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STUDY ON THE FLEXURAL VIBRATION OF RECTANGULAR THIN PLATES WITH FREE BOUNDARY CONDITIONS

L. Shuyu

Applied Acoustics Institute, Shaanxi Normal University, Xian, Shaanxi, 710062, People's Republic of China

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An analytical method is presented for the flexural vibration of rectangular thin plates with free boundary conditions. Based on the apparent elasticity method, the flexural vibration of a rectangular thin plate is reduced to two one-dimensional flexural vibrations of slender rods. Then the two-dimensional flexural vibration of a rectangular thin plate with free boundary conditions is considered as the coupling of these two equivalent flexural vibrations with different equivalent elastic constants. It should be noted that these two equivalent flexural vibrations are different from the traditional one-dimensional flexural vibrations of slender rods. They are coupled to each other by the introduced mechanical coupling coefficient. The analytical solutions for the isotropic rectangular thin plate in flexural vibration are derived and the resonance frequency equation is obtained. The natural vibrational mode is analyzed and the frequency spectra are calculated. It is found that the normal modes and the natural frequencies of the rectangular thin plate in flexural vibration are abundant. Theoretical analyses show that one-dimensional flexural vibration of a slender rod based on the classical elementary flexural theory, as well as the stripe mode vibration of a rectangular thin plate is a limiting vibrational mode of rectangular thin plates. Experiments show that the measured resonance frequencies are in good agreement with the calculated results, and the displacement nodal line pattern is also observed experimentally.

1. INTRODUCTION

Plates are elements of practical importance in many engineering applications. The natural modes and frequencies of flexural vibration of rectangular thin plates have been of interest to structural engineers. In recent years, circular and rectangular thin plates have been used as ultrasonic radiators in ultrasonic ranging, ultrasonic levitation, and ultrasonic drying. The reason is that the radiators of thin plates in flexural vibration can improve the acoustic impedance matching between the piezoelectric transducers and the air medium. Classical analytical methods have been used to deal with the flexural vibration of thin plates with different edge conditions [1–4]. However, for rectangular plates with free boundary conditions, there is no exact analytical solution to the governing differential equation of motion. Therefore, the natural frequencies cannot be found analytically. A number of approximate analytical methods have been developed for the vibration analyses of plates [5, 6], but are limited to plates of specific boundary conditions. Numerical methods are the most powerful tools in dealing with complicated plate problems [7–9], but a large system of algebraic equations and data need to be processed.

In this paper, the apparent elasticity method previously developed for the coupled vibration problem of short columns and other vibrating systems [10–12] is extended to the

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free vibration of rectangular thin plates with free edge conditions. In this method, the concept of the equivalent elastic constants is introduced. The vibration of plates is reduced to the coupling of two flexural vibrations of slender rods with rectangular cross-section. These two flexural vibrations of rods have different equivalent elastic constants and they are coupled to each other by the introduced mechanical coupling coefficient. Using the solutions to flexural vibrations of slender rods with free edge conditions, the natural frequency equation is derived. The resonant frequencies of rectangular thin plates with free boundary conditions are computed and compared with the measured results.

2. FREE VIBRATION OF A RECTANGULAR THIN PLATE WITH FREE BOUNDARY CONDITIONS

In this paper, the free vibration of a rectangular isotropic thin plate with free edge conditions is considered. The length, width and thickness of the plate are L, W, and T. Their directions are consistent with those of the co-ordinate axes of X, Y, and Z, respectively, as shown in Figure 1.

2.1. FREQUENCY EQUATION OF RECTANGULAR THIN PLATES WITH FREE BOUNDARY CONDITIONS IN FLEXURAL VIBRATION

According to the classical thin plate theory, the shear strain and torsion are ignored, the relation between axial strains and stresses in the plate can be expressed as

$$\varepsilon_x = (\sigma_x - v\sigma_y)/E, \tag{1}$$

$$\varepsilon_y = (\sigma_y - v\sigma_x)/E, \tag{2}$$

where ε_x, ε_y and σ_x, σ_y are axial strains and stresses along the co-ordinate axes X and Y; E and v are Young's modulus and Poisson's ratio of the plate material. Letting $n = \sigma_x/\sigma_y$, which is known as the mechanical coupling coefficient, from equations (1) and (2), we have

$$E_x = E/(1 - v/n), \tag{3}$$

$$E_y = E/(1 - vn), \tag{4}$$

where $E_x = \sigma_x/\varepsilon_x$, $E_y = \sigma_y/\varepsilon_y$, which are defined as the equivalent elastic constants. According to the flexural theory of thin plates, the axial stresses σ_x and σ_y make the plate bend around the Y- and X-axis respectively. Therefore, the vibration of the rectangular thin

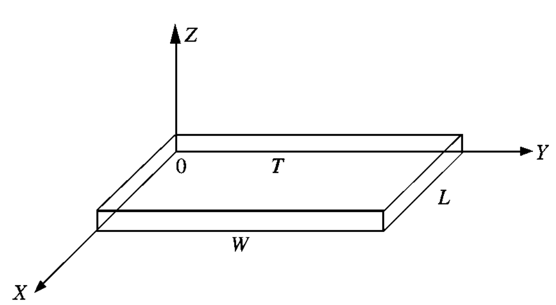

Figure 1. A geometrical diagram of a rectangular thin plate in flexural vibration.

plate can be approximately regarded as a coupling of two equivalent flexural vibrations of slender rods with the same rectangular cross-section. One is the flexural vibration of a slender rod of length L with equivalent elastic constant E_x; the other is that of a slender rod of length W with equivalent elastic constant E_y. These two equivalent flexural vibrations of slender rods are coupled to each other by the mechanical coupling coefficient. According to the classical flexural theory of slender rods [4], the frequency equations of rectangular thin plates with free edge conditions can be derived from the classical theory of slender rods,

$$\tan[\omega L/(2V_x)] = \pm\, th\{\omega L/(2V_x)], \tag{5}$$

$$\tan[\omega W/(2V_y)] = \pm\, th\{\omega W/(2V_y)], \tag{6}$$

where $V_x = (\omega C_x R_x)^{1/2}$, $V_y = (\omega C_y R_y)^{1/2}$, $C_x = (E_x/\rho)^{1/2}$, $C_y = (E_y/\rho)^{1/2}$, $R_x = R_y = T/\sqrt{12}$, V_x, V_y and C_x, C_y are the equivalent sound velocities of flexural and longitudinal vibrations in slender rods, R_x and R_y are the gyration radii of slender rods with rectangular cross-section. From equations (5) and (6) we have

$$\omega L/(2V_x) = P(i), \quad i = 0, 1, 2, 3, \dots, \tag{7}$$

$$\omega W/(2V_y) = Q(j), \quad j = 0, 1, 2, 3, \dots, \tag{8}$$

where $P(i)$ and $Q(j)$ are roots of equations (5) and (6); $P(0) = Q(0) = 0$. When i and j are not equal to zero and are large, $P(i) = \pi(2i + 1)/4$, $Q(j) = \pi(2j + 1)/4$. Every combination of i and j corresponds to one kind of flexural vibrational mode of rectangular thin plates. For flexural vibration of rectangular thin plates, i and j cannot be zero at the same time. Otherwise, there will be no flexure. From equations (3), (4) and (7), (8) using the expressions of V_x, V_y and C_x, C_y, after some transformations, the equivalent natural frequency and the mechanical coupling coefficient can be obtained according to the following equations,

$$vP^4(i)W^4n^2 + [Q^4(j)L^4 - P^4(i)W^4]n - vQ^4(j)L^4 = 0, \tag{9}$$

$$(1 - v^2)A^2 - [R_x^2P^4(i)/L^4 + R_y^2Q^4(j)/W^4]A + R_x^2R_y^2P^4(i)Q^4(j)/(L^4W^4) = 0, \tag{10}$$

where $A = \omega^2/(16C^2)$, $\omega = 2\pi f$, $C^2 = E/\rho$, C is the velocity of sound of longitudinal vibration in slender rods. It can be seen from equations (9) and (10) that when the material, the dimensions and the vibration mode (i and j) are determined, the coupling coefficient and two equivalent natural frequencies can be obtained analytically. From equation (10), these two roots can be derived as

$$A_1 = \frac{R_x^2P^4(i)}{(1 - v^2)L^4}, \quad A_2 = \frac{R_y^2Q^4(j)}{(1 - v^2)W^4}. \tag{11}$$

From equation (11), two equivalent natural frequencies can be obtained:

$$f_1 = \frac{2CR_xP^2(i)}{\pi L^2\sqrt{1 - v^2}}, \quad f_2 = \frac{2CR_yQ^2(j)}{\pi W^2\sqrt{1 - v^2}}. \tag{12}$$

It should be noted that these two frequencies have no practical meaning. However, the actual resonance frequencies of rectangular thin plates in flexural vibration can be determined by these two equivalent natural frequencies. Therefore, equation (10) is the equivalent natural frequency equation of rectangular thin plates with free boundary conditions. The analysis for the actual resonance frequencies of the rectangular thin plate will be explained in the following sections.

2.2. ANALYSIS OF TWO LIMITING FLEXURAL VIBRATIONAL MODES OF RECTANGULAR THIN PLATES

From the above analyses, two limiting vibrational modes can be obtained, which are the flexural vibrations of classical slender rods with free boundary conditions.

2.2.1. $L/W \to 0$

In this case, the plate becomes a slender rod whose length W is much larger than its width L. From equations (3), (4) and (9), we have

$$n = 0, \quad E_x = 0, \quad E_y = E. \tag{13}$$

The natural frequency of the slender rod of length W can be obtained from equation (8),

$$f = 2CR_y Q^2(j)/\pi W^2. \tag{14}$$

This is the resonance frequency of slender rods with free boundary conditions according to the classical slender rod flexural theory. Therefore, the flexural vibration of a classical slender rod is one kind of limiting vibrational modes of the rectangular thin plate when its dimensions satisfy certain conditions.

2.2.2. $L/W \to \infty$

In this case, the rectangular thin plate becomes a slender rod of length L and width W. Using similar procedures, we have

$$n = \infty, \quad E_x = E, \quad E_y = 0 \tag{15}$$

$$f = 2CR_x P^2(i)/\pi L^2. \tag{16}$$

This vibrational mode is the same as the vibration of a slender rod of length L according to the classical flexural theory of slender rods.

From the above analyses, it can be seen that the flexural vibrations of rectangular thin plates become the vibrations of slender rods when the dimensions satisfy certain conditions.

2.3. ANALYSIS OF NATURAL FREQUENCIES OF RECTANGULAR THIN PLATES IN FLEXURAL VIBRATION

It can be seen from equation (10) that two equivalent frequencies can be obtained which represent the two natural frequencies of the equivalent flexural vibrations of slender rods in the directions of the X and Y co-ordinate axes. Since the flexural vibration of a rectangular thin plate is a coupled one of these two equivalent flexural vibrations, and these two equivalent one-dimensional flexural vibrations are perpendicular to each other, the resonance frequency of a rectangular thin plate in flexural vibration should be expressed as the sum of two vectors in frequency domain that correspond to the two equivalent flexural vibrations. Let the two equivalent natural frequencies be f_{ix} and f_{jy}, then the natural frequencies of rectangular thin plates in flexural vibration mode of order i and j can be expressed as

$$f_{ij} = (f_{ix}^2 + f_{jy}^2)^{1/2}, \tag{17}$$

where f_{ij} is the natural frequency of a rectangular thin plate in flexural vibration with free boundary conditions. In this case, there are $i+1$ and $j+1$ nodal lines parallel to the Y and X co-ordinate axes. According to the above analyses, three different vibrational modes of rectangular thin plates can be discussed as follows.

2.3.1. $i \neq 0, j = 0$

For this vibrational mode, the plate in flexural vibration has only nodal lines parallel to the Y-axis. From equations (7) and (8), we have $P(i) \neq 0$, $Q(j) = 0$. Using equations (3), (4) and (9), it can be obtained that $n = 1/v$, $E_x = E/(1 - v^2)$ and $E_y = \infty$. From equation (7) or (10), the natural frequency of the plate is expressed as

$$f_{i0} = \frac{\pi T}{8L^2} \times \left[\frac{E}{12\rho(1 - v^2)}\right]^{1/2} \times (2i + 1)^2. \tag{18}$$

Equation (18) is also the same as that of equation (12). Let $C_D = [E/12\rho(1 - v^2)]^{1/2}$, $N = i + 1$, where N is the number of nodal lines parallel to the Y-axis. Equation (18) can be rewritten as

$$f_{i0} = \pi T C_D (N - \tfrac{1}{2})^2/(2L^2). \tag{19}$$

It can be seen that this is the natural frequency of rectangular thin plates in stripe mode in the literature [13], where the stripe vibrational mode of rectangular thin plates was studied. Therefore, it is obvious that the stripe vibrational mode is one of the vibrational modes of rectangular thin plates in flexural vibration. For this kind of vibrational mode of plates, there is only flexure around the Y-axis, while in the Y direction, there is no flexure around the X-axis. The reason is that $E_y = \infty$ and the plate is rigid in the direction of the Y-axis.

2.3.2. $i = 0, j \neq 0$

Using similar procedures, we have $P(i) = 0$, $Q(j) \neq 0$, $n = v$, $E_x = \infty$, $E_y = E/(1 - v^2)$. The natural frequency is

$$f_{0j} = \pi T C_D (N - 1/2)^2/(2W^2), \tag{20}$$

where $N = j + 1$. In this case, the plate has only flexure around the X-axis, and no flexure around the Y-axis. The reason is that $E_x = \infty$ and the rectangular plate is rigid in the direction of the X-axis. For this vibrational mode of plates, there are only nodal lines parallel to the X-axis, no nodal lines parallel to the Y-axis exist. This is the stripe mode of rectangular plates in the direction of the Y-axis, and it is similar to the stripe mode in the direction of the X-axis discussed in the above section.

2.3.3. $i \neq 0, j \neq 0$

It is obvious that for this kind of vibrational mode, the rectangular thin plate has not only nodal lines parallel to the Y-axis but also nodal lines parallel to the X-axis, i.e., there are perpendicular nodal lines on the surface of the plate. The natural frequency of this vibration mode can be obtained from equation (17). Because i and j are arbitrary positive integers, there are many natural frequencies for the flexural vibration of the rectangular plate.

3. FLEXURAL DISPLACEMENT DISTRIBUTION OF RECTANGULAR THIN PLATES

According to the above analyses, the flexural vibration of a rectangular thin plate in flexural vibration with free boundary conditions can be regarded as the coupling vibration of two equivalent flexural vibrations of slender rods with the same rectangular cross section. Therefore, the flexural displacement of the rectangular plate can be expressed as the product of two displacement distribution functions of slender rods whose lengths are L and W, respectively,

$$\eta_{ij}(x, y) = \eta_i(x)\eta_j(y), \tag{21}$$

$$\eta_i(x) = A_i \operatorname{ch} u + B_i \operatorname{sh} u + C_i \cos u + D_i \sin u, \tag{22}$$

$$\eta_j(y) = A_j \operatorname{ch} v + B_i \operatorname{sh} v + C_j \cos v + D_j \sin v, \tag{23}$$

where $u = \omega x/V_x = 2P(i)x/L$, $v = \omega y/V_y = 2Q(j)y/W$. It should be noted that equation (21) appears to have separable solutions. Actually, the solutions are not separable, they are coupled to each other by the coupling coefficient. When the edges of the plate are free, equations (22) and (23) can be rewritten as

$$\eta_i(x) = A_i[\operatorname{ch} u + \cos u - P(\operatorname{sh} u + \sin u)], \tag{24}$$

$$\eta_j(y) = A_j[\operatorname{ch} v + \cos v - Q(\operatorname{sh} v + \sin v)], \tag{25}$$

$$\eta_{ij}(x, y) = A_i A_j[\operatorname{ch} u + \cos u - P(\operatorname{sh} u + \sin u)]\,[\operatorname{ch} v + \cos v - Q(\operatorname{sh} v + \sin v)]. \tag{26}$$

In equations (24) and (25)

$$P = \frac{\operatorname{sh}[2P(i)] - \sin[2P(i)]}{\operatorname{ch}[2P(i)] + \cos[2P(i)]}, \quad Q = \frac{\operatorname{sh}[2Q(j)] - \sin[2Q(j)]}{\operatorname{ch}[2Q(j)] + \cos[2Q(j)]},$$

A_i and A_j are constants. For different vibrational modes, we can obtain the displacement distribution of flexural vibration of rectangular thin plates with free boundary conditions. For example, when $i \neq 0$, $j = 0$, we can get the displacement distribution of the plate,

$$\eta_{i0}(x, y) = 2A_i A_j[\operatorname{ch} u + \cos u - P(\operatorname{sh} u + \sin u)], \tag{27}$$

as $u = 2P(i)x/L$, $\eta_{i0}(x, y)$ depends only on x. Therefore, there are only nodal lines parallel to the Y-axis, and this is just the stripe mode in the direction of the X-axis.

4. EXPERIMENTS

4.1. MEASUREMENT OF RESONANT FREQUENCIES OF RECTANGULAR THIN PLATES IN FLEXURAL VIBRATIONS

According to the resonance frequency equation (10), some rectangular thin plates are designed and machined. The material is stainless steel. The standard material parameters are as follows: $v = 0{\cdot}28$, $E = 19{\cdot}5 \times 10^{10}$ N/m^2, $\rho = 7{\cdot}80 \times 10^3$ kg/m^3. The resonance frequencies of the plates are measured using the emitting–receiving method as shown in

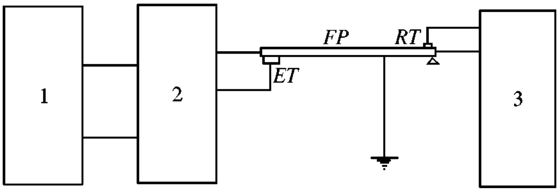

Figure 2. Experimental set-up for the resonance frequency measurement of rectangular thin plates in flexural vibration 1, frequency meter; 2, signal generator; 3, oscilloscope.

TABLE 1

Measured and calculated resonance frequencies of rectangular thin plates in flexural vibration with free boundary conditions

Mode (i, j)	L (mm)	W (mm)	T (mm)	f(Hz)	f_m (Hz)	Δ (%)
(10, 0)	200	80	3	19 528	19 117	2·15
(0, 4)	200	80	3	22 418	21 789	2·89
(9, 3)	200	80	3	20 963	20 257	3·49
(13, 0)	250	100	3	20 660	19 972	3·44
(0, 5)	250	100	3	21 433	20 876	2·67
(11, 4)	250	100	3	20 751	20 113	3·17
(15, 0)	240	120	2	19 702	19 218	2·52
(0, 8)	240	120	2	23 699	22 892	3·53
(13, 6)	240	120	2	20 382	19 616	3·90

Figure 2. In the experiments, the plate FP is excited to vibrate flexurally by an emitting transducer ET at one end of the plate. At the other end, the plate is supported using a supporter. Some sponge is put between the plate and supporter to satisfy the free boundary condition. The exciting transducer also acts as a supporter. The receiving transducer RT is put on the upper surface of the plate at the opposite end of the actuator. To reduce the effect of the receiving transducer on the measured flexural resonance frequencies of the plate, the receiving transducer must be small. The resonance frequencies of the emitting and receiving transducers must be much higher than those of the rectangular plate to be measured. The measuring principle is described as follows. Change the frequency of the input signal of the generator that is connected to the emitting transducer until the output of the receiving transducer that is connected to the oscilloscope has a maximum. The frequency corresponding to this maximum output is the resonance frequency of the plate in flexural vibration. The measured resonance frequencies of the plates are shown in Table 1, where f and f_m are the calculated and measured resonance frequencies, $\Delta = |f - f_m|/f_m$. For the frequency measurement error, the following factors should be taken into account. First, the free boundary condition is not satisfied completely in the experiment. Second, the standard material parameters are different from the practical values of the plate material. Third, for the plate to be measured the emitting and receiving transducers are external loads, this will cause error.

4.2. OBSERVATION OF THE FLEXURAL DISPLACEMENT DISTRIBUTION OF RECTANGULAR THIN PLATES

Some fine powders are put on the radiating surface of the rectangular thin plate to measure the displacement distribution. When the plate is excited using a longitudinal transducer that is attached to the center of the plate, it will vibrate. If the frequency of the exciting transducer is equal to the natural frequency of the plate, the plate will resonate and its vibration is intense. Therefore, the powders will concentrate at the flexural displacement node. The powder distribution pattern is the distribution of nodal lines of the plate. In the experiment, the exciting transducer is a sandwich longitudinal vibrator whose resonance frequency is close to that of the rectangular thin plate in a certain flexural vibrational mode. It is illustrated that when the plate vibrates, the powders on the surface form different distribution patterns according to the exciting frequency and the vibrational mode. The stripe mode and the other vibrational modes, which have both the parallel and perpendicular nodal lines on the surface, have been observed. The observed displacement distribution is consistent with the theoretically predicted results.

5. SUMMARY AND CONCLUSIONS

An analytical method is presented for predicting the natural frequency of flexural vibration of rectangular thin plates with free boundary conditions. The natural frequency equation is derived which can be used to calculate the natural frequencies when the material parameters and the dimensions are given. At the same time, the analytical expression of normal functions of flexural displacement distribution is also obtained. To sum up the above analyses, the following conclusions can be drawn:

(1) Compared with numerical methods, the analytical method presented in this paper is simple; no computer simulation is required. On the other hand, the physical meaning is concise.
(2) As i and j are arbitrary positive integers, there are many vibrational modes for the flexural vibration of rectangular thin plates; therefore, the natural frequencies are very abundant.
(3) As the rectangular thin plate has a large radiating surface, it is predicted that it will find wide applications in ultrasonic ranging, ultrasonic levitation, and ultrasonic drying.
(4) In this paper it is assumed that the shear deformation and the rotary inertia could be ignored. The plate must be thin, its length and width must be much larger than its thickness. On the other hand, for higher frequencies of the thin plates, the shear deformation could not be ignored, so the theory in this paper is suitable for thin plates and vibrational orders of low natural frequency. In other cases, the difference between the theoretically predicted frequencies and the measured results could not be ignored.

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