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ESTIMATION OF SEA COUPLING LOSS FACTORS USING A DUAL FORMULATION AND FEM MODAL INFORMATION, PART I: THEORY

L. Maxit and J.-L. Guyader

INSA de Lyon (*National Institute of Applied Sciences*), *L.V.A.* (*Vibrations—Acoustic Laboratory*), *Bat*. 303, 20, *avenue Albert Einstein*, 69621 *Villeurbanne Cedex*, *France*.
E-mail: *maxit@lva.insa-lyon.fr*; *guyader@lva.insa-lyon.fr*

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The theoretical approach presented in this paper allows SEA coupling loss factors for subsystems to be modelled with FEM. It is then possible to take into account the complicated substructure that can be encountered in practical industrial application. The technique relies on the basic SEA relation for coupled oscillators and the use of dual modal formulation to describe vibration of coupled subsystems. With this approach, the boundary conditions of uncoupled subsystems are clearly defined and, as assumed in SEA, no modal coupling exists in a subsystem. Modes of two different subsystems are coupled together by gyroscopic elements and the coupling strength is related to eigenfrequencies of the uncoupled subsystems and mode shapes through the interaction modal works. A general expression for CLF has been obtained, and it allows CLF to be determined only from the knowledge of the modes of the uncoupled subsystems and the modal damping. Finite element model can be used to calculate the modal information in the case of complex substructures. It is possible to treat the case of heterogeneous subsystems having three-dimensional vibration motions without difficulty. Contrary to the classical approach which is based on SEA inverse matrix and numerical experiments which necessitate calculations of subsystem energies for the coupled structures for many excitation points, this technique calculates CLF directly from the governing equations without solving them. In a companion paper, the present approach is applied to a simple example to illustrate and validate the approach.

1. INTRODUCTION

Statistical energy analysis (SEA) allows the vibro-acoustic behaviour of complex structures in mid- and high-frequency range to be predicted. The method relates the power flow exchanged by two-coupled subsystems to total subsystem energies by the coupling loss factor (CLF). Writing the power balance for stationary motion in each subsystem produces a linear equation system where the unknowns are the total energies of subsystems. Then, the difficulty in applying SEA is not due to solving complicated equations, but to the evaluation of coupling loss factors.

Several techniques have been developed to determine CLF. The travelling wave approach is the most popular to obtain a theoretical expression in simple cases of coupled beams, plates, and shells (see reference [1], Chapter 10). Based on the evaluation of the wave transmission coefficient, this approach is easy to use. However, it can lead to mistakes for a system having a low modal overlap (see reference [2]). CLF can be only

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calculated for theoretical substructures which limits the application in the case of manufacturing structures. Different experimental approaches have been elaborated to evaluate CLF by measurement [3] using measured point mobility [4]; with the concept of energetic mobility [5]; and from measured impulse responses in the coupled system [6–8]; based upon inversion of the SEA equation. The latter, called the power injection method, is the most popular. The difficulty of the method relies on the number of measurements of transfer functions that are necessary and also the evaluation of the spatial averaged energy from some measurement points in the case of heterogeneous subsystems. These experimental techniques are useful in an industrial context but they necessitate having the mechanical structure in place, so that it is not a predictive method. To solve this problem, the possibility of using a Finite Element Method to calculate numerically the response of two-coupled subsystems and then to identify CLF has been used [9–11]. This type of approach is predictive, and has been used to study the validity of some SEA assumptions. The difficulties of this approach are similar to experimental approach for heterogeneous substructures. The limits of application are, of course, the frequency range to use FEM, and the difficulty to give different damping loss factors to the various subsystems because global modes are used. The approach presented in reference [12] differs from previous methods by the use of the numerical Green functions of the un-coupled subsystems and a receptance-based approach. It is then possible to take into account different damping for various subsystems and to increase, in some cases, the frequency range considered because the FEM calculations are made for individual uncoupled subsystems. However, it is necessary to include sufficient uncoupled modes to constitute the Green function which can be a frequency limitation.

The approach presented in this paper allows CLF to be calculated directly from subsystem modal equations. The method uses the modal definitions of CLF which are established in the basic SEA formulation (see reference [1], Chapter 3, [13,14]) and reviewed in Section 2. The expression of the power flow exchanged by two oscillators coupled by a gyroscopic element will be used to calculate the coupling loss factor by summation of the different intermodal coupling factors.

Section 3, describes the major theoretical contribution of this paper. The purpose of this section is to propose one general approach allowing modal equations of motion to be obtained which can be represented as set of oscillators (modes) coupled by gyroscopic elements. These modal equations will enable the modal coupling coefficients necessary to calculate intermodal coupling factors to be identified and thus, coupling loss factors. The formulation will be presented in the general case of the two-coupled continuous three-dimensional elasto-dynamic systems. This approach is based on two subsystem modes definitions, and on the use of the dual modal formulation (DMF). It is analogous to the approach used to describe the mechanical structure–cavity coupling [15–17]. Therefore, a generalization similar to that suggested in reference [18] is presented here.

In the final section, DMF is applied to a discretized system to determine the modal coupling coefficients. Then, in the cases of complex subsystems, FEM can be used to calculate the modes of each uncoupled subsystem and to deduce the coupling loss factor.

SEA provides statistical estimates of energy for an ensemble average of systems whereas FEM gives the deterministic response for a system whose characteristics are known exactly. In consequence, the direct use of finite elements results gives only an estimate of CLF based on one system. However, one advantage of the proposed method is the possibility of deriving a statistical estimate of CLF by introducing a random distribution of eigenfrequencies, and calculating the associated average CLF.

2. CLF EXPRESSION DEDUCED FROM MODAL SEA FORMULATION

2.1. POWER FLOW EXCHANGED BY TWO OSCILLATORS COUPLED BY GYROSCOPIC ELEMENT

Two oscillators coupled via a gyroscopic element (see Figure 1) are considered. M_1, M_2 are the masses, and K_1, K_2 are the stiffnesses of the oscillators. The natural angular frequencies of each uncoupled oscillator are thus $\omega_1 = \sqrt{K_1 M_1^{-1}}$ for oscillator 1 and $\omega_2 = \sqrt{K_2 M_2^{-1}}$ for oscillator 2. (A list of nomenclature is given in Appendix A.)

Each oscillator is damped by a viscous absorber of damping coefficient: Δ_1 for oscillator 1, and Δ_2 for oscillator 2. The coupling forces transmitted through the coupling of constant G_C are proportional to the mass velocities, $\dot{y}_1$ and $\dot{y}_2$. It is assumed that when the velocity of oscillator 1 is positive, the force applied on oscillator 2 is negative and when the velocity of oscillator 2 is positive, the force applied on oscillator 1 is positive. The equations of motion for the two coupled oscillators excited by external forces F_1 and F_2 are then given by

$$\ddot{y}_1(t) + \Delta_1 \dot{y}_1(t) + \omega_1^2 y_1(t) - \sqrt{M_1^{-1} M_2}\gamma \dot{y}_2(t) = F_1(t),$$

$$\ddot{y}_2(t) + \Delta_2 \dot{y}_2(t) + \omega_2^2 y_2(t) + \sqrt{M_1 M_2^{-1}}\gamma \dot{y}_1(t) = F_2(t), \tag{1}$$

where the gyroscopic modal coupling coefficient, γ, is

$$\gamma = G_C/\sqrt{M_1 M_2}. \tag{2}$$

Now, it is assumed that external excitations are independent (uncorrelated), stationary and to have a constant PSD (white noise). It has been demonstrated in this case [1] that the time-averaged power flow from oscillators 1 to 2, P_{12}, is proportional to the difference between the time-averaged total energies of the oscillators $(E_1 - E_2)$,

$$P_{12} = \beta(E_1 - E_2), \tag{3}$$

where the coefficient β is expressed by

$$\beta = \frac{\gamma^2(\Delta_1\omega_2^2 + \Delta_2\omega_1^2)}{(\omega_1^2 - \omega_2^2)^2 + (\Delta_1 + \Delta_2)(\Delta_1\omega_2^2 + \Delta_2\omega_1^2)}. \tag{4}$$

Note that the coefficient β depends on the normal angular frequencies of the uncoupled oscillators, the damping constants and the coupling coefficient γ.

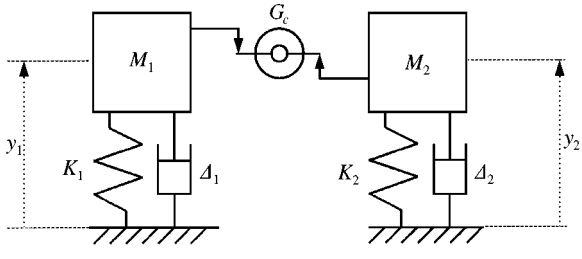

Figure 1. Two oscillators coupled by gyroscopic element.

2.2. FROM TWO-COUPLED OSCILLATORS TO TWO COUPLED SUBSYSTEMS

Now, consider two-coupled mechanical subsystems. The interaction between these subsystems may correspondingly be studied by investigating the interaction between two sets of resonant modes in the frequency band considered. This assertion, which is given little attention in the literature, will be the subject of a detailed study in Section 3. It will be shown that for an appropriate choice of modes and using dual modal formulation, the modal equations of motions have the form

$$\ddot{c}_p^1(t) + \Delta_p^1 \dot{c}_p^1(t) + (\omega_p^1)^2 c_p^1(t) = \frac{F_p^1}{(\omega_p^1)^2 M_p^1} + \sum_{r=1}^{N_2} \left(\sqrt{\frac{M_r^2}{(\omega_p^1)^2 M_p^1}}\, \gamma_{pr}^{12} \dot{a}_r^2(t) \right) \quad \forall p \in [1, N_1],$$

$$\ddot{a}_q^2(t) + \Delta_q^2 \dot{a}_q^2(t) + (\omega_q^2)^2 a_q^2(t) = \frac{F_q^2}{M_q^2} - \sum_{m=1}^{N_1} \left(\sqrt{\frac{(\omega_m^1)^2 M_m^1}{M_q^2}}\, \gamma_{mq}^{12} \dot{c}_m^1(t) \right) \quad \forall q \in [1, N_2], \tag{5}$$

where c_p^1, M_p^1, ω_p^1 and a_q^2, M_q^2, ω_q^2 are respectively the modal amplitudes, generalized masses, and natural frequencies of mode p of subsystem 1 and those of mode q of subsystem 2, F_p^1 and F_q^2 are the generalized 'forces' applied on respectively subsystem 1 and subsystem 2, and γ_{pq}^{12} are the modal coupling coefficients between the pair of modes (p, q).

In the frequency band, is considered to include N_1 resonant modes for subsystem 1 and N_2 resonant modes for subsystem 2. The choice of modes of Section 3 will go in this direction, which suggests that these resonant modes can approximately represent the dynamic behaviour of the coupled subsystems in the frequency band considered.

Then, equations (5) can be schematically represented as Figure 2; one mode of one subsystem is not coupled with modes of the same subsystem but is coupled by gyroscopic elements with the modes of the other subsystem.

Isolating, in the equations system, the coupling between the mode p of subsystem 1 and the mode q of subsystem 2, gives

$$\ddot{c}_p^1(t) + \Delta_p^1 \dot{c}_p^1(t) + (\omega_p^1)^2 c_p^1(t) - \sqrt{\frac{M_q^2}{M_p^1 (\omega_p^1)^2}}\, \gamma_{pq}^{12} \dot{a}_q^2(t) = \frac{L_{1pq}}{M_p^1 (\omega_p^1)^2},$$

$$\ddot{a}_q^2(t) + \Delta_q^2 \dot{a}_q^2(t) + (\omega_q^2)^2 a_q^2(t) + \sqrt{\frac{M_p^1 (\omega_p^1)^2}{M_q^2}}\, \gamma_{pq}^{12} \dot{c}_p^1(t) = \frac{L_{2pq}}{M_q^2}, \tag{6}$$

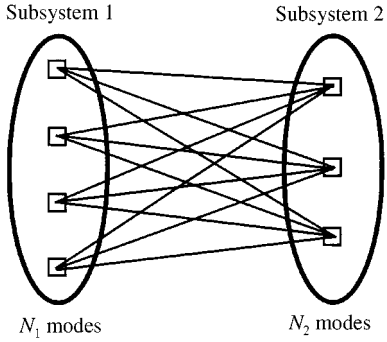

Figure 2. Illustration of coupling of N_1 modes of subsystem 1 with N_2 modes of subsystem 2.

where the terms L_{1pq} and L_{2pq} contain the generalized forces and the interactions forces with all the others modes.

Then, by analogy with the equations of motions of two coupled oscillators (1), the power flow, Π_{pq}^{12} exchanged by these two modes can be deduced,

$$\Pi_{pq}^{12} = \beta_{pq}^{12}(E_p^1 - E_q^2), \tag{7}$$

where E_p^1 and E_q^2 are the modal energies of mode p of subsystem 1 and mode q of subsystem 2, β_{pq}^{12}, called the intermodal coupling factor (ICF), can be determined by analogy with equation (4) as a function of the natural angular frequencies ω_p^1, ω_q^2, the modal damping bandwidth Δ_p^1, Δ_q^2 and the modal coupling coefficient γ_{pq}^{12}:

$$\beta_{pq}^{12} = \frac{(\gamma_{pq}^{12})^2(\Delta_p^1(\omega_q^2)^2 + \Delta_q^2(\omega_p^1)^2)}{((\omega_p^1)^2 - (\omega_q^2)^2)^2 + (\Delta_p^1 + \Delta_q^2)(\Delta_p^1(\omega_q^2)^2 + \Delta_q^2(\omega_p^1)^2)}. \tag{8}$$

The application of equation (3) for the coupling of these two modes implies that the forces L_{1pq} and L_{2pq} are uncorrelated, stationary, and have flat spectra in the frequency band. The condition of validity of this assumption has not been clearly established in the literature. It implies a certain independence of modal amplitudes (see reference [1, pp. 60–61]). According to Fahy [19, see pp. 434–436], the forces due to coupling with other modes will be relatively flat if modal overlap is sufficient and their correlation is small if the coupling between subsystems is weak.

The power flow exchanged by the two subsystems Π_{1-2} is the sum of all the individual mode-to-mode power flows. Then

$$\Pi_{1-2} = \sum_{p=1}^{N_1}\sum_{q=1}^{N_2} \Pi_{pq}^{12} = \sum_{p=1}^{N_1}\sum_{q=1}^{N_2} \beta_{pq}^{12}(E_p^1 - E_q^2). \tag{9}$$

Modal energy equipartition assumption is made,

$$E_p^1 = e_1, \quad \forall p \in [1, \ldots, N_1], \quad E_q^2 = e_2 \quad \forall q \in [1, \ldots, N_2]. \tag{10}$$

where e_1 and e_2 are two constants.

With the orthogonality property of modes

$$\xi_1 = \sum_{p=1}^{N_1} E_p^1, \quad \xi_2 = \sum_{q=1}^{N_2} E_q^2, \tag{11}$$

where ξ_1 and ξ_2 represent the time-averaged total energies of subsystems 1 and 2.

Therefore, using equations (10) and (11), one can write

$$E_p^1 = \xi_1/N_1, \qquad E_q^2 = \xi_2/N_2, \tag{12}$$

and this can be put in equation (9). Finally,

$$\Pi_{1-2} = \omega_c \eta_{12}\left(\xi_1 - \frac{N_1}{N_2}\,\xi_2\right), \tag{13}$$

where the coupling loss factor η_{12} is expressed by

$$\eta_{12} = \frac{\sum_{p=1}^{N_1} \sum_{q=1}^{N_2} \beta_{pq}^{12}}{N_1 \omega_c}, \tag{14}$$

with ω_c the central angular frequency of the frequency band of interest.

This final expression allows the coupling loss factor from the intermodal coupling factors to be estimated which depend on subsystems modes and on modal coupling coefficients, γ_{pq}^{12}.

SEA provides statistical estimates of energy for an ensemble average of subsystems whereas equation (14) is derived from one individual system. As in reference [20], the coefficient obtained from (14), which concerns one individual case, can be distinguished from the traditional CLF that is used for the SEA ensemble. One advantage of the proposed method is the possibility of deriving a statistical estimate of CLF by introducing a random distribution of eigenfrequencies, and by calculating the associated ensemble average CLF. To obtain this statistical estimate of CLF, the intermodal coupling factors would have to be averaged over a population of structures. Lyon [1] gives a mean value of the ICF for a particular population of structures such that the natural frequencies of each subsystem are random variables with values uniformly probable over the frequency band of interest, $\Delta\omega$:

$$\langle \beta_{pq}^{12} \rangle_{\omega_p^1 \omega_q^2} = \frac{\pi}{2\Delta\omega} (\gamma_{pq}^{12})^2. \tag{15}$$

This formula could replace the deterministic ICF in equation (13) in order to obtain the ensemble-averaged coupling loss factor:

$$\langle \eta_{12} \rangle_e = \frac{\sum_{p=1}^{N_1} \sum_{q=1}^{N_2} \langle \beta_{pq}^{12} \rangle_{\omega_p^1 \omega_q^2}}{N_1 \omega_c}. \tag{16}$$

This approach produces simple results but the population of structures considered can be unrealistic. A second approach consists of replacing the deterministic eigenfrequencies by Gaussian random variables centred on the corresponding eigenfrequencies. Then one can get an ensemble-averaged CLF from a Monte Carlo simulation.

With regard to the equipartition assumption, it can be noted that with the approach presented in reference [21] and with the calculation of the intermodal coupling factors by the present approach, it is possible to extend SEA to the case of non-modal energy equipartition.

The boundary conditions necessary to establish equations (5) and how to evaluate the modal coupling coefficients have not yet been defined. It is the aim of the next section.

3. DEFINITION OF SUBSYSTEM MODES AND USE OF THE DUAL MODAL FORMULATION TO OBTAIN CLF

The approach presented here is not classical for mechanical coupled systems. However, the dual formulation is the standard approach used to study the coupling cavity–structure (see references [15–17]).

Therefore, for each time possible, a reference will be made, to this case, to facilitate comprehension.

The present formulation is based on the work of Karnopp. Indeed, in reference [18]† he initiated the use of dual formulation (enunciated in reference [22]) to the coupling of mechanical subsystems and applies the method for the coupling of two rods of identical section. A generalization of this approach for the coupling of two continuous mechanical systems is presented here.

3.1. STRUCTURE DESCRIPTION

Two elastic continuous mechanic systems are considered which are rigidly coupled on $S_{Coupling}$ as shown in Figure 3. V^1 and V^2 represent the volumes occupied by subsystem 1 and subsystem 2. $S^1_{Clamped}$ and S^1_{Free}, and $S^2_{Clamped}$ and S^2_{Free} are respectively the boundary surfaces with blocked displacements and with free displacements for subsystem 1 and subsystem 2. Initially, free vibration of elastic conservative systems is considered to find modal equations of motion of the coupled subsystems.

In Lagrangian coordinates x_i, $(i = 1, 2, 3)$, is defined respectively on the surfaces of volume V^1 and V^2 and the unit vectors n^1 and n^2 along the outer normals of the volumes. The variables W^1_i and W^2_i represent displacements in V^1 and V^2 respectively. σ^1_{ij} and σ^2_{ij} are the stress tensors, ε^1_{ij} and ε^2_{ij} are the strain tensors, and S^1_{ijkl} and S^2_{ijkl} are the compliance tensors associated to materials of V^1 and V^2 respectively. ρ_1 and ρ_2 are the mass densities and are independent of time. The dynamical behaviour of the structure between time t_0 and t_1 is studied.

The continuity conditions on $S_{Coupling}$ can be expressed by

$$W^1_i = W^2_i \quad \text{on } S_{Coupling} \times]t_0, t_1[, \tag{17}$$

$$\sigma^1_{ij} n^1_j + \sigma^2_{ij} n^2_j = 0 \quad \text{on } S_{Coupling} \times]t_0, t_1[. \tag{18}$$

n^1_i and n^2_j are the jth components of the outer normal vectors n^1 and n^2.

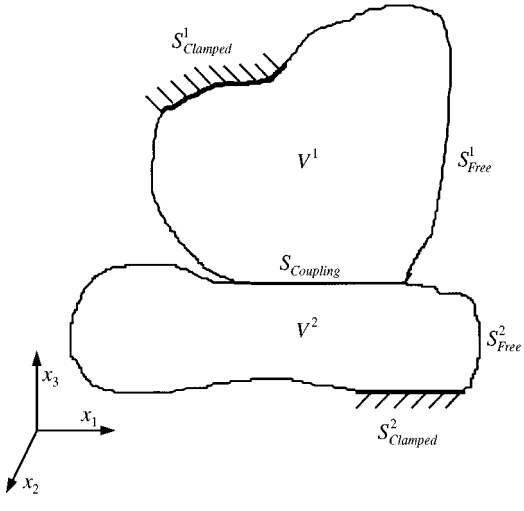

Figure 3. Representation of the coupling of the two elastic continuum systems.

† It may be noted that several typing errors and one error of sign are present in the equations of this paper.

3.2. DEFINITION OF SUBSTRUCTURES

For the substructuring of the problem, it is now necessary to separate these two subsystem, and to prescribe displacements or forces on $S_{Coupling}$ for each subsystem. $S^1_{Coupling}$ and $S^2_{Coupling}$ are defined (as the coupling surfaces $S_{Coupling}$ in subsystem 1 and subsystem 2) respectively.

Displacements on $S^1_{Coupling}$ are prescribed for the subsystem 1 and forces on $S^2_{Coupling}$ for the subsystem 2 (see Figure 4). Then, for subsystem 1, the boundary condition on $S^1_{Coupling}$ is assumed to be

$$W_i^1 = \bar{W}_i^c \quad \text{on } S^1_{Coupling} \times]t_0, t_1[, \tag{19}$$

and, for subsystem 2, the boundary condition on $S^2_{Coupling}$ to be

$$\sigma_{ij}^2 n_j^2 = \vec{F}_i^c \quad \text{on } S^1_{Coupling} \times]t_0, t_1[. \tag{20}$$

Note: For the cavity–structure coupling, the boundary conditions for the cavity coupled with the structure are the displacements imposed by the structure (like subsystem 1) and the boundary conditions for the structure coupled with the cavity are the pressures imposed by the cavity (like subsystem 2).

Then, from the fundamental equations and principles of continuum mechanics, the linear field equations describing the dynamics of both subsystems can be expressed.

For subsystem 1, the following equations constitute an initial problem called problem 1:

Equations of motion:

$$\rho_1 \frac{\partial^2 W_i^1}{\partial t^2} = \sigma_{ij,\,j}^1 \quad \text{in } V^1 \times]t_0, t_1[. \tag{21}$$

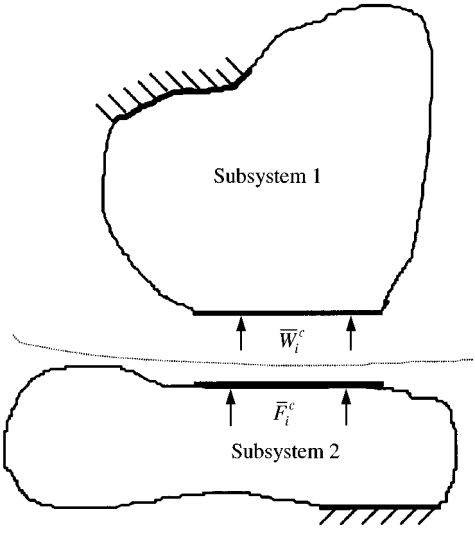

Figure 4. Illustration of the fictive separation of subsystems.

Constitutive equations for Hookean linear elastic solid:

$$\varepsilon_{ij}^1 = \tfrac{1}{2}(W_{i,j}^1 + W_{j,i}^1) = S_{ijkl}^1 \sigma_{kl}^1 \quad \text{in } V^1 \times]t_0, t_1[. \tag{22}$$

Boundary conditions:

$$\sigma_{ij}^1 n_j^1 = 0 \quad \text{on } S_{Free}^1 \times]t_0, t_1[, \tag{23}$$

$$W_i^1 = 0 \quad \text{on } S_{Clamped}^1 \times]t_0, t_1[, \tag{24}$$

$$W_i^1 = \bar{W}_i^c \quad \text{on } S_{Coupling}^1 \times]t_0, t_1[. \tag{25}$$

For subsystem 2, one has problem 2.

Equations of motion:

$$\rho_2 \frac{\partial^2 W_i^2}{\partial t^2} = \sigma_{ij,j}^2 \quad \text{in } V^2 \times]t_0, t_1[. \tag{26}$$

Constitutive equations for Hookean linear elastic solid:

$$\varepsilon_{ij}^2 = \tfrac{1}{2}(W_{i,j}^2 + W_{j,i}^2) = S_{ijkl}^2 \sigma_{kl}^2 \quad \text{in } V^2 \times]t_0, t_1[. \tag{27}$$

Boundary conditions:

$$\sigma_{ij}^2 n_j^2 = 0 \quad \text{on } S_{Free}^2 \times]t_0, t_1[, \tag{28}$$

$$W_i^2 = 0 \quad \text{on } S_{Clamped}^2 \times]t_0, t_1[, \tag{29}$$

$$\sigma_{ij}^2 n_j^2 = \bar{F}_i^c \quad \text{on } S_{Coupling}^2 \times]t_0, t_1[. \tag{30}$$

3.3. VARIATIONAL FORMULATION OF THE PROBLEMS USING THE REISSNER PRINCIPLE

For each subsystem, it is possible to express the dynamic problem by using the Reissner principle. This formulation will enable the use of the modal expansions.

For problem 1, the associate Reissner functional $\Psi_R^1(W_i^1, \sigma_{ij}^1)$ can be expressed by (see references [23, 24] for the second form of the Reissner functional)

$$\Psi_R^1(W_i^1, \sigma_{ij}^1)\colon \Omega_R^1 \times \Sigma_R^1 \to \Re,$$

$$(W_i^1, \sigma_{ij}^1) \to \Psi_R^1(W_i^1, \sigma_{ij}^1),$$

$$\Psi_R^1(W_i^1, \sigma_{ij}^1) = \int_{t_0}^{t_1} \left\{ \int_{V^1} \left[\frac{1}{2} \rho_1 \left(\frac{\partial^2 W_i^1}{\partial t^2} \right) + \sigma_{ij,j}^1 W_i^1 + \frac{1}{2} \sigma_{ij}^1 S_{ijkl}^1 \sigma_{kl}^1 \right] \mathrm{d}V \right.$$

$$\left. - \int_{S_{Coupling}^1} \bar{W}_i^c \sigma_{ij}^1 n_j^1 \, \mathrm{d}S \right\} \mathrm{d}t. \tag{31}$$

The associate admissible spaces are

$$\Omega_R^1(V^1 x]t_0, t_1[) = \{W_i^1 / W_i^1 \in H^1(V^1); \quad W_i^1 = 0 / S_{Clamped}^1\}, \tag{32}$$

$$\Sigma_R^1(V^1 x]t_0, t_1[) = \{\sigma_{ij}^1 / \sigma_{ij}^1 = \sigma_{ji}^1; \sigma_{ij}^1 \in L^2(V^1); \quad \sigma_{ij,j}^1 \in L^2(V^1); \quad \sigma_{ij}^1 n_j^1 = 0 / S_{Free}^1\}. \tag{33}$$

The problem constituted by equations (21–25) can be replaced by searching for W_i^1 and σ_{ij}^1 in their respective admissible spaces Ω_R^1 and Σ_R^1 which render the functional $\Psi_R^1(W_i^1, \sigma_{ij}^1)$ stationary.

In the same way, for problem 2, the Reissner functional $\Psi_R^2(W_i^2, \sigma_{ij}^2)$ can be expressed by (see the first form of the Reissner functional in references [23, 24])

$$\Psi_R^2(W_i^2, \sigma_{ij}^2): \Omega_R^2 \times \Sigma_R^2 \to \Re,$$

$$(W_i^2, \sigma_{ij}^2) \to \Psi_R^2(W_i^2, \sigma_{ij}^2),$$

$$\Psi_R^2(W_i^2, \sigma_{ij}^2) = \int_{t_0}^{t_1} \left\{ \int_{V^1} \left[\frac{1}{2} \rho_2 \left(\frac{\partial^2 W_i^2}{\partial t^2} \right) - \sigma_{ij}^2 \frac{1}{2} (W_{i,j}^2 + W_{j,i}^2) + \frac{1}{2} \sigma_{ij}^2 S_{ijkl}^2 \sigma_{kl}^2 \right] \mathrm{d}V \right.$$

$$\left. - \int_{S_{Coupling}^1} \bar{F}_i^c W_i^2 \, \mathrm{d}S \right\} \mathrm{d}t. \tag{34}$$

The associate admissible spaces are

$$\Omega_R^2(V^2 x]t_0, t_1[) = \{W_i^2 / W_i^2 \in H^1(V^2); \quad W_i^2 = 0 / S_{Clamped}^2\}, \tag{35}$$

$$\Sigma_R^2(V^2 x]t_0, t_1[) = \{\sigma_{ij}^2 / \sigma_{ij}^2 = \sigma_{ji}^2; \sigma_{ij}^2 \in L^2(V^2); \quad \sigma_{ij}^2 \cdot n_j^2 = 0 / S_{Free}^2\}. \tag{36}$$

In this case, the problem constituted by equations (26–30) can be replaced by searching for W_i^2 and σ_{ij}^2 in their respective admissible spaces Ω_R^2 and S_R^2 which render the functional $\Psi_R^2(W_i^2, \sigma_{ij}^2)$ stationary.

3.4. MODES OF UNCOUPLED SUBSYSTEMS

3.4.1. *Definition of subsystem modes*

Initially, modal expansion will be used to solve the two previous variational problems separately. The sets of mode shapes which will be considered generate subspaces of admissible spaces associated with each variational problem. The approximate solutions of these problems will then be found by considering these subspaces. The definition of the modes of each subsystem must be judicious in order to link the two problems in a second step. The coupling of the problems will be possible if modal expansion of the stress field of subsystem 1 allows the force excitation of subsystem 2 to be determined and "inversely", if modal expansion of the displacement field of subsystem 2 allows the displacement excitation of subsystem 1 to be calculated. The subspaces of admissible spaces, and consequently, the mode' definitions must be chosen appropriately.

Modes of subsystem 1 are then defined with blocked displacements on $S_{Coupling}^1$ (see Figure 5(a)). These are called the modes of the uncoupled-blocked subsystem or, in short,

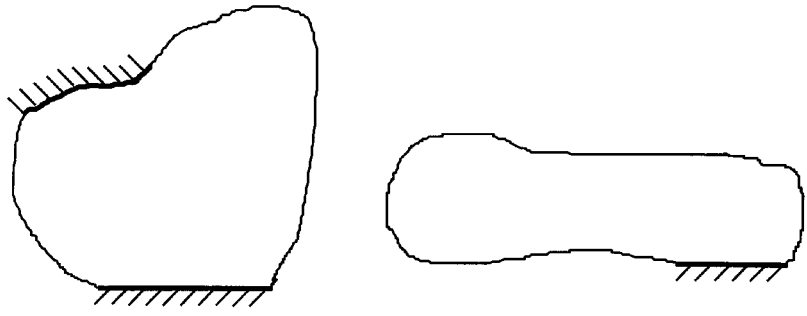

Figure 5. Subsystem definition: (a) uncoupled-blocked subsystem 1; (b) uncoupled-free subsystem 2.

the blocked modes. These modes comply with the admissibility conditions on stresses for the problem 1. (In the case of cavity–structure coupling, the modes of the cavity are the blocked modes because they are determined with rigid walls.)

On the other hand, modes of subsystem 2 are calculated with null stresses on $S^2_{Coupling}$ (see Figure 5(b)). These are called the modes of the uncoupled-free subsystem or free modes. These modes comply with the kinematic admissible conditions for the problem 2. (They correspond to the modes of the *in vacuo* structure in the case of cavity–structure coupling.)

3.4.2. *Calculations and properties of modes*

The eigenvalue problems independently compiled with each field can be obtained from equations of motion, *constitutive law* and boundary conditions expressed in mixed variables.

For the displacement field of subsystem $\alpha(\alpha = 1, 2)$, the eigenvalue problem called the primal problem can be written after a separation of time and space, $W_i^\alpha(M, t) = \tilde{W}_i^\alpha(M)\mathrm{e}^{\mathrm{j}\omega_\alpha t}$:

$$-\rho_\alpha(\omega_\alpha)^2 \tilde{W}_i^\alpha = \tfrac{1}{2} S^{-1}_{ijkl}(\tilde{W}^\alpha_{k,l} + \tilde{W}^\alpha_{l,k})_{,j} \quad \text{in } V^\alpha + \text{boundary conditions.} \tag{37}$$

For the stress field of subsystem α, the eigenvalue problem called the dual problem can be written with $\sigma_{ij}^\alpha(M, t) = \tilde{\sigma}_{ij}^\alpha(M)\,\mathrm{e}^{\mathrm{j}\omega_\alpha t}$:

$$-\rho_\alpha(\omega_\alpha)^2 \tilde{\sigma}_{ij}^\alpha = S^{\alpha^{-1}}_{ijkl}(\tilde{\sigma}^\alpha_{km,m,l} + \tilde{\sigma}^\alpha_{lm,m,k}) \quad \text{in } V^\alpha + \text{boundary conditions.} \tag{38}$$

These two problems (primal and dual) are extracted from the same problem expressed in mixed variables. In general, the problem with mixed variables is respected by the couple of solutions $(\omega_\alpha, \tilde{W}_i^\alpha, \tilde{\sigma}_{ij}^\alpha)$. However, it can be noted that one solution of the dual problem having a null eigenvalue can exist and does not have equivalence in the primal problem. This solution does not then appear in mixed variables. For a cavity, it corresponds to the Helmholtz mode that is found in terms of pressure and not in terms of displacement. This solution having a null eigenfrequency will have a negligible contribution in the modal expansion (because it is non-resonant) as soon as other resonant modes participate to the response. As the CLF calculation takes into account only the resonant modes in a frequency band, the Helmholtz mode contribution can be neglected. Thus, the problem in mixed variables can be used to define the stress and displacement shapes of each mode.

One can define, for the uncoupled-blocked subsystem 1, ω_p^1, the natural angular frequency of mode p; $\tilde{W}_i^{1p}$, the displacement mode shape; and $\tilde{\sigma}_{ij}^{1p}$, the stress mode shape with the following equations (for $\omega_p^1 \neq 0$):

$$-\rho_1(\omega_p^1)^2\,\tilde{W}_i^{1p} = \tilde{\sigma}_{ij,j}^{1p} \quad \text{in } V^1, \tag{39}$$

$$\tfrac{1}{2}(\tilde{W}_{i,j}^{1p} + \tilde{W}_{j,i}^{1p}) = S_{ijkl}^1 \tilde{\sigma}_{kl}^{1p} \quad \text{in } V^1, \tag{40}$$

$$\tilde{\sigma}_{ij}^{1p} n_j^1 = 0 \quad \text{on } S_{Free}^1, \qquad \tilde{W}_i^{1p} = 0 \quad \text{on } S_{Clamped}^1, \qquad \tilde{W}_i^{1p} = 0 \quad \text{on } S_{Coupling}^1 \tag{41}$$

Similarly one can define for the uncoupled-free subsystem 2, ω_q^2, the natural angular frequency of mode q; $\tilde{W}_i^{2q}$, the displacement mode shape; and $\tilde{\sigma}_{ij}^{2q}$, the stress mode shape with the following equations (for $\omega_q^2 \neq 0$):

$$-\rho_2(\omega_q^2)^2\,\tilde{W}_i^{2q} = \tilde{\sigma}_{ij,j}^{2q} \quad \text{in } V^2, \tag{42}$$

$$\tfrac{1}{2}(\tilde{W}_{i,j}^{2q} + \tilde{W}_{j,i}^{2q}) = S_{ijkl}^2 \tilde{\sigma}_{kl}^{2q} \quad \text{in } V^2, \tag{43}$$

$$\tilde{\sigma}_{ij}^{2q} n_j^2 = 0 \quad \text{on } S_{Free}^2, \qquad \tilde{W}_i^{2q} = 0 \quad \text{on } S_{Clamped}^2, \qquad \tilde{\sigma}_{ij}^{2q} n_j^2 = 0 \quad \text{on } S_{Coupling}^2. \tag{44}$$

These modes have the following orthogonality properties (see reference [25]) for subsystem $\alpha(\alpha = 1, 2)$:

$$\int_{V^\alpha} \omega_p^\alpha \omega_q^\alpha \rho_\alpha \tilde{W}_i^{\alpha p} \tilde{W}_i^{\alpha q}\,\mathrm{d}V = K_p^\alpha \delta_{pq}, \tag{45}$$

$$\int_{V^\alpha} \tilde{\sigma}_{ij}^{\alpha p} S_{ijkl}^\alpha \tilde{\sigma}_{kl}^{\alpha q}\,\mathrm{d}V = K_p^\alpha \delta_{pq}, \tag{46}$$

$$\int_{V^\alpha} \tilde{\sigma}_{ij,\,j}^{\alpha p} \tilde{W}_i^{\alpha q}\,\mathrm{d}V = K_p^\alpha \delta_{pq}, \tag{47}$$

$$\int_{V^\alpha} \tilde{\sigma}_{ij}^{\alpha p}\, \tfrac{1}{2}(\tilde{W}_{i,j}^{\alpha q} + \tilde{W}_{j,i}^{\alpha q})\,\mathrm{d}V = K_p^\alpha \delta_{pq}, \tag{48}$$

where K_p^α is the modal stiffness of mode p of subsystem α, and δ_{pq} is the Kronecker symbol ($\delta_{pq} = 0$ if $p \neq q$; $\delta_{pq} = 1$ if $p = q$).

The natural angular frequencies can be expressed by

$$\omega_p^\alpha = \sqrt{K_{p/}^\alpha / M^\alpha}, \quad \alpha = 1, 2, \tag{49}$$

where M_p^α is the modal mass of mode p of subsystem α.

Note: The blocked modes of subsystem 1 comply with the admissibility conditions on stresses of the variational problem 1. Thus the set of the stress mode shapes $\{\tilde{\sigma}_{ij}^{1p}(M), p = 1, 2, \ldots, +\infty\}$ (taking into account the stress mode of null frequency if it exists) constitutes a base of the admissible space Σ_R^1. In a same way, the uncoupled-free modes of subsystem 2 comply with the kinematic admissible conditions of the variational

problem 2. The set of the displacement mode shapes $\{\tilde{W}_i^{2q}(M), q = 1, 2, \ldots, +\infty\}$ constitutes a base of Ω_R^2 (the displacement kinematically admissible space). This will enable the stress field for subsystem 1 and the displacement field for subsystem 2 on the coupling boundary to be calculated from the modal expansion.

3.5. MODAL EXPANSION OF THE SOLUTION

Expanding displacements and stresses of each subsystem in the modal bases yields

$$W_i^1(M,t) = \sum_{n=1}^{\infty} a_n^1(t)\,\tilde{W}_i^{1n}(M), \quad \sigma_{ij}^1(M,t) = \sum_{m=1}^{\infty} b_m^1(t)\,\tilde{\sigma}_{ij}^{1m}(M), \tag{50}$$

$$W_i^2(M,t) = \sum_{r=1}^{\infty} a_r^2(t)\,\tilde{W}_i^{2r}(M), \quad \sigma_{ij}^2(M,t) = \sum_{s=1}^{\infty} b_s^2(t)\,\tilde{\sigma}_{ij}^{2s}(M). \tag{51}$$

The expansion allows the weak solutions of the two problems (21–25), (26–30) to be found by using the Reissner principle and subspaces of admissible spaces. These solutions are weak solutions because the subspace generated by mode shapes will not permit strict verification of equation (25) for problem 1 and equation (30) for problem 2.

Upon introducing expansions (50, 51) into the variational principles (31, 34), and using the modes' orthogonality properties (45–48), the solution of each problem is then obtained by finding modal amplitudes that give stationary the Reissner's Functional. That is, modal amplitudes satisfying the Euler equation (52) associated to a_n^1, b_m^1 and a_p^2, b_q^2 for subsystem 1 and subsystem 2, are respectively,

$$/q_i: \frac{\partial}{\partial q_i}(F(q_i(t), \dot{q}_i(t))) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{q}_i}(F(q_i(t), \dot{q}_i(t)))\right) = 0, \tag{52}$$

where $F(q_i(t), \dot{q}_i(t))$ is the considered functional which depends on $q_i(t), \dot{q}_i(t)$.

Finally, for subsystem 1

$$/a_p^1: \; -\frac{K_p^1}{(\omega_p^1)^2}\,\ddot{a}_p^1(t) - K_p^1 b_p^1(t) = 0, \tag{53}$$

$$/b_p^1: \; -K_p^1 a_p^1(t) + K_p^1 b_p^1(t) - \int_{S^1_{Coupling}} \bar{W}_i^c \tilde{\sigma}_{ij}^{1p} n^1 \,\mathrm{d}S = 0, \tag{54}$$

and for subsystem 2.

$$/a_q^2: \; -\frac{K_q^2}{(\omega_q^2)^2}\,\ddot{a}_q^2(t) - K_q^2 b_q^2(t) + \int_{S^2_{Coupling}} \bar{F}_i^c \tilde{W}_i^{2q} \,\mathrm{d}S = 0, \tag{55}$$

$$/b_q^2: \; -K_q^2 a_q^2(t) + K_q^2 b_q^2(t) = 0. \tag{56}$$

According to the dual formulation used by Karnopp [18], the behaviours of subsystem 1 with stress modal amplitudes, $b_p^1(t)$, and subsystems 2 with displacement modal amplitudes $a_q^2(t)$ must be described. It is analogous to the coupling cavity–structure where

the pressure is the descriptive variable for the subsystem with blocked modes (cavity), and the displacements are the descriptive variables for the subsystem with free modes (structure). Therefore, combining time second derivative of equation (54) with equation (53), and equation (55) with equation (56) gives

$$\ddot{b}_p^1(t) + (\omega_p^1)^2 b_p^1(t) = \frac{1}{(\omega_p^1)^2 M_p^1} \int_{S^1_{Coupling}} \ddot{\bar{W}}_i^c \tilde{\sigma}_{ij}^{1p} n_j^1 \, \mathrm{d}S, \tag{57}$$

$$\ddot{a}_q^2(t) + (\omega_q^2)^2 a_q^2(t) = \frac{1}{M_q^2} \int_{S^1_{Coupling}} \bar{F}_i^c \tilde{W}_i^{2q} \, \mathrm{d}S. \tag{58}$$

3.6. FREE VIBRATIONS OF COUPLED SUBSYSTEMS

Equations (57) and (58) describe the coupling through specified displacements on subsystem 1 and forces on subsystem 2. To express coupling conditions, the specified force acting on subsystem 2 is opposite of the stress boundary vector of subsystem 1 (see equation (18)), and, the prescribed displacements on subsystem 1 are the displacements of subsystem 2 on the coupling surface (see equation (17)). It is now easy to express the accelerations and the forces on the coupling surface by using modal expansions:

$$\bar{F}_i^c = -\sum_{m=1}^{\infty} b_m^1(t) \tilde{\sigma}_{ij}^{1m} n_j^1 \quad \text{on } S_{Coupling}, \tag{59}$$

$$\ddot{\bar{W}}_i^c = \sum_{r=1}^{\infty} \ddot{a}_r^2(t) \tilde{W}_i^{2r} \quad \text{on } S_{Coupling}. \tag{60}$$

Putting equation (60) in equation (57), and equation (59) in equation (58), yields the following system of equations

$$\ddot{b}_p^1(t) + (\omega_p^1)^2 b_p^1(t) = \frac{1}{(\omega_p^1)^2 M_p^1} \sum_{r=1}^{\infty} \ddot{a}_r^2(t) \int_{S_{Coupling}} \tilde{W}_i^{2r} \tilde{\sigma}_{ij}^{1p} n^1 \, \mathrm{d}S \quad \forall p \in [1, \ldots, \infty[,$$

$$\ddot{a}_q^2(t) + (\omega_q^2)^2 a_q^2(t) = -\frac{1}{M_q^2} \sum_{m=1}^{\infty} b_m^1(t) \int_{S_{Coupling}} \tilde{W}_i^{2q} \tilde{\sigma}_{ij}^{1m} n^1 \, \mathrm{d}S \quad \forall q \in [1, \ldots, \infty[. \tag{61}$$

To have equations analogous to equations (5), it is necessary to carry out the change of variable

$$b_p^1(t) = \dot{c}_p^1(t). \tag{62}$$

For the cavity–structure problem, the pressure is replaced by the acoustic–fluid velocity potential, which, with the Euler law in the fluid medium, represents a change of variable similar to equation (62).

Finally, the following equations governing free vibrations of coupled substructures are obtained:

$$\ddot{c}_p^1(t) + (\omega_p^1)^2 c_p^1(t) = \frac{1}{(\omega_p^1)^2 M_p^1} \sum_{r=1}^{\infty} \dot{a}_r^2(t) \int_{S_{Coupling}} \tilde{W}_i^{2r} \tilde{\sigma}_{ij}^{1p} n_j^1 \, \mathrm{d}S \quad \forall p \in [1, \ldots, \infty[,$$

$$\ddot{a}_q^2(t) + (\omega_q^2)^2 a_q^2(t) = -\frac{1}{M_q^2} \sum_{m=1}^{\infty} \dot{c}_m^1(t) \int_{S_{Coupling}} \tilde{W}_i^{2q} \tilde{\sigma}_{ij}^{1m} n_j^1 \, \mathrm{d}S \quad \forall q \in [1, \ldots, \infty[. \tag{63}$$

3.7. FORCED VIBRATION OF COUPLED SUBSYSTEMS

By introducing excitation and damping in equations (63), the system of equations describing the forced response of the coupled subsystems from the modal amplitudes of the modes of the uncoupled subsystem is obtained as

$$\begin{aligned}
&\ddot{c}_p^1(t) + \Delta_p^1 \dot{c}_p^1(t) + (\omega_p^1)^2 c_p^1(t) - \frac{1}{(\omega_p^1)^2 M_p^1} \sum_{r=1}^{\infty} \dot{a}_r^2(t) \int_{S_{Coupling}} \tilde{W}_i^{2r} \tilde{\sigma}_{ij}^{1p} n_j^1 \, \mathrm{d}S \\
&\quad = \frac{F_p^1}{(\omega_p^1)^2 M_p^1} \quad \forall p \in [1, \ldots, \infty[, \\
&\ddot{a}_q^2(t) + \Delta_q^2 \dot{a}_q^2(t) + (\omega_q^2)^2 a_q^2(t) + \frac{1}{M_q^2} \sum_{m=1}^{\infty} \dot{c}_m^1(t) \int_{S_{Coupling}} \tilde{W}_i^{2q} \tilde{\sigma}_{ij}^{1m} n_j^1 \, \mathrm{d}S \\
&\quad = \frac{F_q^2}{M_q^2} \quad \forall q \in [1, \ldots, \infty[.
\end{aligned} \tag{64}$$

Here modal viscous damping has been introduced through the modal damping bandwidths Δ_p^1 and Δ_q^2, and external excitations through the generalized terms F_p^1 and F_q^2.

To identify the modal coupling coefficients as defined in the basic modal of two-coupled oscillators, only the coupling of the pth mode of subsystem 1 and the qth mode of subsystem 2 is considered:

$$\begin{cases}
\ddot{c}_p^1(t) + \Delta_p^1 \dot{c}_p^1(t) + (\omega_p^1)^2 c_p^1(t) \\
\quad - \sqrt{\dfrac{M_q^2}{(\omega_p^1)^2 M_p^1}} \left[\dfrac{1}{\sqrt{(\omega_p^1)^2 M_p^1 M_q^2}} \displaystyle\int_{S_{Coupling}} \tilde{W}_i^{2q} \tilde{\sigma}_{ij}^{1p} n_j^1 \, \mathrm{d}S \right] \dot{a}_q^2(t) \\
\quad = \dfrac{L_{1pq}}{(\omega_p^1)^2 M_p^1} \quad \forall (p, q) \in ([1, \ldots, \infty[)^2, \\
\ddot{a}_q^2(t) + \Delta_q^2 \dot{a}_q^2(t) + (\omega_q^2)^2 a_q^2(t) \\
\quad + \sqrt{\dfrac{(\omega_p^1)^2 M_p^1}{M_q^2}} \left[\dfrac{1}{\sqrt{(\omega_p^1)^2 M_p^1 M_q^2}} \displaystyle\int_{S_{Coupling}} \tilde{W}_i^{2q} \tilde{\sigma}_{ij}^{1p} n_j^1 \, \mathrm{d}S \right] \dot{c}_p^1(t) = \dfrac{L_{2pq}}{M_q^2}.
\end{cases} \tag{65}$$

Here L_{1pq} and L_{2pq} contain the generalized 'forces' and the interaction 'forces' with all the other modes.

The modal coupling coefficient between the pth mode and the qth mode can be deduced directly by comparison with equation (6):

$$\gamma_{pq}^{12} = \frac{1}{\sqrt{(\omega_p^1)^2 M_p^1 M_q^2}} \int_{S_{Coupling}} \tilde{W}_i^{2q} \tilde{\sigma}_{ij}^{1p} n_j^1 \, \mathrm{d}S. \tag{66}$$

Physically, it is important to note that the integral represents the interaction modal work exchanged by the pth blocked mode of subsystem 1 and the qth free mode of subsystem 2. The interaction modal work is given by

$$\mathbf{W}_{pq}^{12} = \int_{S_{Coupling}} \tilde{W}_i^{2q} \tilde{\sigma}_{ij}^{1p} n_j^1 \, \mathrm{d}S. \tag{67}$$

This is demonstrated in the general case of three-dimensional continuum mechanical subsystems, and thus can be accepted as a basic principle that can be applied in simplified models of beams, plates and shells. In these cases, the work associated with dual variables introduced in 1-D or 2-D models (force–displacement, moment–rotation, etc.) has to be considered.

3.8. DISCUSSION

It is an advantage of the dual formulation, as compared to displacement formulation and free modes, that there are no 'direct couplings' between modes of the same subsystem (see reference [26]). Indeed, from a formal point of view, mode p of subsystem 1 and mode q of subsystem 2 are coupled with the modes of subsystem 2 and subsystem 1, respectively, but are not directly coupled with the other modes of subsystem 1 and subsystem 2, respectively. When using classical displacement formulation, also the introduction of blocked modes gives no direct coupling for modes of same subsystem (see reference [1, p. 61]). However, the modes of different subsystems are coupled by mass, stiffness and damping elements. The damping oscillator/mode coupling is not taken into account in classical SEA, and that poses a problem; the present approach has no such problem, the modal coupling being only gyroscopic.

In some structures, coupling dissipative joints are present. This situation can be schematically represented as in Figure 6 where one has introduced one stiffness coupling element and one dissipative element between the two subsystems.

In this case, the uncoupled-subsystems shown in Figure 7 can be used; subsystem 1 is blocked and subsystem 2 is free. The uncoupled-blocked subsystem takes into account the stiffness and dissipative elements. The modes of this uncoupled-subsystem are obtained by considering the associated conservative structure. The damping coupling element is taken into account by the damping loss factor of subsystem 1. Of course, as for the general case, the modal couplings are gyroscopics. Then, the dual modal formulation allows dissipative coupling between subsystem to be taken into account without modification of SEA relations.

As mentioned by Karnopp [18] for two identical coupled rods, the dual modal formulation can converge even if the subsystems have equivalent impedance. For this particular case, it could be necessary to consider a large number of modes to converge, whereas for cases which one subsystem is stiffer that the other, it is only necessary to

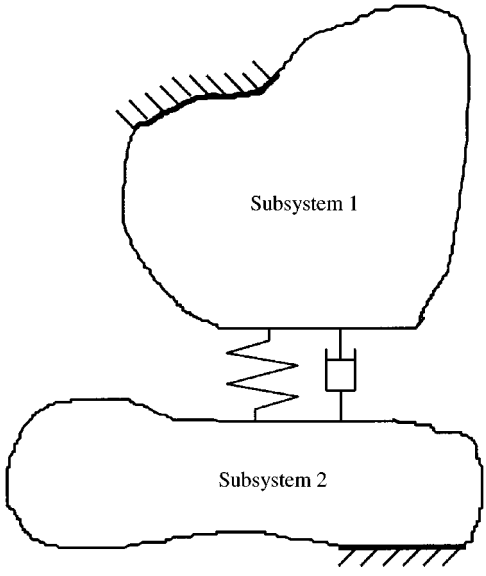

Figure 6. Two subsystems coupled by stiffness and dissipative elements.

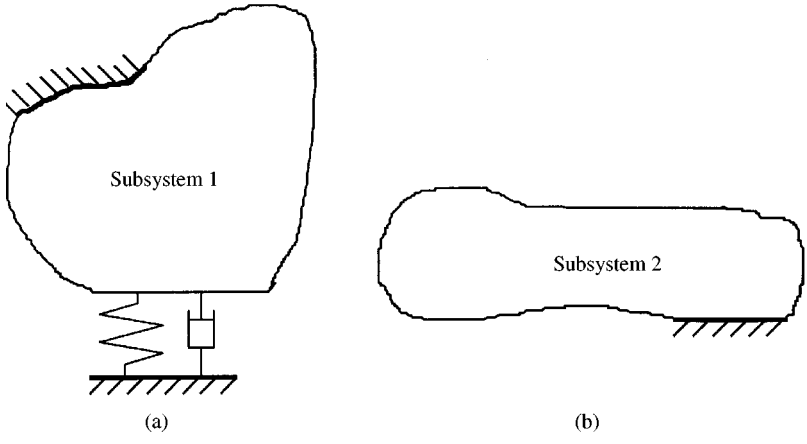

Figure 7. Subsystem definition. (a) uncoupled-blocked subsystem 1; (b) uncoupled-free subsystem 2.

consider a few modes of one subsystem to be coupled with a few modes of the other. In section 2, the interactions between the subsystem modes which have their eigenfrequencies in the frequency band have been considered as SEA. The present approach, considering resonant modes and using blocked modes for one subsystem and free modes for the other, is then well suited to coupling in which one subsystem is stiffer that the other, but leads to approximation when both subsystem tends to have some stiffness. Obviously, the soft subsystem has to be blocked and the stiff subsystem has to be considered to be free. The influence of a poor choice of modes on CLF results will be presented in the companion paper. In the particular case of equally stiff subsystems on the coupling boundary, the choice is arbitrary but the prediction is not so good as in the case of impedance rupture.

3.9. CALCULATION OF COUPLING LOSS FACTORS

Combining equations (8), (14), (66) and (67), a general expression is obtained which allows the coupling loss factors to be calculated from subsystem modal information:

$$\eta_{12} = \frac{1}{N_1 \omega_q} \sum_{p=1}^{N_1} \sum_{q=1}^{N_2} \left[\frac{(\mathbf{W}_{pq}^{12})^2}{(\omega_p^1)^2 M_p^1 M_q^2} \left(\frac{[\Delta_p^1 (\omega_q^2)^2 + \Delta_q^2 (\omega_p^1)^2]}{((\omega_p^1)^2 - (\omega_q^2)^2)^2 + (\Delta_p^1 + \Delta_q^2)(\Delta_p^1 (\omega_q^2)^2 + (\Delta_q^2 (\omega_p^1)^2)} \right) \right]. \tag{68}$$

3.10. ENERGY PROPERTIES

The equation system (64) can be interpreted as the coupling between a set of oscillators associated to subsystem 1 with another set of oscillators associated to subsystem 2. The interest here is in evaluating the energy properties of these associated oscillators. In a first step, the relations between subsystem energies and modal energies are established and secondly, the links between the energies associated to amplitudes $a_q^2(t)$ and $c_p^1(t)$, and the modal energies are evaluated.

3.10.1. *Subsystem energy and modal energy*

The instantaneous kinetic energy of subsystem $\alpha (= 1, 2)$ is expressed by

$$E_K^\alpha = \int_{V^\alpha} \frac{1}{2} \rho_\alpha \left(\frac{\partial W_i^\alpha}{\partial t} \right)^2 \mathrm{d}V. \tag{69}$$

Using the displacement modal expansion and taking into account the modal orthogonality property (45), gives

$$E_K^\alpha(t) = \frac{1}{2} \sum_{n=1}^{\infty} M_n^\alpha (\dot{a}_n^\alpha(t))^2. \tag{70}$$

Therefore,

$$E_K^\alpha(t) = \sum_{n=1}^{\infty} E_K^{\alpha n}(t), \tag{71}$$

where $E_K^{\alpha n}(t) = \frac{1}{2} M_n^\alpha (\dot{a}_n^\alpha(t))^2$ is the modal kinetic energy of mode n of subsystem α.

In the same manner, the instantaneous potential energy of subsystem α is expressed by

$$E_P^\alpha(t) = \frac{1}{2} \int_{V^\alpha} \sigma_{ij}^\alpha S_{ijkl}^\alpha \sigma_{kl}^\alpha \, \mathrm{d}V. \tag{72}$$

Using the stress modal expansion, and taking into account the modal orthogonality property (46), gives finally

$$E_P^\alpha(t) = \frac{1}{2} \sum_{n=1}^{\infty} K_n^\alpha (b_n^\alpha(t))^2. \tag{73}$$

$$E_P^\alpha(t) = \sum_{n-1}^{\infty} E_P^{\alpha n}(t), \tag{74}$$

where $E_P^{\alpha n}(t) = \frac{1}{2} K_n^\alpha (b_n^\alpha(t))^2$ is the modal potential energy of mode n of subsystem α.

In conclusion, the total energy of a subsystem is equal to the sum of the modal total energies of all modes.

3.10.2. *Energy properties associated to amplitudes* $a_q^2(t)$ *and* $c_p^1(t)$

From the form of equation (64), the model amplitudes $a_q^2(t)$ can be associated with the displacement amplitude of an oscillator whose mass is the generalized mass and its stiffness the generalized stiffness of mode q. It is well known that the kinetic (potential) energy of the oscillator represents the mode q kinetic energy (potential energy, upon taking into account equation (56)). Thus, the total energy of the oscillator is the modal total energy.

For subsystem 1, the parameters, which must be associated to the oscillator, are not classical (as for subsystem 2). Calculations are then necessary to identify what is represented by kinetic energy and what is represented by the potential energy of the corresponding oscillator.

By identification from the form of equation (64), the modal parameters of the pth blocked mode of subsystem 1 are associated with the oscillator parameters as shown in Figure 8. The oscillator mass, M, is equal to the modal stiffness, K_p^1, and the oscillator stiffness, K, is equal to $(K_p^1)^2/M_p^1$. The kinetic energy of the oscillator at any time can be expressed by

$$E_K(t) = \tfrac{1}{2} M(\dot{y}(t))^2 = \tfrac{1}{2} K_p^1(\dot{c}_p^1(t))^2. \tag{75}$$

Introducing relation (62) gives:

$$E_K(t) = \tfrac{1}{2} K_p^1(b_p^1(t))^2. \tag{76}$$

Thus, the kinetic energy of the associated oscillator is the potential energy of the pth blocked mode.

The potential energy of the oscillator, is given by

$$E_p(t) = \frac{1}{2} K(y(t))^2 = \frac{1}{2} \frac{(K_p^1)^2}{M_p^1} (c_p^1(t))^2. \tag{77}$$

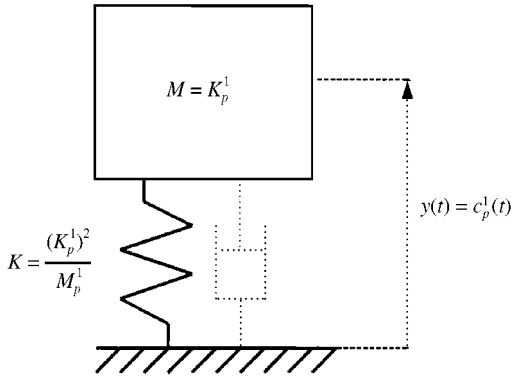

Figure 8. Illustration of the oscillator associated to the pth blocked modes.

Putting equation (62) in equation (53), and then calculating the integral over the time (taking into account that one considers stationary motion), gives

$$c_p^1(t) = -\frac{M_p^1}{K_p^1}\dot{a}_p^1(t). \tag{78}$$

Putting this into equation (77), gives

$$E_p(t) = \tfrac{1}{2} M_p^1 (\dot{a}_p^1(t))^2. \tag{79}$$

Thus, the potential energy of the associated oscillator is the kinetic energy of the pth blocked mode. It is, however, important to note that the total energy of the associated oscillator is equal to the total energy of the pth blocked modes.

4. CLF CALCULATION WITH FEM DATA

4.1. INTRODUCTION

The goal of the present approach is to apply SEA to complicated substructures. In that case only a finite element model of each subsystem can be used. The advantages for the proposed method of calculating CLF in this case are as follows: (1) Subsystem boundary conditions are clearly defined (the uncoupled modes are clearly defined). (2) Heterogeneous subsystems having three-dimensional vibration motions can be treated without any difficulty. This can be quite difficult to do with the classical numerical experiment and the inverse SEA technique due to the choice of the position and type of force which should be applied. (3) The present technique involves a short computing time, because the solution of the equation of motion is not necessary. The CLF are calculated directly from the coefficients of modal equations.

4.2. INTERACTION MODAL WORK FOR DISCRETIZED SYSTEM

In this approach, the CLFs are calculated from the interaction modal work of couple of modes. For a FEM discretized system, the expression for the interaction modal work can be determined directly from the nodal variables. For the node i, the displacement variables are the three displacements (u_k^i, $k = 1, 2, 3$) and the three rotations (u_k^i, $k = 4, 5, 6$); and the dual variables are the three forces (f_k^i, $k = 1, 2, 3$), and the three moments (f_k^i, $k = 4, 5, 6$).

A system composed of two vibrating subsystems discretized by finite element and coupled together is considered. The two subsystems have some common nodes called a "coupling node set" on the coupling boundary. As in section 3, it can be supposed that subsystem 1 is the uncoupled-blocked subsystem and that subsystem 2 is the uncoupled-free subsystem. Then, subsystem 1 must be described by the nodal forces (f_k^{1i}, $k = 1, \ldots, 6$) and subsystem 2 by the nodal displacement (u_k^{2i}, $k = 1, \ldots, 6$). From the physical interpretation it can be deduced that the interaction modal work between the pth mode of subsystem 1 and the qth mode of subsystem 2 is expressed by

$$W_{pq}^{12} = \sum_{i \in \left\{ \begin{matrix} \text{Coupling} \\ \text{node set} \end{matrix} \right\}} \sum_{k=1}^{6} \tilde{f}_{pk}^{1i} \tilde{u}_{qk}^{2i}, \tag{80}$$

where $\tilde{f}_{pk}^{1i}$ is the kth component force of node i of the pth mode of subsystem 1, and $\tilde{u}_{qk}^{2i}$ is the kth component displacement of node i of the qth mode of subsystem 2.

Note: All nodal variables must be described in the same global co-ordinate system for the two subsystems.

In conclusion, in order to calculate the coupling loss factor of a complicated system only the FEM eigenvalue problem must be solved for each uncoupled sybsystem. The information which must be extracted include the natural angular frequencies, the generalized masses and the mode shapes (nodal forces or nodal displacement) on the coupling boundary (coupling node set). Equation (80) permits the modal interaction works to be determined. Then expression (68) can be applied directly to calculate the CLF between the two subsystems.

5. CONCLUSIONS

A technique is presented here to calculate SEA coupling loss factors for complicated subsystem modelled with FEM. The technique relies on the basic modal formulation of SEA and the use of a formulation called the dual modal formulation. The use of the DMF presents some advantages: (a) the modal equations have no direct coupling between modes of the same subsystem (as assumed in SEA); (b) the subsystem modes considered are the physical local modes when there is a rupture of impedance between the two subsystems. Therefore, these resonant modes are able to represent the behaviour of the structure in a frequency bandwidth; (c) the natural frequencies which must be considered to determine the CLF are the natural frequencies of the uncoupled (free or blocked) subsystems; (d) there is not the problem of stored energy in the coupling between modes because it is only gyroscopic (no mass and stiffness coupling that store energy are necessary with the present approach); (e) the method is simple to apply because it is possible to use directly the physical interpretation of the interaction modal work.

The final expression for CLF which has been obtained enables it to be determined only from the knowledge of the modes of the uncoupled-subsystems and the modal damping. The finite element method can be used to calculate the modal information in the case of complex subsystems that allows this technique to be applied to industrial structures. Contrary to the classical SEA matrix-inversion technique, the present method does not require the solution of equations of motion for many excitations. The CLFs are directly obtained by equation (68) without solution of an equation. This saves much computing time. In addition, different damping loss factors can be used for a subsystems without difficulty. In the numerical simulation technique and SEA matrix inversion, there is often the technical difficulty that FEM codes use a global damping loss factor for the considered structure. It is then impossible to use different damping loss factors for substructures.

It should be noted that it is also possible to establish an ensemble averaged estimate of the CLF by replacing deterministic intermodal coupling factors by an ensemble average of the intermodal coupling factors.

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REFERENCES

1. R. H. LYON and R. G. DEJONG 1995 *Theory and Application of Statistical Energy Analysis*. London: Butterworth-Heineman.
2. F. F. YAP and J. WOODHOUSE 1996 *Journal of Sound and Vibration* **197**, 351–371. Investigation of damping effects on statistical energy analysis of coupled structures.
3. C. CACCIOLATI and J. L. GUYADER 1994 *Philosophical Transactions of Royal Society of London* **346**, 465–475. Measurement of SEA coupling loss factors using point mobilities.
4. G. OREFICE 1997 *Ph.D. Thesis, Institut National des Sciences Appliquées de Lyon, France*. Description par Mobilités Energétiques des échanges vibratoires dans les systèmes couplés.
5. K. O. LUNDBERG 1997 *Acustica* **83**, 605–610. A method for estimating SEA loss factors.
6. F. J. FANY 1998 *Journal of Sound and Vibration* **214**, 261–267. An alternative to the SEA coupling loss factor: rationale and method for experimental determination.
7. D. A. BIES and S. HAMID 1980 *Journal of Sound and Vibration* **70**, 187–204. In situ determination of coupling loss factors by the power injection method.
8. N. LALOR 1997 *IUTAM Symposium on SEA. ISVR, University of Southampton. U.K. 8–11 July 1997* (F. J. FAHY and W. G. PRICE editors), Dordrecht: Kluwer Academic Publishers. The practical implementation of SEA.
9. J. A. STELL and R. J. M. CRAIK 1994 *Journal of Sound and Vibration* **178**, 553–561. Statistical energy analysis of structure-borne sound-transmission by finite-element methods.
10. C. SIMMONS and C. FREDO 1990 *Internoise*'90. *International Conference on Noise Control Engineering, Gothenburg, Sweden, August*. Vibration analysis of coupled stiffened plates using the finite element method.
11. C. R. FREDO 1997 *Journal of Sound and Vibration* **199**, 645–666. A SEA-like approach for the derivation of energy flow coefficients with a finite element model.
12. K. SHANKAR and A. J. KEANE 1997 *Journal of Sound and Vibration* **201**, 491–513. Vibrational energy flow analysis using a substructure approach: the application of receptance theory to FEA and SEA.
13. G. MAIDANIK and J. DICKEY 1987 *The Winter Annual Meeting of the American Society of Mechanical Engineers, MA, U.S.A.*, 13–18 *December*. Modal and wave approaches to the statistical energy analysis (SEA).
14. E. E. UNGAR 1966 *Fundamentals of Statistical Energy Analysis of Vibrating Systems, U.S. Air Force AFFDL-TR* 66-52, April.
15. F. J. FAHY 1969 *Journal of Sound and Vibration* **10**, 490–512. Vibration of containing structures by sound in the contained fluid.
16. F. J. FAHY 1970 *Journal of Sound and Vibration* **13**, 171–194. Response of a cylinder to random sound in the contained fluid.
17. R. H. LYON and G. MAIDANIK 1962 *The Journal of the Acoustical Society of America* **34**, 623–639. Power flow between linearly coupled oscillators.
18. D. KARNOPP 1966 *The Journal of the Acoustical Society of America* **40**, 380–384. Coupled vibratory-system analysis, using the dual formulation.
19. F. J. FAHY 1994 *Philosphical Transactions of Royal Society of London* **A346**, 431–447. Statistical energy analysis: a critical overview.
20. C. FREDO 1995 *Ph.D. Thesis, Chalmers University of Technology, Sweden*. Statistical energy analysis and the individual case.
21. L. MAXIT and J. L. GUYADER 1999 *Sixth International Congress on Sound and Vibration, Copenhagen, Denmark, 5–8 July* 1999. Statistical modal energy distribution analysis (SmEdA).
22. S. H. CRANDALL, D. C. KARNOPP, E. F. KURTZ, JR. and D. C. PRIDMORE-BROWN 1968 *Dynamics of Mechanical and Electromechanical Systems* (S. H, Crandall editor). New York: McGraw-Hill Book Company.
23. C. L. DYM and I. H. SHAMES 1973 *Solid Mechanics, a Variational Approach*. New York: McGraw-Hill Book Company.
24. R. M. DESEIGNE 1983 *Ph.D. Thesis, Institut National des Sciences Appliquèes de Lyon, France*. Méthodes variationnelles mixtes en élastostatique, application à des structures composites: plaques multicouches et composites plans colles.
25. J. L. GUYADER 1984 *Thése de doctorat d'Etat es-science, institut National des Sciences Appliquées de Lyon and University Claude Bernard Lyon I, France*. Méthode d'étude des vibrations des milieux continus imparfaitement caractérisés.

26. S. H. CRANDALL and R. LOTZ 1971 *The Journal of the Acoustical Society of America* **49**, 352–356. On the coupling loss factor in statistical energy analysis.

APPENDIX A. NOMENCLATURE

$a_p^1(t)$, $a_q^2(t)$	modal amplitude of displacement expansion
$b_p^1(t)$, $b_q^2(t)$	modal amplitude of stress expansion
$c_p^1(t)$, $c_q^2(t)$	modal amplitude defined by equation (62)
E_1, E_2	time-averaged total energy of oscillator
E_K^{1p}, E_K^{2q}	instantaneous modal kinetic energy
E_P^{1p}, E_P^{2q}	instantaneous modal potential energy
E_p^1, E_q^2	time-averaged modal total energy
E_K^1, E_K^2	instantaneous subsystem kinetic energy
E_P^1, E_P^2	instantaneous subsystem potential energy
f_k^{1i}, f_k^{2i}	force nodal variable
$\tilde{f}_{pk}^{1i}$, $\tilde{f}_{qk}^{2i}$	force nodal mode shape
F_1, F_2	external force
F_p^1, F_q^2	generalized "force"
$\bar{F}_i^c$	specified forces on $S_{Coupling}^2$
G_c	gyroscopic constant
K_1, K_2	oscillator's stiffness
K_p^1, K_q^2	modal stiffness
L_{pq}^1, L_{pq}^2	generalized "forces" and interaction "forces" with others modes
m, n, p, q, r, s	modal order
M_1, M_2	oscillator's mass
M_p^1, M_q^2	modal mass
n_j^1, n_i^2	outer normal vector component
N_1, N_2	number of resonant modes in the considered frequency bandwidth
P_{12}	time-averaged power flow from oscillator 1 to oscillator 2
S_{ijkl}^1, S_{ijkl}^2	compliance tensor
S_{Free}^1, S_{Free}^2	boundary surface with free displacement
$S_{Clamped}^1$, $S_{Clamped}^2$	boundary surface with blocked displacement
$S_{Coupling}^1$, $S_{Coupling}^2$	coupling boundary surface
t_0, t_1, t	time
u_k^{1i}, u_k^{2i}	displacement nodal variable
$\tilde{u}_{pk}^{1i}$, $\tilde{u}_{qk}^{2i}$	displacement nodal mode shape
V^1, V^2	volume occupied by subsystem
$\mathbf{W}_{pq}^{12}$	interaction modal work between mode p of subsystem 1 and mode q of subsystem 2
W_i^1, W_i^2	displacement vector
$\tilde{W}_i^{1p}$, $\tilde{W}_i^{2q}$	displacement mode shape
$\bar{W}_i^c$	specified displacements on $S_{coupling}^1$
y_1, y_2	oscillator amplitude
β	proportional constant between P_{12} and $(E_1 - E_2)$
β_{pq}^{12}	intermodal coupling factor (ICF) between mode p of subsystem 1 and mode q of subsystem 2
$\Delta\omega$	angular frequency bandwidth of interest
Δ_1, Δ_2	oscillator's damping bandwidth
Δ_p^1, Δ_q^2	modal damping bandwidth
ε_{ij}^1, ε_{ij}^2	strain tensor
η_{12}	coupling loss factor (CLF)
γ	gyroscopic coupling coefficient between two oscillators
γ_{pq}^{12}	gyroscopic modal coupling coefficient between mode p of subsystem 1 and mode q of subsystem 2
Π_{12}	time-averaged power flow exchanged by two subsystems
Π_{pq}^{12}	time-averaged power flow from mode p of subsystem 1 to mode q of subsystem 2

ω_c	central angular radian frequency of the frequency bandwidth $\Delta\omega$
ω_1, ω_2	oscillator's natural angular frequency
ω_p^1, ω_q^2	mode's natural angular frequency
Ω_R^1, Ω_R^2	displacement admissible space
ρ_1, ρ_2	mass density
σ_{ij}^1, σ_{ij}^2	stress tensor
$\tilde{\sigma}_{ij}^{1p}$, $\tilde{\sigma}_{ij}^{2q}$	stress mode shape
Σ_R^1, Σ_R^2	stress admissible space
ξ_1, ξ_2	time-averaged total energy of subsystem
ψ_R^1, ψ_R^2	Reissner functional