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ON THE SPURIOUS SOLUTIONS IN COMPLEX ENVELOPE DISPLACEMENT ANALYSIS

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Complex envelope displacement analysis seems to be a promising approach to analyze high-frequency structural problems and is expected to be useful to study structural-acoustic coupled problems. However, in the original derivation of the method it was shown the presence of a troublesome spurious solution consisting of a high wavenumber component that would make the envelope approach inefficient. The elimination of this term is quite simple for one-dimensional systems but, in view of more complex developments, the problem deserves a serious investigation to explain the origin of this contribution and to introduce a simple and general approach to cancel this term in more general applications. In the present paper both these aspects are carefully considered and successfully developed.

1. INTRODUCTION

In reference [1] a complex envelope displacement analysis (CEDA) was developed, able to predict an envelope solution for high-frequency structural problems with a low computational burden, typical of low-frequency dynamics, so that it appears to be quite promising for further extensions in studying structural–acoustic coupled problems. However, a pitfall of CEDA relies in the presence of a spurious solution [1]. Actually, this is not a problem for one-dimensional structures because it can be simply eliminated, but it represents real trouble for two- and three-dimensional systems. In fact, while the envelope solution is a low-frequency response that can be determined by using a coarse discretization mesh—it is a quasi-static response—the spurious solution has a high-frequency content that, if not eliminated, make unsuccessful the whole envelope approach.

In reference [2] Verbeek *et al.* developed a specific finite element code for CEDA of longitudinal rods, that provides very good results even for damped structures. The very interesting point of this paper is that the authors stress that, during their developments and computations, they never found any spurious solution.

Solicited by their results, we reanalysed the whole problem of the spurious solution: carefully we considered the reason of their origin and finally obtained few though convincing conclusions indicating how it can be eliminated.

2. THE CEDA FORMULATION: BOUNDARY CONDITIONS AND SPURIOUS SOLUTION

The basic steps leading to the complex envelope displacement formulation are first presented, focusing the attention on the problem of the boundary conditions that are strongly related to the presence of the spurious solution.

The equation of motion of a longitudinal rod subjected to a harmonic load is given by

$$w'' + k_0^2 w = f(x) = \frac{F_0}{EA}\,\delta(x-a) = P_0\,\delta(x-a), \tag{1}$$

in which F_0 is the external point load applied in $x = a$ and $k_0 = \omega_0/c$ is the carrier wavenumber corresponding to the harmonic frequency of excitation ω_0.

The same second order equation holds for a flexural beam provided that Langley's approximation [3] is introduced, with the forcing term given by $f = -F_0\,\delta(x-a)/2EIk_0^2$.

By applying the Hilbert transform to any term of equation (1), one obtains

$$\tilde{w}'' + k_0^2\tilde{w} = \tilde{f}. \tag{2}$$

By multiplying equation (2) by the imaginary unit j and summing it to the equation of motion, one obtains

$$(w + \mathrm{j}\tilde{w})'' + k_0^2(w + \mathrm{j}\tilde{w}) = f + \mathrm{j}\tilde{f},$$

where $\tilde{w}$ is the Hilbert transform of the physical displacement w.

The last equation can be written in more compact form by introducing the analytic displacement $\hat{w} = [w + \mathrm{j}\tilde{w}]$.†

$$\hat{w}'' + k_0^2\,\hat{w} = \hat{f}. \tag{3}$$

The complex envelope is defined in reference [1] as

$$\overleftarrow{w}(x) = \hat{w}(x)\,\mathrm{e}^{-\mathrm{j}k_0 x} \tag{4}$$

and admits the inverse relationship

$$w(x) = \mathrm{Re}\{\overleftarrow{w}(x)\,\mathrm{e}^{\mathrm{j}k_0 x}\}. \tag{5}$$

The remarkable property of $\overleftarrow{w}$ is that its wavenumber spectrum is mostly concentrated around the origin of the wavenumber axis. In fact, the Fourier transform in the wavenumber domain of the physical response to a harmonic point load $P_0\,\delta(x-a)$ is $P_0\,\mathrm{e}^{\mathrm{j}ka}/(k_0^2 - k^2)$. While the wavenumber content of the point load extends over the complete wavenumber range, the denominator tends to concentrate the energy of the whole signal around $\pm k_0$, as shown in Figure 1. Since the Fourier transform of the analytic signal is $\hat{W}(k) = W(k) + \mathrm{sign}(k)\,W(k)$, the wavenumber spectrum of the analytic response is one-sided, i.e., the spectrum on the negative wavenumbers is cancelled. Finally, when performing the operation $\overleftarrow{w} = \hat{w}\mathrm{e}^{-\mathrm{j}k_0 x}$, the high wavenumber spectrum is shifted toward the wavenumber's origin. Therefore, the envelope solution presents low wavenumber oscillations in space, and it can be used conveniently as a field descriptor in the context of a high frequency analysis. Moreover a one-to-one correspondence exists between the physical signal and the envelope, so that, if needed, the physical variable can be reconstructed from the envelope one.

† An analytic signal is a complex signal where the imaginary part is the Hilber transform of the real part.

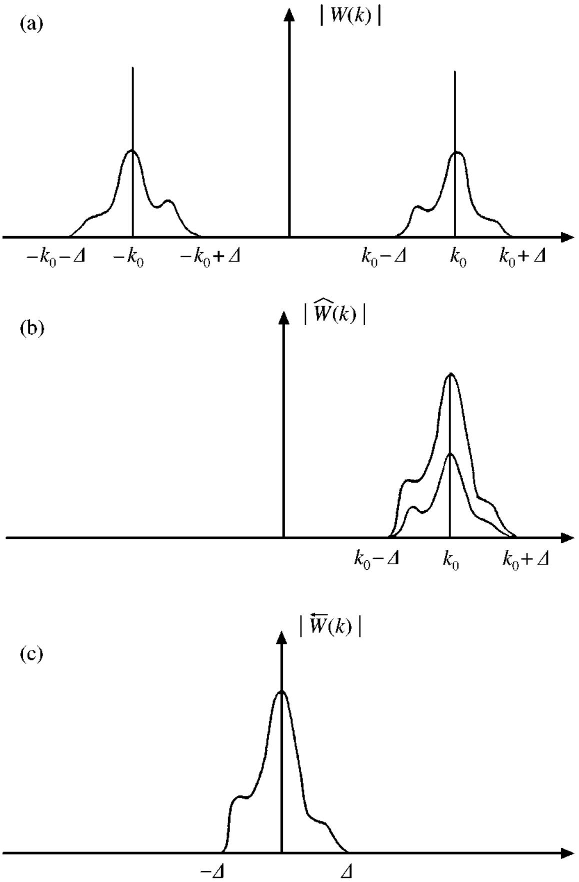

Figure 1. Fourier transforms: (a) physical displacement; (b) analytical displacement; (c) complex envelope displacement.

Because of the link $\hat{w} = \overleftarrow{w}\,\mathrm{e}^{\mathrm{j}k_0 x}$ between the analytic and complex envelope displacement, it is possible to derive the complex envelope equation. The subsequent derivatives of the analytic displacement are

$$\hat{w} = \overleftarrow{w}\,\mathrm{e}^{\mathrm{j}k_0 x}, \quad \hat{w}' = (\overleftarrow{w}' + \mathrm{j}k_0\,\overleftarrow{w})\,\mathrm{e}^{\mathrm{j}k_0 x}, \quad \hat{w}'' = (\overleftarrow{w}'' + 2\mathrm{j}k_0\,\overleftarrow{w}' + k_0^2\overleftarrow{w})\,\mathrm{e}^{\mathrm{j}k_0 x}.$$

By substituting $\hat{w}''$ and $\hat{w}$ into equation (3) and applying the envelope operator to the external force $\hat{f}$, one has

$$(\overleftarrow{w}'' + 2\mathrm{j}k_0\,\overleftarrow{w}' - k_0^2\,\overleftarrow{w}) + k_0^2\,\overleftarrow{w} = \overleftarrow{f}, \tag{6}$$

i.e., the final form of the complex envelope equation is determined as

$$\overleftarrow{w}'' + 2\mathrm{j}k_0\,\overleftarrow{w}' = \overleftarrow{f} = P_0\,\overleftarrow{\delta}\,(x-a). \tag{7}$$

$\overleftarrow{\delta}$ is the complex envelope of the Dirac function, i.e.,

$$\overleftarrow{\delta}(x-a) = [\delta(x-a) + \mathrm{j}\tilde{\delta}(x-a)]\,\mathrm{e}^{-\mathrm{j}k_0 x} = \left[\delta(x-a) + \mathrm{j}\,\frac{1}{\pi(x-a)}\right]\mathrm{e}^{-\mathrm{j}k_0 x}.$$

Note that the point force is transformed by the envelope operator into a distributed force.

The general solution of equation (7) is

$$\overleftarrow{w} = P_0 \overleftarrow{g}_f + A + B\,\mathrm{e}^{-2\mathrm{j}k_0 x}, \tag{8}$$

where A and B are complex constants that must be determined through the boundary conditions and $\overleftarrow{g}_f$ (Green function) is the complex envelope response of an infinite beam to the forcing envelope load $\overleftarrow{\delta}$, i.e.,

$$\overleftarrow{g}_f = \mathscr{F}^{-1}\left\{-\frac{\mathscr{F}[P_0 \overleftarrow{\delta}]}{k^2 + 2kk_0}\right\},$$

where $\mathscr{F}$ denotes the Fourier transform (wavenumber domain) and $\mathscr{F}^{-1}$ the inverse Fourier transform. Since A and B are complex constants, four real conditions are necessary to solve the problem. However, the physical problem (1) provides only two physical boundary conditions. For fixed ends, for example, they are $w|_{(0,\,L)} = 0$. If one wishes to solve the envelope equation (7) by these physical conditions, one must use the relationship between $\overleftarrow{w}$ and w, i.e., equation (4), that leads, for the above physical conditions, to the relationships

$$w|_{0,L} = \mathrm{Re}\{\overleftarrow{w}\,\mathrm{e}^{\mathrm{j}k_0 x}\}_{(0,L)} = 0.$$

It is easy to realize that the two available physical (real) conditions are not sufficient to determine the complex constants A and B, but the two missing conditions are not specified. If these missing conditions are not chosen appropriately, the general solution contains the terms A and $B\,\mathrm{e}^{-2\mathrm{j}k_0 x}$. The term A is not a problem but the presence of the second term is really troublesome in this context, because it would make useless the whole CEDA approach, in that a very fast oscillating solution (wavenumber $2k_0$ doubling the physical wavenumber k_0) is present. Therefore, the term $B\,\mathrm{e}^{-2\mathrm{j}k_0 x}$ in equation (8) must be eliminated. In reference [1] this term was called *spurious* to mean that it is certainly unwelcome and, in some way, unexpected.

Let us analyse more deeply this point.

Focussing on the procedure to determine the CEDA equation, the two basic elements are (i) the physical equation of motion (1) and (ii) the same equation after the Hilbert transform is applied, i.e., equation (2). By performing a linear combination of these two equations, equation (3) is obtained from which the CEDA equation is derived once the substitution $\hat{w} = \overleftarrow{w}\,\mathrm{e}^{\mathrm{j}k_0 x}$ is performed.

The spurious solution appears slyly into the problem. The real and imaginary parts of the complex equation (3) are equations (1) and (2), respectively, i.e., they appear to be independent. Now, the physical boundary conditions suffice equation (1) to be solved, but nothing is prescribed about equation (2). However, one is easily convinced that the related constraints cannot be assigned arbitrarily.

In fact, the physical solution of equation (1) is

$$w = w_p + a\,\mathrm{e}^{\mathrm{j}k_0 x} + a^*\,\mathrm{e}^{-\mathrm{j}k_0 x}, \tag{9}$$

where $*$ denotes complex conjugate. The constant a is determined by the physical boundary conditions and w_p is a particular solution.

On the other hand, the solution to equation (2) is

$$\tilde{w} = \tilde{w}_p + b\,\mathrm{e}^{\mathrm{j}k_0 x} + b^*\,\mathrm{e}^{-\mathrm{j}k_0 x},$$

but no information is explicitly available for b.

By Hilbert transforming the solution of equation (1), i.e., equation (9), one obtains

$$\tilde{w}_p - \mathrm{j}\, a \mathrm{e}^{\mathrm{j}k_0 x} + j\, a^* \,\mathrm{e}^{-jk_0 x}.$$

Thus, the solution of equation (2) does not provide, in general, the Hilbert transform of the physical solution. This happens only if $b = \mathrm{j}a$, i.e., by imposing an additional complex condition that introduces the missing information to solve the complex CEDA equation. If this condition is not suitably imposed, equation (2) does not provide the Hilbert transform of the physical displacement w: consequently, the solution of equation (3) is not an analytic signal; i.e., its imaginary part is not the Hilbert transform of the real part. Finally, the solution of the CEDA equation is not a complex envelope and contains the spurious high wavenumber term $B\,\mathrm{e}^{-2\mathrm{j}k_0 x}$.

In conclusion, the requirement $B = 0$ is equivalent to the requirement that $b = -\mathrm{j}a$. This note, hopefully, explains how the spurious term arises, why it must be cancelled and how its elimination provides the closure condition to solve the CEDA equation.

3. POSSIBLE APPROACHES TO AVOID THE SPURIOUS SOLUTION

Different procedures to eliminate the spurious solution are presented.

3.1. THE COMPLEX CONSTANT TECHNIQUE

In the original developments of the envelope method [1], the problems of the boundary conditions and the spurious solution were solved separately, by using a first order formulation of the CEDA equation. Those developments are hereby resumed.

The elimination of the spurious solution is first considered.

The integral of the first order envelope equation derived from equation (5): i.e.,

$$\overleftarrow{w}' + 2\,\mathrm{j}k_0\,\overleftarrow{w} = \int \overleftarrow{f}\,\mathrm{d}x$$

can be formally expressed by

$$\overleftarrow{w} = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\overleftarrow{F}(k)}{k(k+2k_0)}\,\mathrm{e}^{\mathrm{j}kx}\,\mathrm{d}k + B\,\mathrm{e}^{-2\mathrm{j}k_0 x},$$

where $\overleftarrow{F}(k)$ is the Fourier transform in the wavenumber domain of the envelope load $\overleftarrow{f}(x)$. The requirement $B = 0$ can be achieved by imposing, for $x = 0$,

$$\overleftarrow{w}(0) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\overleftarrow{F}(k)}{k(k+2k_0)}\,\mathrm{d}k.$$

Actually, the last expression can be replaced by $\overleftarrow{w}(0) = 0$ because the right-hand side is a small quantity when point loads are considered. The first order CEDA equation is then solved by using only this simple end condition, and its solution is free from the spurious term. This solution, say $\overleftarrow{w}_0$, does not satisfy the physical boundary condition. However, this necessary result is simply obtained (see reference [1] for more details) by adding a complex constant γ to $\overleftarrow{w}_0$, i.e., one obtains

$$w(x) = \mathrm{Re}\,\{(\overleftarrow{w}_0 + \gamma)\,\mathrm{e}^{\mathrm{j}k_0 x}\}.$$

where γ is determined by imposing the physical boundary conditions on $w(x)$.

By this procedure the solution $\overleftarrow{w}_0$ of the CEDA equation and the fitting of the physical boundary conditions are obtained in separate phases. In some way, the satisfaction of the end constraint becomes a simple post-processing of the numerical solution.

Even if this procedure is quite simple, on the other hand, a limitation exists in extending the method to damped as well as to higher-dimensional problems, especially when a finite element approach must be used. Therefore, the proposed strategy is not of practical interest and must be revisited.

3.2. THE EXTENDED FIELD APPROACH

In reference [2] Verbeek *et al.*, in proposing a finite element formulation for the complex envelope displacement of a rod, developed a particular approach to deal with the boundary problem of CEDA. They wrote both the physical and the CEDA equations by introducing the constraint forces at the rod's ends on the right-hand side, i.e.,

$$w'' + 2\,k_0^2\, w = f + f_0\, \delta(x) + f_L\, \delta(x - L)$$

and

$$\overleftarrow{w}'' + 2\mathrm{j}k_0\overleftarrow{w} = \overleftarrow{f} + f_0\, \overleftarrow{\delta}(x) + f_L\overleftarrow{\delta}(x - L),$$

where f_0 and f_L are the (physical) unknown constraint forces. Due to the effect of the $\overleftarrow{\delta}$'s the authors stressed that the constraint forces spread over inside and outside the waveguide. To account for this, they extended the physical domain of analysis. In particular, if the waveguide is defined in the interval $[0, L]$, two new domain are added, say $[-L_E, 0]$ and $[L, L + L_E]$. In this way the constraint forces are included in the new domain $x \in [-L_E, L + L_E]$ as external forces. By this formulation the problem to be solved needs six real conditions (two complex constants for the equation plus other two, real, to determine the unknown forces at the constraints). Two conditions are provided, as usual, by the physical end constraints. The remaining four conditions are not yet explicitly established.

In the finite element formulation of CEDA presented in reference [2], the complex envelope of the normal force along the waveguide ($N = EAw'$), i.e., $\overleftarrow{N} = (N + \mathrm{j}\tilde{N})\, \mathrm{e}^{\mathrm{j}k_0 x} = EA\,(w' + \mathrm{j}\tilde{w}')\, \mathrm{e}^{\mathrm{j}k_0 x}$, is determined from the following steps:

$$\overleftarrow{w} = (w + \mathrm{j}\, \tilde{w})\, \mathrm{e}^{-\mathrm{j}k_0 x} \Rightarrow \overleftarrow{w}' = (w' + \mathrm{j}\tilde{w}')\, \mathrm{e}^{-\mathrm{j}k_0 x} - \mathrm{j}k_0\, (w + \mathrm{j}\tilde{w})\, e^{-\mathrm{j}k_0 x},$$

$$(w' + \mathrm{j}\tilde{w}')\, \mathrm{e}^{-\mathrm{j}k_0 x} = \overleftarrow{w}' + \mathrm{j}k_0\overleftarrow{w} \;\Rightarrow\; \overleftarrow{N} = EA\,(w' + \mathrm{j}\tilde{w}')\, \mathrm{e}^{-\mathrm{j}k_0 x} = EA\,(\overleftarrow{w}' + \mathrm{j}k_0\, \overleftarrow{w}).$$

Although the authors introduced some arguments to establish that $\overleftarrow{N}$ must vanish in $x = -L_E$ and $x = L + L_E$, if L_E is long enough, and used this result as a trivial asymptotic simplification, they actually imposed the conditions $\overleftarrow{N}(x = -L_E) = 0$ and $\overleftarrow{N}(x = L + L_E) = 0$. These conditions represent silently their boundary conditions and the obtained results are not affected by the spurious solution. Although it is not easy to show why this technique allows to avoid the spurious solution, it has been proven numerically that they have just the effect of reducing the high wavenumber contribution to a negligible disturbance. However, at present, this way of managing the boundary problem seems redundant. In fact, it is not necessary to perform the analysis on the extended domain to solve the CEDA solution effectively. Moreover, the extension of the domain introduces some numerical troubles. It will be now discussed how a simpler approach can be used to avoid the spurious solution.

3.3. A SIMPLER APPROACH TO AVOID THE SPURIOUS SOLUTION

Consider the second order CEDA equation, i.e.

$$\overleftarrow{w}'' + 2\mathrm{j}k_0\,\overleftarrow{w} = \overleftarrow{f}.$$

Irrespective of the boundary conditions of the physical problem, it was shown that the term responsible for the spurious solution is $B\,\mathrm{e}^{-2\mathrm{j}k_0 x}$. To eliminate this unwanted term, it can be easily verified that it is sufficient either to impose at the end $x = 0$ the (complex) condition $\overleftarrow{w}' = 0$ or to impose at each end ($x = 0$ and $x = L$) the conditions $Re\{\overleftarrow{w}'\} = 0$. In fact, the general solution of the CEDA problem is

$$\overleftarrow{w} = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\overleftarrow{F}(k)}{k(k+2k_0)}\mathrm{e}^{\mathrm{j}k_0 x}\,\mathrm{d}k + A + B\,\mathrm{e}^{-2\mathrm{j}k_0 x} = \overleftarrow{w}_p + A + B\,\mathrm{e}^{-2\mathrm{j}k_0 x}.$$

When imposing $\overleftarrow{w}' = 0$ at the origin, one obtains

$$\overleftarrow{w}'(0) = \overleftarrow{w}_p'(0) - 2\mathrm{j}k_0 B = 0.$$

The particular solution $\overleftarrow{w}_p$ is a complex envelope. It is rather flat (both the real and imaginary parts) all along the guide, except that in correspondence to the external force, when a sudden jump is observed. Therefore, its derivative is almost zero everywhere except nearby the point force, where a spike arises. Therefore, the previous written condition simply leads to $B \cong 0$ thus almost eliminating the spurious solution and providing the required smooth result.

Instead of imposing $\overleftarrow{w}' = 0$ at $x = 0$ only, one can reach the same result, i.e. elimination of the spurious solution, by imposing $\mathrm{Re}\{\overleftarrow{w}'\}$ at the two ends $x = 0$ and $x = L$.

In fact, if the condition $\mathrm{Re}\{\overleftarrow{w}'\}$ is imposed at both ends $x = 0$ and $x = L$, one has

$$\mathrm{Re}\{\overleftarrow{w}\} = \mathrm{Re}\{\overleftarrow{w}_p\} + A_R + B_R\cos 2k_0 x - B_I \sin 2k_0 x.$$

Thus,

$$\mathrm{Re}\{\overleftarrow{w}'(0)\} = \mathrm{Re}\{\overleftarrow{w}_p\}'(0) - B_I = 0 \;\Rightarrow\; B_I = 0$$

and

$$\mathrm{Re}\{\overleftarrow{w}'(L)\} = \mathrm{Re}\{\overleftarrow{w}_p\}'(L) - B_R \sin 2k_0 L = 0 \;\Rightarrow\; B_R = 0,$$

so that, in conclusion one has, as wished, $B = 0$. This procedure was implemented in a finite element code [4], following the lines presented in reference [2], and the results were very satisfactory.

4. NUMERICAL RESULTS

Some numerical results related to a steel longitudinal rod of dimensions $1 \times 0{\cdot}1 \times 0{\cdot}1$ m, are presented. The force has frequency 3×10^5 rad/s and amplitude 1000 N. The first case is for clamped ends and the force location at $x = 0{\cdot}7$ m. The envelope solution obtained by the extended field technique and the physical solution obtained by a classical approach are shown in Figure 2. The solution domain is $x \in [-0{\cdot}2, 1{\cdot}2]$. The same case is presented in Figure 3, but the envelope solution is determined by the simpler technique proposed in this

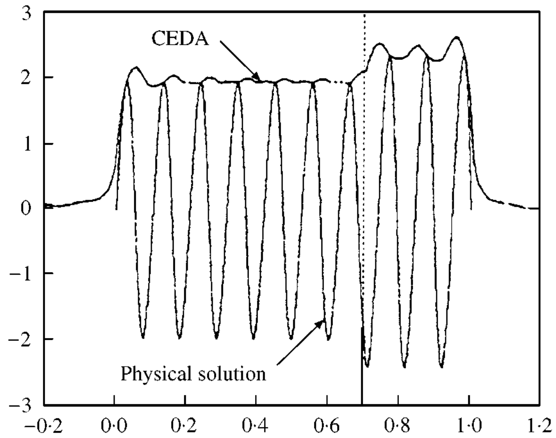

Figure 2. Comparison between the physical and envelope solution, obtained by the extended field technique [2], for a clamped–clamped rod (excitation located at $x = 0{\cdot}7$).

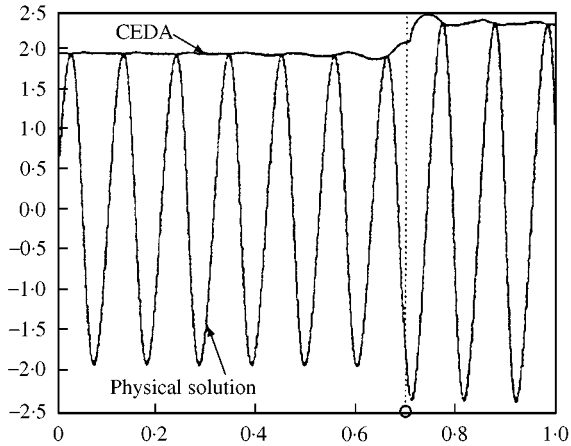

Figure 3. Comparison between the physical and envelope solution, obtained by the method presented in this paper, for a clamped–clamped rod (excitation located at $x = 0{\cdot}7$).

paper. In both cases the results are very satisfactory. It is important to notice that once the complex envelope is known, not only the envelope but even the complete physical oscillating solution can be recovered.

In Figure 4 the case with clamped-free ends is presented, where the external force is located at $x = 0{\cdot}2$ m. The translation of the boundary conditions in terms of CEDA constraints are

$$\left\{\begin{matrix} w(0) = 0 \\ w'(L) = 0 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} \mathrm{Re}\,\{\overleftarrow{w}(0)\} = 0 \\ \mathrm{Im}\,\{\overleftarrow{w}(L)\}\cos k_0 L + \mathrm{Re}\,\{\overleftarrow{w}(L)\}\sin k_0 L = 0 \end{matrix}\right\}.$$

In addition, the condition $\overleftarrow{w}'(0) = 0$ is used to eliminate the spurious solution.

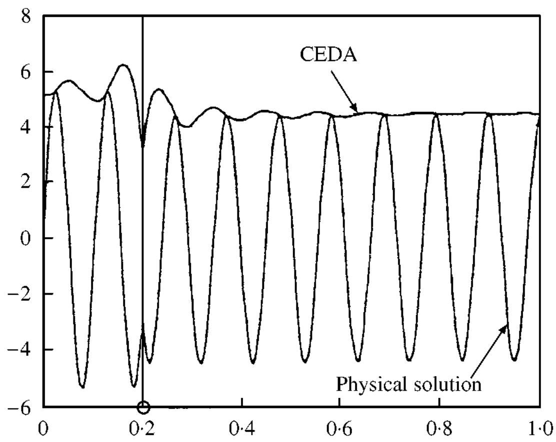

Figure 4. Comparison between the physical and envelope solution, obtained by the method presented in this paper, for a clamped-free rod (excitation located at $x = 0{\cdot}2$).

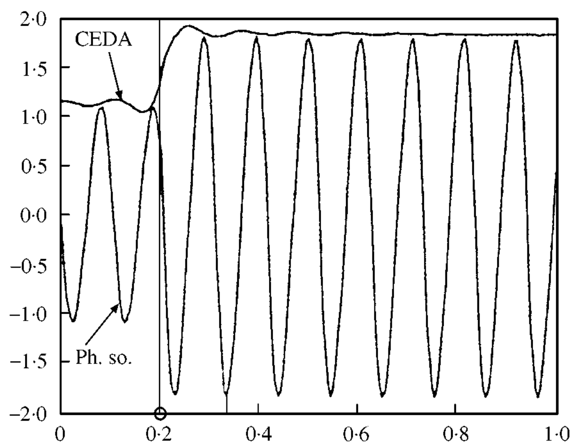

Figure 5. Comparison between the physical and envelope solution, obtained by the method presented in this paper, for a rod clamped at one end and a spring on the opposite side (excitation located at $x = 0{\cdot}2$).

The last case has a clamped end and a spring of constant k_{end} at the opposite side. The complete set of additional conditions are in this case:

$$\left\{ \begin{matrix} w(0) = 0 \\ N(L) = EAw'(L) = k_{end}\, w(L) \\ \overleftarrow{w}'(0) = 0 \end{matrix} \right\}$$

$$\Downarrow$$

$$\left\{ \begin{matrix} \mathrm{Re}\,\{\overleftarrow{w}(0)\} = 0 \\ [EAk_0 \sin k_0 L + k_{end} \cos k_0 L]\, \mathrm{Re}\,\{\overleftarrow{w}(L)\} = [-EAk_0 \cos k_0 L + k_{end} \sin k_0 L]\, \mathrm{Im}\,\{\overleftarrow{w}(L)\} \\ \overleftarrow{w}' = 0 \end{matrix} \right\}$$

The comparison shown in Figure 5 is quite good even in this case.

5. CONCLUDING REMARKS

A complete explanation of the origin of the spurious solution encountered in formulating the complex envelope displacement analysis is provided. This was a serious drawback of this promising approach.

It is shown that the envelope formulation can originate the spurious solution depending on whether the analytic equation (6), associated to the physical one to provide the envelope equation, generates either analytic or non-analytic solutions. In order to avoid the spurious solution two suitable boundary conditions must be added, besides the physical ones. To this aim, an extended field technique was originally proposed in reference [2], in the framework of a finite element formulation. In the present paper a different procedure is proposed to rationalize the elimination of the spurious solution.

The results are very satisfactory so that the envelope approach can be thought with much more confidence for the solution of complex systems.

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