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LETTERS TO THE EDITOR

PERIOD PRESERVING SCHEMES FOR THE NUMERICAL INTEGRATION OF THE EQUATION OF MOTION

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1. INTRODUCTION

A good scheme for the numerical solution [1] of the differential equation of motion must be, at least globally, energy preserving. A computational scheme that conceals a numerically engendered energy sink or spurious viscosity will suck energy out of, an otherwise conservative, periodically moving system driving down the amplitude into exponential demise. On the other hand, a numerical integration scheme that harbours a spurious energy source will pump phantom energy into the system fueling an explosive growth in the amplitude and an early numerical catastrophic end to the computation. Computational phase errors have a cumulative, long-term effect of placing the computed body at a wrong, and ever more doubtful, position within its orbit, a numerical aberration that could seriously compromise the usefulness of such a scheme for long-term computations as of planetary motion.

It is the purpose of this note to present an explicit numerical integration scheme for the solution of linear as well as non-linear second order initial value problems, that at least for a standard simple model, preserves cyclic energy and the period of motion.

The price to pay for period conservation is in some coefficients being time-step dependent.

2. LINEAR PREDICTION

Consider the initial value problem $x' = -y$, $y' = x$, $x_0 = x(0) = 1$, $y_0 = y(0) = 0$, where $x = x(t)$, $y = y(t)$, $t > 0$, and where $(\ )'$ means differentiation with respect to time t. This initial value problem, that coincides with the single second order problem $x'' + x = 0$, $x(0) = 1$, $x'(0) = 0$, is solved by $x = \cos t$, $y = \sin t$ representing a constant circular motion of period $T = 2\pi$.

We propose to solve the initial value problem with the explicit scheme

$$x_1 = x_0 + \tau x_0', \quad y_1 = y_0 + \tau(\alpha_0 y_0' + \alpha_1 y_1'), \tag{1}$$

in which τ is the time step, where $x_1 = x(\tau)$ and $y_1 = y(\tau)$, approximately, and with the coefficients α_0, α_1 to be determined by stability and accuracy considerations.

With $x' = -y$, $y' = x$, system (1) becomes

$$x_1 = x_0 - \tau y_0, \quad y_1 = y_0 + \tau(\alpha_0 x_0 + \alpha_1 x_1). \tag{2}$$

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that explicitly produces x_1 and y_1 out of x_0 and y_0, and then x_2 and y_2 out of x_1 and y_1, and so on up to x_n and y_n. System (2) is solved by $x_n = z^n x_0$, $y_n = z^n y_0$ for the magnification factor z that satisfies the pair of linear equations

$$\begin{aligned} zx_0 &= x_0 - \tau y_0, \\ zy_0 &= y_0 + \tau(\alpha_0 x_0 + \alpha_1 z x_0) \end{aligned} \tag{3}$$

for any x_0 and y_0. Equation (3) is recast in matrix form to become

$$\begin{bmatrix} z-1 & \tau \\ -\tau(\alpha_0 + \alpha_1 z) & z-1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \mathrm{o} \tag{4}$$

and the condition for it to have a non-trivial solution is that its determinant be zero, leading to the characteristic equation

$$z^2 + 2(-1 + \tfrac{1}{2}\alpha_1\tau^2)z + 1 + \alpha_0\tau^2 = 0 \tag{5}$$

for z. The periodic nature of the solution to the initial value problems dictates that z be complex. Let $|z|$ be the modulus of complex z. If $|z| < 1$, then $|z|^n \to 0$ as $n \to \infty$, and if $|z| > 1$, then $|z|^n \to \infty$ as $n \to \infty$. To avoid these undesirable eventualities of an artificial energy sink and an artificial energy source we select $\alpha_0 = 0$ in equation (2), and are left with the characteristic equation

$$z^2 + 2(-1 + \tfrac{1}{2}\alpha_1\tau^2)z + 1 = 0 \tag{6}$$

that possesses two complex roots z_1 and z_2 such that $z_1 z_2 = |z|^2 = 1$.

In fact,

$$z = 1 - \tfrac{1}{2}\alpha_1\tau^2 \pm \mathrm{i}\tau\sqrt{\alpha_1 - \tfrac{1}{4}\alpha_1^2\tau^2}, \tag{7}$$

where $\mathrm{i}^2 = -1$, and z is complex if

$$\alpha_1 > 0, \quad 4 - \alpha_1\tau^2 > 0. \tag{8}$$

By the fact that $|z| = 1$, the complex solutions to equation (6) can be written as

$$z_1 = \cos\theta - \mathrm{i}\sin\theta, \quad z_2 = \cos\theta + \mathrm{i}\sin\theta \tag{9}$$

with

$$\cos\theta = 1 - \tfrac{1}{2}\alpha_1\tau^2, \quad \sin\theta = \tau\sqrt{\alpha_1 - \tfrac{1}{4}\alpha_1^2\tau^2}. \tag{10}$$

Now, x_n and y_n are generally written as

$$x_n = c_1 z_1^n + c_2 z_2^n, \quad y_n = c_1' z_1^n + c_2' z_2^n \tag{11}$$

with c_1, c_2, c_1', c_2' determined by the initial conditions. Given $x_0 = 1$, $y_0 = 0$ we get from equation (2) $x_1 = 1$, $y_2 = \alpha_1\tau$. Writing x_n and y_n in equation (11) for $n = 0$ and 1 we obtain the two systems of linear equations

$$\begin{bmatrix} 1 & 1 \\ z_1 & z_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ z_1 & z_2 \end{bmatrix}\begin{bmatrix} c'_1 \\ c'_2 \end{bmatrix} = \alpha_1 \tau \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{12}$$

readily solved for c_1, c_2, c'_1, c'_2 as

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{z_2 - z_1}\begin{bmatrix} z_2 - 1 \\ -z_1 + 1 \end{bmatrix}, \quad \begin{bmatrix} c'_1 \\ c'_2 \end{bmatrix} = \frac{\alpha_1 \tau}{z_2 - z_1}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{13}$$

in which $z_1 = \cos\theta - i\sin\theta$, $z_2 = \cos\theta + \mathrm{i}\sin\theta$, $z_2 = z_1 = 2\mathrm{i}\sin\theta$. Writing z_1 and z_2 in terms of θ reshapes equation (11) into the form

$$x_n = (c_1 + c_2)\cos n\theta + \mathrm{i}(c_2 - c_1)\sin n\theta, \quad y_n = (c'_1 + c'_2)\cos n\theta + \mathrm{i}(c'_2 - c'_1)\sin n\theta. \tag{14}$$

We have from equation (13) that

$$\begin{aligned} c_1 + c_2 = 1, \quad & c_2 - c_1 = -\tfrac{1}{2}\mathrm{i}(\sin\theta)^{-1}\alpha_1\tau^2, \\ c'_1 + c'_2 = 0, \quad & c'_2 - c'_1 = -\mathrm{i}(\sin\theta)^{-1}\alpha_1\tau, \end{aligned} \tag{15}$$

with which we finally get

$$x_n = \cos n\theta + \tfrac{1}{2}\alpha_1\tau^2(\sin\theta)^{-1}\sin n\theta, \quad y_n = \alpha_1\tau(\sin\theta)^{-1}\sin n\theta \tag{16}$$

as the general numerical solution to our initial value problem.

3. PERIOD PRESENTATION

A cycle is completed when $\sin n\theta = 0$ or $n\theta = 2\pi$. Then, according to equation (16) $y_n = 0$ and $x_n = 1$. From $\tau n\theta = 2\pi\tau$ and $T = n\tau$ we obtain the computed period as

$$T = 2\pi(\tau/\theta) \tag{17}$$

and to retain $T = 2\pi$ we select α_1 in equation (1) so as to guarantee $\tau = \theta$ or $\sin\tau = \sin\theta$. This condition becomes, in view of equation (10),

$$\sin\tau = \tau\sqrt{\alpha_1 - \tfrac{1}{4}\alpha_1^2\tau^2}, \tag{18}$$

leading to the quadratic equation

$$\tfrac{1}{4}\tau^2\alpha_1^2 - \alpha_1 + \frac{\sin^2\tau}{\tau^2} = 0 \tag{19}$$

for α_1 and resulting in

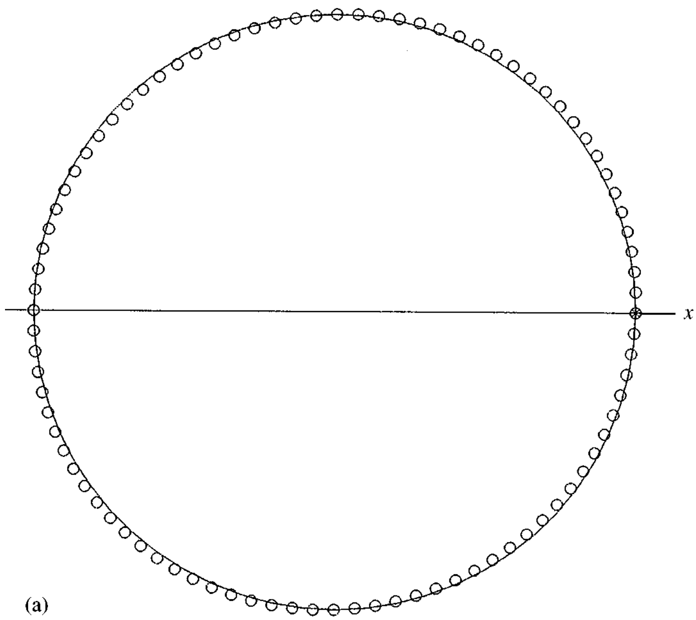
$$\alpha_1 = 2(1 - \cos\tau)/\tau^2 \tag{20}$$

or

$$\alpha_1 = 1 - \tfrac{1}{12}\tau^2 + \tfrac{1}{360}\tau^4, \tag{21}$$

if τ is small.

Figure 1(a) describes a uniform lunar motion computed by scheme (2) with $\alpha_0 = 0$ and $\alpha_1 = \alpha$ given in equation (20) for $\tau = 2\tau/90$. The approximate nature of the computation manifests itself by an apparent wobbling of the moon in and out of its theoretically perfectly circular orbit, yet the moon seems to be exactly in the same computed spot period after



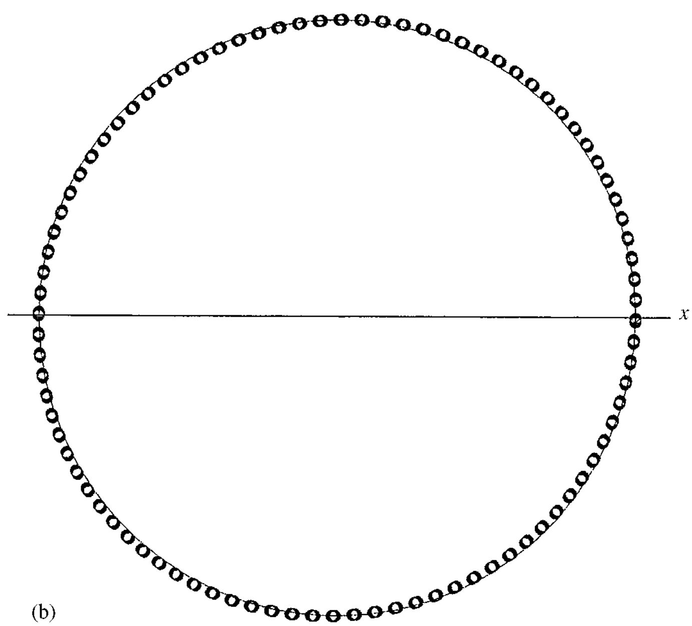
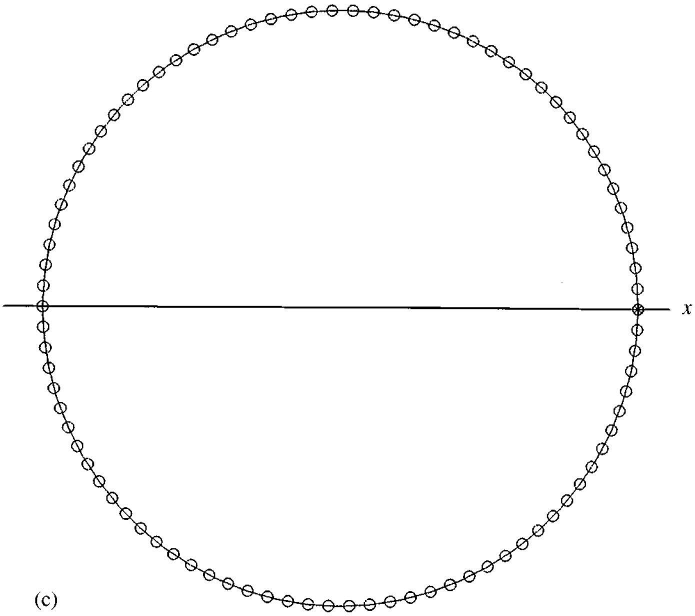


Figure 1. Computed location of a planet moving in a theoretically circular orbit of period $T = 2\pi$: (a) linear prediction, $\tau = 2\pi/90$, $\alpha = 0{\cdot}999594$, 124 periods; (b) linear prediction $\tau = 2\pi/90$, $\alpha = 0{\cdot}999$, 8 periods; (c) quadratic prediction $\tau = 2\pi/90$, $\alpha = 0{\cdot}99918782$, 124 periods. Asterisk marks the last computed position.



(c)

Figure 1. Continued.

period. The last computed position of the moon after 124 periods is marked by an asterisk that appears to accurately fall at $x = 1$, $y = 0$.

Figure 1(b) shows the results of the same computation done now with $\alpha = \alpha_1 = 0{\cdot}999$ which is slightly less than the value suggested by equation (20). A small apparent retardation is noticed for the moon that fails to reach now the position it occupied in the previous cycle.

4. QUADRATIC PREDICTION

Inclusion of the acceleration in the prediction for x_1 suggests the higher order scheme

$$x_1 = x_0 + \tau x_0' + \tfrac{1}{2}\tau^2 x_0'', \quad y_1 = y_0 + \tfrac{1}{2}\tau(\alpha_0 y_0' + \alpha_1 y_1'). \tag{22}$$

that becomes for $x' = -y, \quad y' = x, \quad x'' = -x$:

$$x_1 = x_0 - \tau y_0 - \tfrac{1}{2}\tau^2 x_0, \quad y_1 = y_0 + \tfrac{1}{2}\tau(\alpha_0 x_0 + \alpha_1 x_1). \tag{23}$$

Substitution of $x_1 = zx_0$, $y_1 = zy_0$ into equation (23) results in the system

$$\begin{bmatrix} z - 1 + \tfrac{1}{2}\tau^2 & \tau \\ -\tfrac{1}{2}\tau(\alpha_0 + \alpha_1 z) & z - 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \text{o} \tag{24}$$

from which we obtain the quadratic characteristic equation

$$z^2 + 2(-1 + \tfrac{1}{4}\tau^2 + \tfrac{1}{4}\alpha_1\tau^2)z + 1 + \tfrac{1}{2}\tau^2(\alpha_0 - 1) = 0 \tag{25}$$

for magnification factor z. To assure $|z| = 1$ for the complex roots of equation (25) we set $\alpha_0 = 1$ and are left with

$$z^2 + 2(-1 + \tfrac{1}{4}\tau^2\beta)z + 1 = 0, \tag{26}$$

where $\beta = 1 + \alpha_1$. The two roots of equation (26) are

$$z = 1 - \tfrac{1}{4}\tau^2\beta \pm i\tau\sqrt{\tfrac{1}{2}\beta - \tfrac{1}{16}\tau^2\beta^2} \tag{27}$$

and z is complex if

$$\beta > 0, \quad 8 - \tau^2\beta > 0. \tag{28}$$

Because $|z| = 1$ we may write the complex roots of equation (26) as

$$z = \cos\theta \pm i\sin\theta, \quad \cos\theta = 1 - \tfrac{1}{4}\tau^2\beta, \quad \sin\theta = \tau\sqrt{\tfrac{1}{2}\beta - \tfrac{1}{16}\tau^2\beta^2}. \tag{29}$$

According to the analysis of section 2 the numerical scheme is period conserving if $\tau = \theta$ or $\sin\theta = \sin\tau$. This is assured, according to equation (29), if β is such that

$$\sin\tau = \tau\sqrt{\tfrac{1}{2}\beta - \tfrac{1}{16}\tau^2\beta^2}. \tag{30}$$

or

$$\tfrac{1}{16}\tau^2\beta - \tfrac{1}{2}\beta + \left(\frac{\sin\tau}{\tau}\right)^2 = 0, \tag{31}$$

resulting in

$$\beta = 4(1 - \cos\tau)/\tau^2 \tag{32}$$

or

$$\alpha_1 = 1 - \tfrac{1}{6}\tau^2 + \tfrac{1}{180}\tau^4 \tag{33}$$

if τ is small.

Figure 1(c) shows the results of computation done with this higher order scheme for $\alpha = \alpha_1 = \beta - 1$ for β given in equation (32). The spurious periodic wobbling of the moon visually disappeared and the motion is also otherwise apparently perfectly periodic.

REFERENCES

1. FRIED 1979 *Numerical Solution of Differential Equations*. New York: Academic Press.