\mathbf{y}). \quad \text{(B.8)}$$