Journal of Sound and Vibration (2001) **239**(5), 1086–1095
doi:10.1006/jsvi.2000.3095, available online at http://www.idealibrary.com on IDEAL®

INITIAL CONDITION RESPONSE OF A VISCOELASTIC DYNAMICAL SYSTEM IN THE PRESENCE OF DRY FRICTION AND IDENTIFICATION OF SYSTEM PARAMETERS

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(*Received* 24 *May* 2000)

1. INTRODUCTION

Physical systems exhibit several mechanisms for loss of energy. Depending upon acceptable levels of approximations for a particular system, one or more models for different damping mechanisms would need to be incorporated into a model of the system. Viscous and Coulomb friction are among the most commonly used models for damping in elastic solids. Viscoelastic solids, such as foam, are known to exhibit several mechanisms for loss of energy [1]. These include small strain viscoelastic damping of the cellular matrix, pneumatic damping, hysteresis at medium to large deformations and thermal energy dissipation. Also, energy loss occurs because of excitation and emission of elastic waves resulting from interaction of rubbing surfaces in the viscoelastic medium, which can be described by a dry friction model [2].

For a dynamic viscoelastic system, a simple model can be developed by using a linear constitutive relation for the stress–strain behavior of the material. Hereditary models of viscoelasticity are popular candidates for this purpose. In the case of a single-degree-of-freedom dynamic system, the characteristics of the viscoelastic solid can be modelled by means of linear stiffness and viscous damping terms along with certain viscoelastic terms. One such model has been considered by Muravyov and Hutton [3], where the authors have presented a closed-form solution for the response of the system. In the present work, we have used this model to develop a procedure for obtaining the free response of the system in the presence of dry friction.

In the case of elastic systems, the free vibration response can be utilized for estimation of viscous and Coulomb friction effects [4, 5]. Here we utilize the free response solution developed for the viscoelastic system with dry friction, along with a modified Prony series modelling technique [7] for estimating the parameters of the system. The acceleration response of the system to arbitrary initial velocities is used for the estimation procedure, because in an actual experiment with such systems, the acceleration would typically be the measured response variable.

2. INITIAL CONDITION RESPONSE

Consider a linear single-degree-of-freedom dynamic system where the restoring forces are attributed to a linear viscoelastic material. The stress–strain relationship for such a material

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can be expressed as

$$\sigma(t) = E\left\{\varepsilon(t) - \int_0^t \Gamma(t-\tau)\varepsilon(\tau)\,\mathrm{d}\tau\right\}, \tag{1}$$

where $\Gamma(t)$ is the relaxation kernel of the material and E is the instantaneous Young's modulus. A common form of the relaxation kernel often used in the literature is that in the form of a sum of exponentials:

$$\Gamma(t) = \sum_{i=1}^{N} a_i \mathrm{e}^{-\alpha_i t}. \tag{2}$$

The equation of motion of the dynamical system then becomes

$$m\ddot{x} + c\dot{x} + kx - k\int_0^t \sum_{i=1}^{N} a_i \mathrm{e}^{-\alpha_i(t-\tau)} x(\tau)\,\mathrm{d}\tau = -f(t), \tag{3}$$

where x represents the displacement of the mass m from the equilibrium position (x^*), c and k are the viscous damping coefficient and stiffness of the system, and a_i and α_i are the viscoelastic material parameters. The motion is assumed to start at time $t = 0$. The initial displacement $x(0)$ is taken to be zero to avoid ambiguity in the assumed model. $x(0) = 0$ is implied by the lower limit of the integral being zero which represents the fact that $x(t) = 0$, for $t < 0$. $f(t)$ represents the dry friction force, modelled by Coulomb friction as $f(t) = f_0\,\mathrm{sgn}(\dot{x})$, $\dot{x} \neq 0$, where f_0 is a constant representing the kinetic friction force. Note that the uncertainty in the determination of dry friction at zero velocity leads to a corresponding uncertainty in locating the equilibrium point. Consequently, a "sticky" region for the location of the equilibrium positions exists, given by $-x_s \leqslant x^* \leqslant x_s$, where $x_s = f_s/k(1 - \sum_{i=1}^{N} a_i/\alpha_i)$, and f_s is the maximum static friction force.

For $\dot{x} \neq 0$, equation (3) is piecewise solvable, and can be written as

$$m\ddot{x} + c\dot{x} + kx - k\int_0^t \sum_{i=1}^{N} a_i \mathrm{e}^{-\alpha_i(t-\tau)} x(\tau)\,\mathrm{d}\tau = -f_0, \quad \dot{x} > 0, \tag{4}$$

$$m\ddot{x} + c\dot{x} + kx - k\int_0^t \sum_{i=1}^{N} a_i \mathrm{e}^{-\alpha_i(t-\tau)} x(\tau)\,\mathrm{d}\tau = f_0, \quad \dot{x} < 0. \tag{5}$$

Each continuous piece of motion during which velocity does not change sign is called a *half-cycle*. In the absence of dry friction, the solution of equation (3) will consist of $N + 2$ exponentials [3] (except when the system eigenvalues are not all distinct). Dry friction introduces an additional constant term (DC) in the solution. It will be shown below that, in the presence of dry friction, the solution for the ith half-cycle is of the form

$$x(t) = K_i + \sum_{j=1}^{N+2} C_j^i \mathrm{e}^{p_j t}, \quad i = 1, 2, 3, \ldots, \tag{6}$$

where K_i and the coefficients C_j^i are constants, which are different for each half-cycle of the response. Since the homogeneous solution of the above equations remains the same in each half-cycle, the coefficients p_j in the exponential terms do not change from one half-cycle to another. The closed form solution, in different half-cycles, is derived here by substituting the above solution form into the differential equation and using the initial conditions at the start of the half-cycles.

2.1. MOTION DURING THE FIRST HALF-CYCLE

Suppose the initial conditions for the response are such that $\dot{x}(0) > 0$. Then, for the first half-cycle of the motion when $\dot{x}(t) \geqslant 0$, the response is governed by equation (4). Let the solution be

$$x(t) = K_1 + \sum_{j=1}^{N+2} C_j^1 \mathrm{e}^{p_j t}, \quad 0 \leqslant t \leqslant t_1, \tag{7}$$

where the subscript on K and the superscript on C_j indicate correspondence to the first half-cycle, and time t_1 is such that $\dot{x}(t_1) = 0$. The constant K_1 denotes the contribution of the particular solution arising because of friction. Substitution of equation (7) into equation (4) yields

$$m \sum_{j=1}^{N+2} C_j^1 p_j^2 \mathrm{e}^{p_j t} + c \sum_{j=1}^{N+2} C_j^1 p_j \mathrm{e}^{p_j t} + k \left\{ K_1 + \sum_{j=1}^{N+2} C_j^1 \mathrm{e}^{p_j t} \right\}$$
$$- kK_1 \sum_{i=1}^{N} \frac{a_i}{\alpha_i}(1 - \mathrm{e}^{-\alpha_i t}) - k \sum_{j=1}^{N+2} C_j^1 \sum_{i=1}^{N} a_i \frac{\mathrm{e}^{(\alpha_i + p_j)t} - 1}{\alpha_i + p_j} \mathrm{e}^{-\alpha_i t} = -f_0. \tag{8}$$

By regrouping and equating the constants, the coefficients of $\mathrm{e}^{p_j t}$, and the coefficients of $\mathrm{e}^{-\alpha_i t}$ on either sides of equation (8), the following equations are obtained:

$$mp_j^2 + cp_j + k - k \sum_{i=1}^{N} \frac{a_i}{\alpha_i + p_j} = 0, \quad j = 1, 2, \ldots, N+2, \tag{9}$$

$$\frac{K_1}{\alpha_i} + \sum_{j=1}^{N+2} \frac{C_j^1}{\alpha_i + p_j} = 0, \quad i = 1, 2, \ldots, N, \tag{10}$$

$$K_1 - K_1 \sum_{i=1}^{N} \frac{a_i}{\alpha_i} = -\frac{f_0}{k}. \tag{11}$$

The initial conditions yield two additional equations:

$$K_1 + \sum_{j=1}^{N+2} C_j^1 = x(0) = 0, \qquad \sum_{j=1}^{N+2} C_j^1 p_j = \dot{x}(0) = \dot{x}_0. \tag{12, 13}$$

Equation (9) is the characteristic equation of the system, whose roots are the coefficients p_j, $j = 1, 2, \ldots, N+2$, which are the eigenvalues of the system. The solution to equation (11) yields K_1, while the coefficients C_j^1 can be obtained by solving a set of $N+2$ linear equations, given by equations (10), (12) and (13). Here it is assumed that the eigenvalues are all distinct.

2.2. MOTION DURING THE SECOND HALF-CYCLE

In this part of the motion, velocity is negative. The equation of motion thus takes the form

$$m\ddot{x} + c\dot{x} + kx - k \int_0^{t_1} \sum_{i=1}^{N} a_i \mathrm{e}^{-\alpha_i (t-\tau)} x(\tau)\, \mathrm{d}\tau$$
$$- k \int_{t_1}^{t} \sum_{i=1}^{N} a_i \mathrm{e}^{-\alpha_i (t-\tau)} x(\tau)\, \mathrm{d}\tau = f_0, \quad t_1 \leqslant t \leqslant t_2, \tag{14}$$

where $x(\tau)$ in the first integral is already known from the first half-cycle and is given by equation (7). This integral represents the "memory" the system has of its motion during the first half-cycle. Substituting $x(t)$, $t \leqslant t_1$, from equation (7) into equation (14) results in

$$m\ddot{x} + c\dot{x} + kx - k\int_{t_1}^{t} \sum_{i=1}^{N} a_i \mathrm{e}^{-\alpha_i(t-\tau)} x(\tau)\,\mathrm{d}\tau$$

$$= f_0 + k\sum_{i=1}^{N} a_i \left\{ K_1\left(\frac{\mathrm{e}^{-\alpha_i t_1} - 1}{\alpha_i}\right) + \sum_{j=1}^{N+2} C_j^1 \left(\frac{\mathrm{e}^{(\alpha_i + p_j)t_1} - 1}{\alpha_i + p_j}\right)\right\} \mathrm{e}^{-\alpha_i t}. \qquad (15)$$

Thus, the motion of the system during the first half-cycle contributes N extra terms to the forcing function. The most general form of solution will have $2N + 3$ terms and can be written as

$$x(t) = K_2 + \sum_{j=1}^{N+2} C_j^2 \mathrm{e}^{q_j t} + \sum_{l=1}^{N} D_l \mathrm{e}^{-\alpha_l t}, \qquad (16)$$

where the exponents q_j correspond to the homogeneous solution. Substituting equation (16) into equation (15), we get

$$m\left\{ \sum_{j=1}^{N+2} C_j^2 q_j^2 \mathrm{e}^{q_j t} + \sum_{l=1}^{N} D_l \alpha_l^2 \mathrm{e}^{-\alpha_l t} \right\} + c\left\{ \sum_{j=1}^{N+2} C_j^2 q_j \mathrm{e}^{q_j t} - \sum_{l=1}^{N} D_l \alpha_l \mathrm{e}^{-\alpha_l t} \right\}$$

$$+ k\left\{ K_2 + \sum_{j=1}^{N+2} C_j^2 \mathrm{e}^{q_j t} + \sum_{l=1}^{N} D_l \mathrm{e}^{-\alpha_l t} \right\}$$

$$- k\Bigg[\sum_{i=1}^{N} \frac{a_i K_2}{\alpha_i} - \sum_{i=1}^{N} \frac{a_i K_2}{\alpha_i} \mathrm{e}^{\alpha_i t_1} \mathrm{e}^{-\alpha_i t} + \sum_{j=1}^{N+2} C_j^2 \sum_{i=1}^{N} \frac{a_i}{\alpha_i + q_j} \mathrm{e}^{q_j t}$$

$$- \sum_{i=1}^{N} a_i \left\{ \sum_{j=1}^{N+2} \frac{C_j^2}{\alpha_i + q_j} \mathrm{e}^{(\alpha_i + q_j)t_1} \mathrm{e}^{-\alpha_i t} + \sum_{l=1, l \neq i}^{N} \frac{D_l}{\alpha_i - \alpha_l} \mathrm{e}^{(\alpha_i - \alpha_l)t} \right.$$

$$\left. - \sum_{l=1, l \neq i}^{N} \frac{D_l}{\alpha_i - \alpha_l} \mathrm{e}^{(\alpha_i - \alpha_l)t_1} + D_i(t - t_1) \right\} \Bigg]$$

$$= f_0 + k\sum_{i=1}^{N} a_i \left\{ \sum_{j=1}^{N+2} C_j^1 \frac{\mathrm{e}^{(\alpha_i + p_j)t_1} - 1}{\alpha_i + p_j} + K_1 \frac{\mathrm{e}^{\alpha_i t_1} - 1}{\alpha_i} \right\} \mathrm{e}^{-\alpha_i t}. \qquad (17)$$

Comparing coefficients of t on either side of equation (17), it can be seen that $D_l = 0$, $l = 1, 2, \ldots, N$. Also the characteristic equation, obtained by equating the coefficients of $\mathrm{e}^{q_j t}$, is

$$mq_j^2 + cq_j + k - k\sum_{i=1}^{N} \frac{a_i}{\alpha_i + q_j} = 0, \quad j = 1, 2, \ldots, N + 2, \qquad (18)$$

which is same as equation (9). Hence $q_j = p_j$, $j = 1, 2, \ldots, N + 2$. Substituting $D_l = 0$ and $q_j = p_j$ in equation (17) and equating the constants as well as coefficients of $\mathrm{e}^{-\alpha_i t}$, the following equations are obtained:

$$K_2 \frac{e^{\alpha_i t_1}}{\alpha_i} + \sum_{j=1}^{N+2} C_j^2 \frac{e^{(\alpha_i + p_j)t_1}}{\alpha_i + p_j} = \left\{ \sum_{j=1}^{N+2} C_j^1 \frac{e^{(\alpha_i + p_j)t_1} - 1}{\alpha_i + p_j} + K_1 \frac{e^{\alpha_i t_1} - 1}{\alpha_i} \right\}, \quad i = 1, 2, \ldots, N, \tag{19}$$

$$K_2 - K_2 \sum_{i=1}^{N} \frac{a_i}{\alpha_i} = \frac{f_0}{k}. \tag{20}$$

The initial conditions for the motion of the second half-cycle yield the following two equations:

$$K_2 + \sum_{j=1}^{N+2} C_j^2 e^{p_j t_1} = x(t_1) = x_1, \tag{21}$$

$$\sum_{j=1}^{N+2} C_j^2 p_j e^{p_j t_1} = \dot{x}(t_1) = \dot{x}_1 = 0. \tag{22}$$

Equation (20) can be solved for the constant K_2. Then equations (19), (21) and (22) can be used to solve for the coefficient C_j^2. The time t_1 is obtained from the solution of the transcendental equation

$$\sum_{j=1}^{N+2} C_j^1 p_j e^{p_j t_1} = 0. \tag{23}$$

2.3. GENERAL RESPONSE FOR THE Mth HALF-CYCLE

The above results can be generalized for characterizing the system response during the Mth half-cycle, provided the initial displacement for the half-cycle is outside of the displacement interval, $-x_s < x < x_s$. The motion during the Mth half-cycle ($M > 1$) can then be written as

$$x(t) = K_M + \sum_{j=1}^{N+2} C_j^M e^{p_j t}, \quad t_{M-1} \leqslant t \leqslant t_M. \tag{24}$$

The substitution of this solution form into the governing equation of motion yields the following set of equations:

$$mp_j^2 + cp_j + k - k \sum_{i=1}^{N} \frac{a_i}{\alpha_i + p_j} = 0, \quad j = 1, 2, \ldots, N+2, \tag{25}$$

$$K_M \frac{e^{\alpha_i t_{M-1}}}{\alpha_i} + \sum_{j=1}^{N+2} C_j^M \frac{e^{(\alpha_i + p_j)t_{M-1}}}{\alpha_i + p_j}$$
$$= \sum_{m=1}^{M-1} \left\{ K_m \frac{e^{\alpha_i t_m} - e^{\alpha_i t_{m-1}}}{\alpha_i} + \sum_{j=1}^{N+2} C_j^m \frac{e^{(\alpha_i + p_j)t_m} - e^{(\alpha_i + p_j)t_{m-1}}}{\alpha_i + p_j} \right\}, \quad i = 1, 2, \ldots, N, \tag{26}$$

$$K_M = \frac{(-1)^M f_0}{k(1 - \sum_{i=1}^{N} a_i/\alpha_i)}, \tag{27}$$

where $t_0 = 0$. The initial conditions for the Mth half-cycle provide two additional equations:

$$K_M + \sum_{j=1}^{N+2} C_j^M \mathrm{e}^{p_j t_{M-1}} = x(t_{M-1}) = x_{M-1}, \tag{28}$$

$$\sum_{j=1}^{N+2} C_j^M p_j \mathrm{e}^{p_j t_{M-1}} = \dot{x}(t_{M-1}) = \dot{x}_{M-1} = 0. \tag{29}$$

Here, t_{M-1} is the solution of the transcendental equation

$$\sum_{j=1}^{N+2} C_j^{M-1} p_j \mathrm{e}^{p_j t_{M-1}} = 0 \tag{30}$$

and x_{M-1} is given by

$$x_{M-1} = K_{M-1} + \sum_{j=1}^{N+2} C_j^{M-1} \mathrm{e}^{p_j t_{M-1}}. \tag{31}$$

The eigenvalues p_j can be determined from equation (25) by using a root-solving routine (see, e.g., reference [9]) or by reduction to an eigenvalue problem [3, 9]. Equation (27) gives the value of constant K_M for the Mth half-cycle, while equations (26), (28) and (29) yield a linear system of $N + 2$ equations in $N + 2$ unknown complex coefficients C_j^M. In matrix form the equations for C_j^M can be written as

$$\begin{bmatrix} \dfrac{\mathrm{e}^{p_1 t_{M-1}}}{\alpha_1 + p_1} & \dfrac{\mathrm{e}^{p_2 t_{M-1}}}{\alpha_1 + p_2} & \cdots & \dfrac{\mathrm{e}^{p_{N+2} t_{M-1}}}{\alpha_1 + p_{N+2}} \\ \dfrac{\mathrm{e}^{p_1 t_{M-1}}}{\alpha_2 + p_1} & \dfrac{\mathrm{e}^{p_2 t_{M-1}}}{\alpha_2 + p_2} & \cdots & \dfrac{\mathrm{e}^{p_{N+2} t_{M-1}}}{\alpha_2 + p_{N+2}} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\mathrm{e}^{p_1 t_{M-1}}}{\alpha_N + p_1} & \dfrac{\mathrm{e}^{p_2 t_{M-1}}}{\alpha_N + p_2} & \cdots & \dfrac{\mathrm{e}^{p_{N+2} t_{M-1}}}{\alpha_N + p_{N+2}} \\ \mathrm{e}^{p_1 t_{M-1}} & \mathrm{e}^{p_2 t_{M-1}} & \cdots & \mathrm{e}^{p_{N+2} t_{M-1}} \\ p_1 \mathrm{e}^{p_1 t_{M-1}} & p_2 \mathrm{e}^{p_2 t_{M-1}} & \cdots & p_{N+2} \mathrm{e}^{p_{N+2} t_{M-1}} \end{bmatrix} \begin{bmatrix} C_1^M \\ C_1^M \\ \cdots \\ \cdots \\ \cdots \\ C_{N+2}^M \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ \cdots \\ R_N \\ x_{M-1} - K_M \\ \dot{x}_{M-1} \end{bmatrix}, \tag{32}$$

where $\dot{x}_{M-1} = 0$ for $M > 1$, and

$$R_i = -\frac{K_M}{\alpha_i} + \mathrm{e}^{-\alpha_i t_{M-1}} \sum_{m=1}^{M-1} \left\{ K_m \frac{\mathrm{e}^{\alpha_i t_m} - \mathrm{e}^{\alpha_i t_{m-1}}}{\alpha_i} + \sum_{j=1}^{N+2} C_j^m \frac{\mathrm{e}^{(\alpha_i + p_j) t_m} - \mathrm{e}^{(\alpha_i + p_j) t_{m-1}}}{\alpha_i + p_j} \right\}. \tag{33}$$

This completes the process of finding the time response of the system, given the material and other physical parameters. Note that the eigenvalues p_j need to be distinct. If the eigenvalues are not all distinct then the general form of solution cannot be expressed by equation (6) and the proposed method cannot be applied. This case is not discussed in the present work.

3. IDENTIFICATION OF SYSTEM PARAMETERS AND ESTIMATION OF DRY FRICTION

In this section, we consider the inverse problem of estimation of the relevant system parameters: stiffness k, viscous damping c, viscoelastic parameter α_i and a_i and dry friction f_0, if the data from acceleration response $\ddot{x}(t)$ is available from an experiment. It may be noted here that when using acceleration response, the constant component of the half-cycle response K_M is not explicitly known. The acceleration response for the Mth half-cycle is assumed to be

$$\ddot{x}(t) = \sum_{j=1}^{N+2} C_j^M p_j^2 \mathrm{e}^{p_j t}. \tag{34}$$

For convenience, in addition, express K_M as

$$K_M = (-1)^M K_0, \tag{35}$$

where $K_0 = f_0/k(1 - \sum_{i=1}^{N}(a_i/\alpha_i))$. The continuity of displacement $x(t)$ at $t = t_M$ yields

$$x_M = K_M + \sum_{j=1}^{N+2} C_j^M \mathrm{e}^{p_j t_M} = K_{M+1} \sum_{j=1}^{N+2} C_j^{M+1} \mathrm{e}^{p_j t_M} \tag{36}$$

and hence

$$\frac{(-1)^M}{2} \sum_{j=1}^{N+2} \{C_j^{M+1} - C_j^M\} \mathrm{e}^{p_j t_M} - K_0 = 0. \tag{37}$$

For the identification process, the values of the times $t = t_M$, $M = 1, 2, \ldots, \widehat{M}$, must be known accurately. $\widehat{M}$ denotes the total number of half-cycles present in the response, until the motion decays into the equilibrium region. As shown in section 2.3, t_M is obtained by solving a transcendental equation for the zero-velocity condition (e.g., equation (30)), and will, in general, be obtained numerically.

3.1. CALCULATION OF p_j, C_j^M, AND K_0

Prony series [7] is a method that can be used to fit a sum of exponentials to data. The method consists of three stages. In the first stage, the data are used to find the coefficient of a polynomial whose zeros are related to the eigenvalues p_j. The second stage constitutes finding the zeros of the polynomial and hence obtaining the p_j. Since these coefficients are the same in each half-cycle, data from each half-cycle can be combined to determine an overall estimate of the polynomial coefficients, and subsequently the p_j [8]. In the third stage of the Prony series, equation (34) is used to determine the residues C_j^M, and since the p_j are known, this produces a set of linear equations for each half-cycle. However, equation (37) can also be incorporated into this estimation, so that the residues of all the half-cycles C_j^M, $j = 1, 2, \ldots, N+2$, and $M = 1, 2, \ldots, \widehat{M}$, and K_0 can be estimated simultaneously. Furthermore, the condition of zero velocity at the times $t = t_M$ can be imposed as hard constraints on the estimation process, yielding a constrained least-squares estimation problem for the C_j^M, $M = 1, 2, \ldots, \widehat{M}$. These hard constraints, for Mth half-cycle, can be expressed as

$$\sum_{j=1}^{N+2} C_j^M p_j \mathrm{e}^{p_j t_{M-1}} = 0 \tag{38}$$

and

$$\sum_{j=1}^{N+2} C_j^M p_j \mathrm{e}^{p_j t_M} = 0. \tag{39}$$

These modifications would improve the accuracy of estimation in the presence of measurement noise and other uncertainties.

3.2. CALCULATION OF α_i

On substituting $M + 1$ in place of M in equation (26),

$$K_{M+1}\frac{e^{\alpha_i t_M}}{\alpha_i} + \sum_{j=1}^{N+2} C_j^{M+1}\frac{e^{(\alpha_i+p_j)t_M}}{\alpha_i+p_j}$$

$$= \sum_{m=1}^{M}\left\{K_m\frac{e^{\alpha_i t_m}-e^{\alpha_i t_{m-1}}}{\alpha_i} + \sum_{j=1}^{N+2} C_j^m\frac{e^{(\alpha_i+p_j)t_m}-e^{(\alpha_i+p_j)t_{m-1}}}{\alpha_i+p_j}\right\},\quad i = 1, 2, \ldots, N. \tag{40}$$

Subtracting equation (26) from equation (40) and using equation (35) for K_M gives

$$(-)^{M+1}2K_0\frac{e^{\alpha_i t_M}}{\alpha_i} + \sum_{j=1}^{N+2}(C_j^{M+1}-C_j^M)\frac{e^{(\alpha_i+p_j)t_M}}{\alpha_i+p_j} = 0,\quad i = 1, 2, \ldots, N, \tag{41}$$

which can be further simplified to

$$(-)^{M+1}\frac{2K_0}{\alpha_i} + \sum_{j=1}^{N+2}\frac{(C_j^{M+1}-C_j^M)}{\alpha_i+p_j}e^{p_j t_M} = 0,\quad i = 1, 2, \ldots, N. \tag{42}$$

For any i, equation (42) gives rise to an $(N + 2)$th order polynomial-type equation in the unknown parameter α_i. However, the coefficients of the leading two terms are identically zero and the resulting reduced Nth order polynomial equation can be solved for the α_i, $i = 1, 2, \ldots, N$. See Appendix A for a proof of this assertion.

3.3. CALCULATION OF a_i, k, AND c

Using the estimated α_i, $i = 1, 2, \ldots, N$, equation (25) can be used to generate a system of $N + 2$ linear equations, expressed as

$$\begin{bmatrix} p_1 & 1 & \dfrac{-1}{p_1+\alpha_1} & \cdots & \dfrac{-1}{p_1+\alpha_N} \\ p_2 & 1 & \dfrac{-1}{p_2+\alpha_1} & \cdots & \dfrac{-1}{p_2+\alpha_N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{N+2} & 1 & \dfrac{-1}{p_{N+2}+\alpha_1} & \cdots & \dfrac{-1}{p_{N+2}+\alpha_N} \end{bmatrix}\begin{bmatrix} c \\ k \\ \tilde{a}_1 \\ \cdots \\ \tilde{a}_N \end{bmatrix} = -m\begin{bmatrix} p_1^2 \\ p_2^2 \\ \cdots \\ \cdots \\ p_{N+2}^2 \end{bmatrix}. \tag{43}$$

This linear system of equations can be solved to yield estimates of c, k and $\tilde{a}_i$, where $\tilde{a}_i = ka_i$. The coefficients a_i can then be calculated from $\tilde{a}_i$ using the estimated value of the instantaneous stiffness k.

3.4. CALCULATION OF f_0

Knowing the values of α_i, a_i, k and K_0, the dry friction constant f_0 can be determined from

$$f_0 = k\left(1 - \sum_{i=1}^{N} \frac{a_i}{\alpha_i}\right) K_0. \tag{44}$$

4. SUMMARY

A closed-form solution for the initial condition response of a viscoelastic dynamical system in the presence of dry friction has been presented. The response for any half-cycle is found to consist of a sum of exponentials and a constant term. Furthermore, assuming that the acceleration response is available as a sum of exponentials, a method has been suggested to estimate the system parameters including dry friction. The estimation technique is based on utilization of information from (at least) two consecutive half-cycles, although it is contended that an estimation based on utilization of information from many half-cycles would yield better results in the presence of measurement noise and other uncertainties.

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APPENDIX A: SOLUTION OF EQUATION (42) FOR α_i

Equation (42) can be put in the form

$$\frac{A_0\beta^{N+2} + A_1\beta^{N+1} + A_2\beta^N + A_3\beta^{N-1} + \cdots + A_{N+2}}{\beta(\beta+p_1)(\beta+p_2)\cdots(\beta+p_{N+2})} = 0, \tag{A1}$$

where $\beta = \alpha_i$ is the only variable. Hence, α_i can be obtained by finding the roots of

$$A_0\beta^{N+2} + A_1\beta^{N+1} + A_2\beta^N + A_3\beta^{N-1} + \cdots + A_{N+2} = 0. \tag{A2}$$

Using equation (42), the coefficient A_0 is given by

$$A_0 = -2K_M + \sum_{j=1}^{N+2} (C_j^{M+1} - C_j^M)\mathrm{e}^{p_j t_M}. \tag{A3}$$

Using equation (36), the relationship arising from the continuity of displacement at $t = t_M$ is

$$A_0 = -2K_M - K_{M+1} + K_M = 0, \tag{A4}$$

because $K_{M+1} = -K_M$.

The coefficient of α_i^{N+1} in equation (42) is A_1, and is given by

$$\begin{aligned} A_1 &= -2K_M \sum_{j=1}^{N+2} p_j + \sum_{j=1}^{N+2} (C_j^{M+1} - C_j^M)\left(\sum_{l=1, l\neq j}^{N+2} p_l\right)\mathrm{e}^{p_j t_M} \\ &= -2K_M \sum_{j=1}^{N+2} p_j + \sum_{j=1}^{N+2}\left(\sum_{l=1}^{N+2} p_l - p_j\right)C_j^{M+1}\mathrm{e}^{p_j t_M} \\ &\quad - \sum_{j=1}^{N+2}\left(\sum_{l=1}^{N+2} p_l - p_j\right)C_j^{M}\mathrm{e}^{p_j t_M} \\ &= \left(\sum_{j=1}^{N+2} p_j\right)\left(-2K_M + \sum_{j=1}^{N+2} C_j^{M+1}\mathrm{e}^{p_j t_M} - \sum_{j=1}^{N+2} C_j^{M}\mathrm{e}^{p_j t_M}\right) \\ &\quad - \left(\sum_{j=1}^{N+2} (C_j^{M+1} - C_j^M)\, p_j \mathrm{e}^{p_j t_M}\right). \end{aligned} \tag{A5}$$

Using the velocity-continuity relationship

$$\dot{x}_M = \sum_{j=1}^{N+2} p_j C_j^M \mathrm{e}^{p_j t_M} = \sum_{j=1}^{N+2} p_j C_j^{M+1}\mathrm{e}^{p_j t_M} \tag{A6}$$

and the displacement-continuity relationship (36) in equation (A5) yields

$$\begin{aligned} A_1 &= \left(\sum_{j=1}^{N+2} p_j\right)(-2K_M + (x_M - K_{M+1} - (x_M - K_M)) - \dot{x}_M + \dot{x}_M \\ &= \left(\sum_{j=1}^{N+2} p_j\right)(-2K_M - K_{M+1} + K_M) = 0. \end{aligned}$$

Thus the leading two coefficients are zero, which makes equation (A2) of order N. It can further be proved that $A_2 \neq 0$. For example, for the case $N = 1$, $A_2 = (2kK_M/m)(-1 + a/\alpha)$. A_2 will be zero only when $a = \alpha$, which represents the case of marginal stability (zero eigenvalue) for a system with $N = 1$.