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ACTIVE FLOW CONTROL

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This paper considers the two-dimensional problem of a plane vortex sheet disturbed by an impulsive line source. A previous incorrect treatment of this problem is examined in detail. Instabilities of the vortex sheet are triggered by the source and grow exponentially in space and time. The Green function is constructed for the problem and it is shown that a point source properly positioned and delayed will induce a field that cancels the unstable growing modes. The resulting displacement of the vortex sheet is expressed in simple terms. The instabilities are checked by the anti-source which combines with the field of the primary source into a vortex sheet response which decays with time at large time. This paper is a contribution to the study of active control of shear layer instabilities, the main contribution being to clear up a previous paper with peculiar results that are, in fact, wrong.

1. INTRODUCTION

Of course, I accepted the editorial invitation to help celebrate Phil Doak's 80th birthday by contributing a paper in this special issue of the Journal, which has grown from a gleam in Elfyn Richards' eye in the early 1960s to mature into the most authoritative archival publication it is today, largely due to Phil Doak's vision, great care, and hard work. There were times when Phil and I shared different views on the merits of papers submitted to the young journal. Mistakes are bound to be made and in serious analytical texts they are sometimes hard to find. They are particularly hard to find by referees assessing papers too casually. The arguments are often subtle and very complicated. I thought I would contribute to this issue a discourse on what must be mistake in one of my papers. The evidence for there being a mistake had grown strong enough by the time I received the invitation to write that I was happy to spend time in sorting it out; I could not imagine it would be a hard job at all. Since then I am taxing the editorial nerve by delaying submission of the paper until the very last moment, because I am still not sure of the correct answer.

The effort to really understand how sound interacted with the mean flow of a jet had led some 30 years ago to several initial value problems being posed, sound interacting the vortex sheet being the simplest example. Analyses of these problems inevitably involve the use of generalized functions and distributions and Fourier transforms. The Wiener–Hopf problem raised issues of causality. D. S. Jones and his group in Dundee were amongst the pioneers, their analyses appearing regularly in journals respected more for the rigour of the approach than for their recognizable modelling of physical problems. A paper [1] by Jones and Morgan that appeared in 1972 was characteristically elegant, difficult to read, and mysterious. It considered the Green function, the field induced by an impulsive point source adjacent to a vortex sheet. Of course, there was no effect of the sheet until sound had arrived at it, but on arrival, the sound triggered off a response of the sheet that travelled and grew

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and made waves whose strength grew exponentially in time. Many of those interested in the physics of sound interacting with vortex layers, whilst appreciating many of the predicted effects, were inclined to discount the significance of the exponentially growing instability waves. But they could not be avoided and became more and more difficult to understand the more one sought their physical explanation. One of the strangest aspects was that the solution to the problem was perfectly well behaved and modelled by generalized functions but only so till a precise time had elapsed since their activation. Then on the shear layer, at a distance downstream of the source equal to the source's elevation above the layer, after a time equal to that taken by instability waves to travel that distance, the solution changed in character, becoming very much more complex with its modelling not contained in the space of generalized functions. Something more elegant or abstract or erudite, certainly, more complicated, was needed to go further. Jones and Morgan [1] introduced a set of ultra-distributions and showed it to be the proper description of waves beyond this critical time. Not one of my close colleagues had much clue about what ultra-distributions were, or why they were needed, nor about the physics of what was really going on. They attracted a great deal of our attention. In particular, after a meeting in Stanford in 1980 C. C. Chao, David Crighton and I were speculating on what the ultra-distribution could possibly be, and C. C. pressed on me the need to try and understand the problem as he drove me to the airport in San Francisco. I was returning to London, a flight of some 13 h then, long enough for me to believe that I had solved the problem. The need for ultra-distributions was because the vortex sheet was incorrectly modelled on linear theory on the assumption that its displacement was always small. By the critical time identified by Jones and Morgan, the amplitude of the displacement had actually grown to infinity. Certainly, something pretty strange would be needed to represent an infinitely big displacement in terms of infinitesimally small disturbances. That dismissive attitude led me to describe the solution of the Green function for the incompressible vortex sheet in the way I did in my AIAA paper [2]. The problem is defined in equations (39)–(41), and my solution stated in equation (42), without proof. The displacement of the vortex sheet induced by the impulsive point source was given in equation (44) of that paper, again without proof. Of course, I believed that these results were accurate at the time of going to press. I found them very interesting, but hardly anybody else has commented on them to my knowledge.

The more I learn about sounds of aerodynamic origin the more it seems that the distinction between flow and sound, and sound and sources, and sources and flow, are distinctions made on semantic grounds. It is obvious that unstable shear layers support disturbances with many of the characteristics of the large eddies of turbulent layers. If one is clever enough to model or observe the instability waves one also models aspects of the eddies and the unsteady flows adjacent to them, which evolve into sound proper as they escape their source. I think of sound and disturbances that grow on shear layer instabilities as being very much the same thing and hope that one day it will be possible to control both. The technique of anti-sound where secondary waves are deliberately created to interfere destructively with the primary noise might one day be applied to shear layers. Linear control is obviously the simplest to consider and may well be the most useful. Peake and Crighton have summarized this subject in the latest edition of the Annual Reviews of Fluid Mechanics [3]. They refer to an experiment in which Dines devised a sound created with a loudspeaker, that was out of phase with that emerging from a turbulent flame and thereby silenced it. The degree of cancellation depended on the accuracy with which the anti-sound mimicked the original one. The same equipment applied to the sound of the Rijke tube produced an altogether more dramatic effect. It silenced the Rijke tube by avoiding instability; the basic flow coupled with the controller was stable. Disturbances could not grow, so there was nothing to hear. Peake and Crighton [3] give other examples of energetic flows that are calmed by linear controllers, but all those systems are simple ones with very few degrees of freedom.

Instabilities of the vortex sheet are at the heart of the processes that make jets noisy; they may be amenable to active control. The simplest problem, that of the infinite vortex sheet, watched over by a controller that creates disturbances which cancel natural disturbances, has been analyzed and published recently [4]. The two-dimensional inviscid flow is the simplest; it is unstable to all scales of disturbances which grow exponentially as they travel downstream. Ffowcs Williams and Möhring claim that their specified controller has the property of adding a disturbance that cancels a pre-existing instability, making the vortex sheet/controller combination a stable one. Their analysis is not the easiest to follow, using doubly complex analysis involving i and j, both square roots of -1, but not the same square roots. During the development of that paper, it became clear that there was a considerable similarity of the solution with that in my 1982 paper [2]. It is surely the same solution, and equation (44) of reference [2] expresses the displacement of the sheet in simpler terms. The impulsive source drives an impulsive response and induces also an evolving flow that appears to be driven by sources moving steadily in the image space on the other side of the vortex sheet. It seems quite straightforward to see in principle how an impulsive anti-source could be arranged and positioned to obliterate the image source whose approach to the vortex sheet induces the unbounded response. It might be fun to see what the vortex sheet displacement would look like after the application of the anti-source and that was set as a problem for an undergraduate project some two years ago. Simon Coles was the undergraduate that attempted the problem. He succeeded in illustrating the manner in which the growing response was controlled, but his solution had the most alarming property that the vortex sheet, when eventually calmed, took on a rest position in which it was bent—a most unlikely thing. Surely, it is not possible for the vortex sheet to support neutral waves, as this result would imply. Coles' conclusions were therefore that either equation (44) of my paper was wrong, or there was indeed a neutral, indeed several possible neutrally stable solutions to the Kelvin–Helmholtz problem, or three, there were errors in his analysis which neither he nor his supervisor could find.

In honour of Philip Doak, I thought I would sort this problem out, and in what follows I derive and present an expression for the Green function of the Kelvin–Helmholtz problem in terms of functions used in the earlier paper. Unfortunately, it is not the solution given before!

2. SOURCE-LIKE SOLUTIONS TO THE KELVIN–HELMHOLTZ PROBLEM

The inviscid fluid in $y > 0$ is disturbed from rest by a potential

$$\phi^+(x, t) = \mathrm{D}/\mathrm{D}t\,(\Phi(X, Y)) \tag{1}$$

and the fluid in $y < 0$ is disturbed from its state of uniform velocity at speed U in the x direction by a potential

$$\phi^-(x, t) = \partial/\partial t\,(\Phi(X, Y')), \tag{2}$$

$$\mathrm{D}/\mathrm{D}t = \partial/\partial t + U(\partial/\partial x)$$

and

$$X = x - \tfrac{1}{2}Ut, \quad Y = y + d \pm \tfrac{1}{2}Ut, \quad Y' = y - d \mp \tfrac{1}{2}Ut, \tag{3}$$

where $\Phi(X, Y)$ is a potential satisfying Laplace's equation in $y > 0$ but with source-like singularities in $y < 0$. It has at most a logarithmic behavior when $|y|$ or $|x| \to \infty$. It is defined to be an even function of Y, i.e.,

$$\Phi(X, Y) = \Phi(X, -Y). \tag{4}$$

Pressure continuity at $y = 0$ requires

$$\partial\phi^+/\partial t = \mathrm{D}\phi^-/\mathrm{D}t, \tag{5}$$

which can be demonstrated as follows:

$$\frac{\partial\phi^+}{\partial t} = \frac{\partial}{\partial t}\frac{\mathrm{D}}{\mathrm{D}t}\Phi = -\frac{U^2}{4}\left(\frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Y^2}\right)\Phi \tag{6}$$

because

$$\frac{\partial\Phi}{\partial t} = -\frac{U}{2}\left(\frac{\partial}{\partial X} \mp \frac{\partial}{\partial Y}\right)\Phi(X, Y), \quad \frac{\mathrm{D}\Phi}{\mathrm{D}t} = \frac{U}{2}\left(\frac{\partial}{\partial X} \pm \frac{\partial}{\partial Y}\right)\Phi(X, Y). \tag{7}$$

Similarly,

$$\frac{\mathrm{D}\phi^-}{\mathrm{D}t} = \frac{\partial}{\partial t}\frac{\mathrm{D}}{\mathrm{D}t}\Phi(X, Y') = -\frac{U^2}{4}\left(\frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Y'^2}\right)\Phi(X, Y') \tag{8}$$

because

$$\frac{\partial}{\partial t}\Phi(X, Y') = -\frac{U}{2}\left(\frac{\partial\Phi}{\partial X} \pm \frac{\partial\Phi}{\partial Y'}\right), \quad \frac{\mathrm{D}}{\mathrm{D}t}\Phi(X, Y') = \frac{U}{2}\left(\frac{\partial}{\partial X} \mp \frac{\partial}{\partial Y'}\right)\Phi(X, Y'). \tag{9}$$

At $y = 0$, $Y = -Y'$ and the symmetry of Φ about $y = 0$ equates equations (6) and (8). The second vortex sheet boundary condition, that the two sides of the sheet move together requires that

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\partial\phi^+}{\partial y} = \frac{\partial}{\partial t}\frac{\partial\phi^-}{\partial y} \quad \text{at } y = 0. \tag{10}$$

This is also the case for these fields, as can be shown as:

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t}\frac{\partial\phi^+}{\partial y} = \frac{\partial}{\partial y}\frac{\mathrm{D}^2}{\mathrm{D}t^2}\Phi(X, Y) &= \frac{\partial}{\partial y}\left\{\frac{U}{2}\left(\frac{\partial}{\partial X} \pm \frac{\partial}{\partial Y}\right)\right\}^2\Phi \\ &= \frac{\partial}{\partial y}\left\{\frac{U^2}{4}\left(\nabla^2 \pm \frac{2\partial^2}{\partial X\partial Y}\right)\Phi\right\} \\ &= \mp\frac{U^2}{2}\frac{\partial^3\Phi}{\partial X^3}, \quad \text{because } \nabla^2\Phi = 0,\end{aligned} \tag{11}$$

$$\frac{\partial}{\partial t}\frac{\partial \phi^-}{\partial y} = \frac{\partial}{\partial y}\frac{\partial^2}{\partial t^2}\Phi(X, Y') = \frac{\partial}{\partial y}\left\{\frac{-U}{2}\left(\frac{\partial}{\partial X} \pm \frac{\partial}{\partial Y'}\right)\right\}^2 \Phi$$

$$= \frac{\partial}{\partial y}\left\{\frac{U^2}{4}\left(\nabla^2 \pm 2\frac{\partial^2}{\partial X \partial Y'}\right)\Phi\right\}$$

$$= \mp \frac{U^2}{2}\frac{\partial^3 \Phi}{\partial X^3}. \tag{12}$$

Again, the symmetry Φ about $Y = 0$ guarantees equations (11) and (12) are true at the vortex sheet, $y = 0$. All potential fields satisfying the constraints of equations (1)–(4) are solutions of the Kelvin–Helmholtz problem.

3. THE GREEN FUNCTION FOR THE KELVIN–HELMHOLTZ PROBLEM

We will now show that it is possible to satisfy all the constraints of the point-source-driven linear vortex-sheet problem, with a function which is a superposition of four terms of the type considered above, each of which satisfy the homogeneous problem. These functions, based on the geometric terms, r, r_s, θ and θ_s, are defined as:

$$r^2 = (x - \tfrac{1}{2}Ut)^2/d^2 + (y + d - \tfrac{1}{2}Ut)^2/d^2,$$

$$r_s^2 = (x - \tfrac{1}{2}Ut)^2/d^2 + (y + d + \tfrac{1}{2}Ut)^2/d^2,$$

$$\theta = \tan^{-1}(y + d - \tfrac{1}{2}Ut)/(x - \tfrac{1}{2}Ut),$$

$$\theta_s = \tan^{-1}(y + d + \tfrac{1}{2}Ut)/(x - \tfrac{1}{2}Ut),$$

$$r_0 = x^2/d^2 + (y + d)^2/d^2. \tag{13}$$

The two-dimensional source field $\ln r$ and θ, the field of a line vortex, are closely related by what are essentially the Cauchy–Riemann equations for the complex representation, conditions that are easily demonstrated by direct differentiation

$$\partial \ln r/\partial x = \partial\theta/\partial y \quad \text{and} \quad \partial\theta/\partial x = -\partial \ln r/\partial y. \tag{14}$$

Above the vortex sheet, $y > 0$, the potential is made up of a sum of elementary terms, the strength of each term being set by the constants A, B, α and β, which we shall evaluate by satisfying the boundary conditions

$$\phi^+ = \delta \ln r_0 + \mathrm{H}(\mathrm{D}/\mathrm{D}t)\{A\theta + B\theta_s + \alpha \ln r + \beta \ln r_s\}. \tag{15}$$

Below the sheet, $y < 0$

$$\phi^- = \delta \ln r_0 + \mathrm{H}(\partial/\partial t)\{A\theta' + B\theta'_s + \alpha \ln r' + \beta \ln r'_s\}. \tag{16}$$

H is written for the Heaviside function of t whose derivative is δ. r_0 vanishes at the point where the impulsive source fired, $(0, -d)$, i.e., $r_0 = r|_{t=0}$. The primed functions are constructed to have the symmetry properties assumed in equation (4) and are defined as:

$$r'^2 = (x - \tfrac{1}{2}Ut)^2/d^2 + (-y + d - \tfrac{1}{2}Ut)^2/d^2,$$

$$r_s'^2 = (x - \tfrac{1}{2}Ut)^2/d^2 + (-y + d + \tfrac{1}{2}Ut)^2/d^2,$$

$$\theta' = \tan^{-1}(d - y - \tfrac{1}{2}Ut)/(x - \tfrac{1}{2}Ut),$$

$$\theta_s' = \tan^{-1}(d - y + \tfrac{1}{2}Ut)/(x - \tfrac{1}{2}Ut). \qquad (17)$$

Each element of equation (15) satisfies Laplace's equation in $y > 0$. ϕ contains the impulsive inhomogeneity, otherwise, Laplace's equation is satisfied. The field is zero everywhere prior to the action of the source at $t = 0$, so it is causal.

The first boundary condition of pressure continuity at $y = 0$, which is certainly satisfied in $t > 0$ because of equations (6) and (8), must also be satisfied through the instant $t = 0$, a requirement that $\partial\phi^+/\partial t = \mathrm{D}\phi^-/\mathrm{D}t$ at $y = 0$,
i.e.,

$$\delta' \ln r_0 + \delta\frac{\mathrm{D}}{\mathrm{D}t}\{\,\} + \mathrm{H}\frac{\partial}{\partial t}\frac{\mathrm{D}}{\mathrm{D}t}\{\,\} = \delta'(\ln r_0) + \delta U\frac{\partial}{\partial x}\ln r_0 + \delta\frac{\partial}{\partial t}\{\,\}' + \mathrm{H}\frac{\partial}{\partial t}\frac{\mathrm{D}}{\mathrm{D}t}\{\,\}' \quad \text{at } y = 0, \qquad (18)$$

where $\{\,\}$ and $\{\,\}'$ are written, respectively, for the two curly bracket terms in equations (15) and (16).

The δ' terms are identical. The Heaviside terms equate because of equations (6) and (8), leaving the δ function multiplier which can be made to balance by setting values to A, B, α and β. First, we note that

$$\theta = \theta_s = \theta' = \theta_s'$$

and

$$r = r_s = r' = r_s' = r_0 \quad \text{at } y = 0, \quad t = 0 \qquad (19)$$

so that

$$\delta\frac{\partial}{\partial t}\{\,\} = \delta\frac{\partial}{\partial t}\{\,\}' \quad \text{at } y = 0 \qquad (20)$$

The remaining terms in equation (18) then reduce to the condition

$$\delta U\frac{\partial}{\partial x}\{\,\} = \delta U\frac{\partial \ln r_0}{\partial x}, \qquad (21)$$

i.e.,

$$A\theta + B\theta_s + \alpha\ln r' + \beta\ln r_s' = \ln r_0 \quad \text{at } y = 0, \quad t = 0. \qquad (22)$$

In view of equation (19), this equation requires that

$$A = -B \quad \text{and} \quad \alpha + \beta = 1. \qquad (23)$$

Ensuring that both sides of the vortex sheet move together determines the second boundary condition (10), so

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\partial\phi^+}{\partial y} = \delta'\frac{\partial}{\partial y}\ln r_0 + \delta U\frac{\partial^2}{\partial x\partial y}\ln r_0 + \delta\frac{\partial}{\partial y}\frac{\mathrm{D}}{\mathrm{D}t}\{\,\} + \mathrm{H}\frac{\partial}{\partial y}\frac{\mathrm{D}^2}{\mathrm{D}t^2}\{\,\}, \tag{24}$$

$$\frac{\partial}{\partial t}\frac{\partial\phi^-}{\partial y} = \delta'\frac{\partial}{\partial y}\ln r_0 + \delta\frac{\partial^2}{\partial y\partial t}\{\,\}' + \mathrm{H}\frac{\partial}{\partial y}\frac{\partial^2}{\partial t^2}\{\,\}'. \tag{25}$$

The δ′ terms in equations (24) and (25) are identical, so are the Heaviside function terms, by equations (11) and (12). The two equations balance when

$$\delta U\frac{\partial^2}{\partial x\partial y}\ln r_0 + \delta U\frac{\partial^2}{\partial x\partial y}\{\,\} + \delta\frac{\partial^2}{\partial y\partial t}\{\,\} = \delta\frac{\partial^2}{\partial y\partial t}\{\,\}' \quad \text{on } y=0. \tag{26}$$

Because $\{\,\}$ and $\{\,\}'$ are symmetric about $y = 0$, their gradient is antisymmetric, making

$$\begin{aligned}\frac{\partial^2}{\partial y\partial t}\{\,\} &= -\frac{\partial^2}{\partial y\partial t}\{\,\}' \quad \text{at } y=0,\\ &= \frac{\partial}{\partial y}\frac{\partial}{\partial t}\{A\theta + B\theta_s + \alpha\ln r + \beta\ln r_s\},\end{aligned} \tag{27}$$

and because

$$\frac{\partial}{\partial t}\{\theta \text{ or } \ln r\} = -\frac{U}{2}\left\{\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right\}\{\theta \text{ or } \log r\} \tag{28}$$

and

$$\frac{\partial}{\partial t}\{\theta_s \text{ or } \ln r_s\} = -\frac{U}{2}\left\{\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right\}\{\theta_s \text{ or } \ln r_s\}, \tag{29}$$

$$\frac{\partial}{\partial y}\frac{\partial}{\partial t}\{\,\} = \frac{\partial}{\partial y}\left(-\frac{U}{2}\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right](A\theta + \alpha\ln r) - \frac{U}{2}\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right](B\theta_s + \beta\ln r_s)\right). \tag{30}$$

Equation (14) and its equivalent

$$\frac{\partial\ln r_s}{\partial x} = \frac{\partial\theta_s}{\partial y}, \qquad \frac{\partial\theta_s}{\partial x} = -\frac{\partial\ln r_s}{\partial y} \tag{31}$$

can be used to eliminate the θ terms in equation (30) to make

$$\begin{aligned}\frac{\partial}{\partial y}\frac{\partial}{\partial t}\{\,\} &= -\frac{U}{2}\frac{\partial}{\partial y}\Bigg(-A\frac{\partial\ln r}{\partial y} + A\frac{\partial\ln r}{\partial x} + \alpha\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right]\ln r - B\frac{\partial\ln r_s}{\partial y}\\ &\qquad + \beta\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]\ln r_s - B\frac{\partial\ln r_s}{\partial x}\Bigg)\\ &= -\frac{U}{2}\frac{\partial}{\partial y}\left((A+\alpha)\frac{\partial}{\partial x}\ln r + (\alpha - A)\frac{\partial}{\partial y}\ln r + (\beta - B)\frac{\partial}{\partial x}\ln r_s - (B+\beta)\frac{\partial}{\partial y}\ln r_s\right).\end{aligned} \tag{32}$$

Similarly, using equations (14) and (31), θ can be eliminated to give

$$\frac{\partial^2}{\partial x \partial y}\{\} = \frac{\partial^2}{\partial x \partial y}\{A\theta + B\theta_s + \alpha \ln r + \beta \ln r_s\}$$

$$= A\frac{\partial^2 \ln r}{\partial x^2} + B\frac{\partial^2 \ln r_s}{\partial x^2} + \alpha\frac{\partial^2 \ln r}{\partial x \partial y} + \beta\frac{\partial^2 \ln r_s}{\partial x \partial y}. \quad (33)$$

Equation (26) can now be simplified by using equations (27), (32) and (33) to become

$$\delta U\left[\frac{\partial^2}{\partial x \partial y}\ln r_0 + A\frac{\partial^2 \ln r}{\partial x^2} + B\frac{\partial^2 \ln r_s}{\partial x^2} + \alpha\frac{\partial^2 \ln r}{\partial x \partial y} + \beta\frac{\partial^2 \ln r_s}{\partial x \partial y}\right.$$

$$\left. - (A+\alpha)\frac{\partial^2 \ln r}{\partial x \partial y} + (A-\alpha)\frac{\partial^2 \ln r}{\partial y^2} + (B-\beta)\frac{\partial^2 \ln r_s}{\partial x \partial y} + (B+\beta)\frac{\partial^2 \ln r_s}{\partial y^2}\right]_{y=0} = 0, \quad (34)$$

i.e., given equation (19),

$$(1 - A + B)\frac{\partial^2 \ln r_0}{\partial x \partial y} = 0 \quad \text{and} \quad (\beta - \alpha)\frac{\partial^2 \ln r}{\partial y^2}\bigg|_{y=0} = 0, \quad (35)$$

which, with equation (23), sets

$$A = \tfrac{1}{2}, \quad B = -\tfrac{1}{2}, \quad \alpha = \tfrac{1}{2}, \quad \beta = \tfrac{1}{2}. \quad (36)$$

All the conditions required by the Green function are then satisfied—apart from a normalizing constant, which can be seen by noting that [5, p. 58]

$$\nabla^2 \phi^- = \delta \nabla^2 \ln r_0 = 2\pi \delta(x, y + d, t) \quad (37)$$

larger by the factor 2π than what would be more naturally described as the Green function, but that is largely a matter of semantics.

4. THE VORTEX-SHEET DISPLACEMENT

The boundary condition that the vortex sheet moves with the normal velocity of the fluid equates

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi^+}{\partial y}\bigg|_{y=0}. \quad (38)$$

To integrate this into a description of the vortex-sheet displacement, it is convenient to express the velocity as a differential of something with respect to time. We do this as follows.

First note that

$$\frac{\mathrm{D}}{\mathrm{D}t}(\theta + \ln r) = \frac{U}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)(\theta + \ln r) \quad (39)$$

and, because of equation (14), it equals

$$= -U\frac{\partial \ln r}{\partial y}. \quad (40)$$

Similarly,

$$\mathrm{D}/\mathrm{D}t\,(-\theta_s + \ln r_s) = U/2\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)(-\theta_s + \ln r_s)$$

$$= U\frac{\partial}{\partial y}\ln r_s \quad \text{because of equation (31).} \tag{41}$$

Equations (39) and (41) produce the equation

$$\frac{\partial}{\partial y}\mathrm{D}/\mathrm{D}t\,\{\theta - \theta_s + \ln r + \ln r_s\} = U\frac{\partial^2}{\partial y^2}\ln r_s/r. \tag{42}$$

r and r_s are defined in equation (13), from which it is apparent that

$$\frac{\partial \ln r}{\partial y} = -\frac{2}{U}\frac{\partial \ln r}{\partial t} - \frac{\partial \ln r}{\partial x}$$

and

$$\frac{\partial^2 \ln r}{\partial y^2} = \frac{4}{U^2}\frac{\partial^2 \ln r}{\partial t^2} + \frac{4}{U}\frac{\partial^2 \ln r}{\partial x \partial t} + \frac{\partial^2 \ln r}{\partial x^2}. \tag{43}$$

$\nabla^2 \ln r = 0$, so equation (43) is equal to

$$\frac{\partial^2 \ln r}{\partial y^2} = \frac{2}{U^2}\frac{\partial^2 \ln r}{\partial t^2} + \frac{2}{U}\frac{\partial^2 \ln r}{\partial t \partial x}. \tag{44}$$

Similarly, the fact that

$$\frac{\partial \ln r_s}{\partial y} = \frac{2}{U}\frac{\partial \ln r_s}{\partial t} + \frac{\partial \ln r_s}{\partial x} \tag{45}$$

leads to

$$\frac{\partial^2 \ln r_s}{\partial y^2} = \frac{2}{U^2}\frac{\partial^2 \ln r_s}{\partial t^2} + \frac{2}{U}\frac{\partial^2 \ln r_s}{\partial t \partial x}. \tag{46}$$

Equations (44) and (46) allow equation (42) to be rewritten as

$$\frac{\partial}{\partial y}\mathrm{D}/\mathrm{D}t\,\{\theta - \theta_s + \ln r + \ln r_s\} = U\frac{2}{U}\left\{\frac{1}{U}\frac{\partial^2}{\partial t^2}\ln r_s/r + \frac{\partial^2 \ln r_s/r}{\partial t \partial x}\right\} \tag{47}$$

$$= 2\frac{\partial}{\partial t}\left\{\frac{1}{U}\frac{\partial \ln r_s/r}{\partial t} + \frac{\partial \ln r_s/r}{\partial x}\right\} \tag{48}$$

which can be rewritten using equations (43) and (45) as

$$= \frac{\partial}{\partial t}\left\{\frac{\partial \ln r_s}{\partial y} + \frac{\partial \ln r_s}{\partial x} + \frac{\partial \ln r}{\partial y} - \frac{\partial \ln r}{\partial x}\right\}. \tag{49}$$

Equation (38) can now be integrated trivially, because

$$\frac{\partial\eta}{\partial t} = \delta\frac{\partial}{\partial y}\ln r_0 + \frac{1}{2}\mathrm{H}\frac{\partial}{\partial y}\frac{\mathrm{D}}{\mathrm{D}t}\{\theta - \theta_s + \ln r + \ln r_s\}$$

$$= \delta\frac{\partial}{\partial y}\ln r_0 + \frac{1}{2}\mathrm{H}\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial y}\ln r + \frac{\partial}{\partial y}\ln r_s + \frac{\partial}{\partial x}\ln r_s - \frac{\partial}{\partial x}\ln r\right\}\Bigg|_{y=0} \quad (50)$$

giving, because $r = r_s$ at $y = 0$, $t = 0$,

$$\eta(x, t) = \frac{\mathrm{H}}{2}\left\{\frac{\partial}{\partial y}\ln r + \frac{\partial}{\partial y}\ln r_s + \frac{\partial}{\partial x}\ln r_s - \frac{\partial}{\partial x}\ln r\right\}_{y=0} \quad (51)$$

$$= \frac{\mathrm{H}}{2}\left\{\frac{(d - \frac{1}{2}Ut)}{(x - \frac{1}{2}Ut)^2 + (d - \frac{1}{2}Ut)^2} + \frac{(d + \frac{1}{2}Ut)}{(x - \frac{1}{2}Ut)^2 + (d + \frac{1}{2}Ut)^2}\right.$$

$$\left. \frac{+ (x - \frac{1}{2}Ut)}{(x - \frac{1}{2}Ut)^2 + (d - \frac{1}{2}Ut)^2} - \frac{(x - \frac{1}{2}Ut)}{(x - \frac{1}{2}Ut)^2 + (d + \frac{1}{2}Ut)^2}\right\} \quad (52)$$

This is the form that equation (44) of reference [2] should have taken.

5. THE STABILIZING ANTI-SOURCE

The Green function contains a term which is catastrophic at time $2d/U$, when the residual image source collides with the boundary. The catastrophe could be avoided by a negative source applied after a time delay of τ at the point $x = \frac{1}{2}U\tau$ and $y = -d + \frac{1}{2}U\tau$. That anti-source would generate a vortex sheet displacement given by the negative of equation (52), with x replaced by $x - \frac{1}{2}U\tau$, d replaced by $(d - \frac{1}{2}U\tau)$ and t replaced by $t - \tau$, i.e.,

$$\eta_a(x, t - \tau) = \frac{-\mathrm{H}(t - \tau)}{2}\left\{\frac{d - \frac{1}{2}Ut}{(x - \frac{1}{2}Ut)^2 + (d - \frac{1}{2}Ut)^2} + \frac{d + \frac{1}{2}Ut - U\tau}{(x - \frac{1}{2}Ut)^2 + (d + \frac{1}{2}Ut - U\tau)^2}\right.$$

$$\left. + \frac{(x - \frac{1}{2}Ut)}{(x - \frac{1}{2}Ut)^2 + (d - \frac{1}{2}Ut)^2} - \frac{(x - \frac{1}{2}Ut)}{(x - \frac{1}{2}Ut)^2 + (d + \frac{1}{2}Ut - U\tau)^2}\right\}. \quad (53)$$

The singular terms are cancelled in the double-source response provided $\tau < 2d/U$, leaving a vortex sheet displacement for $t > \tau$ of

$$\eta + \eta_a = \frac{1}{2}\left\{\frac{d + \frac{1}{2}Ut}{(x - \frac{1}{2}Ut)^2 + (d + \frac{1}{2}Ut)^2} - \frac{d + \frac{1}{2}Ut - U\tau}{(x - \frac{1}{2}Ut)^2 + (d + \frac{1}{2}Ut - U\tau)^2}\right.$$

$$\left. - \frac{(x - \frac{1}{2}Ut)}{(x - \frac{1}{2}Ut)^2 + (d + \frac{1}{2}Ut)^2} + \frac{(x - \frac{1}{2}Ut)}{(x - \frac{1}{2}Ut)^2 + (d + \frac{1}{2}Ut - U\tau)^2}\right\}, \quad (54)$$

a finite displacement that eventually settles down to $\eta + \eta_a = 0$.

6. CONCLUSION

We have shown that it is indeed possible to introduce a source whose effect is to neutralize the natural unstable response of a vortex sheet. Unfortunately, we also show that the solution given in reference [2] is wrong.

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