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LETTERS TO THE EDITOR

NON-LINEAR TORSIONAL VIBRATION AND DAMPING ANALYSIS OF SANDWICH BEAMS

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1. INTRODUCTION

Sandwich structures find an increasing application in aerospace, shipbuilding, construction and other industries. The study of dynamic behaviour of such structures requires a knowledge of their damping characteristics. The damping factor plays an important role in controlling the resonant response of the structures and thus in prolonging their service life under periodic loading or impact. It is common practice to use viscoelastic material for the core in sandwich layer arrangements to enhance the damping characteristics of the structures.

A considerable amount of research work has been done on the vibration and damping of beams with constrained layer/sandwich layer arrangements and it has been reviewed by Nakra [1] on the topic dealing with vibration control with viscoelastic material. Some works have also been devoted towards the optimum design of viscoelastic damping layer treatment for beams and plates. The notable contributions are the work of Hajela and Lin [2], and Marcelin *et al.* [3]. All these studies are concerned with the flexural vibration and damping study of the beams with viscoelastic core using analytical/numerical methods.

There are many practical situations in which such structures are subjected to the torsional vibrations. Some studies on the torsional vibrations of isotropic beams with non-circular cross-sections have been carried out by Dimarogonas and Massouros [4], and Christides and Barr [5]. In the case of composite laminates, the static analysis under the torsional loads has received considerable attention in the literature, for instance, the work of Tsai [6], Whitney [7] and Sankar [8]. However, the investigation concerning the torsional vibration and damping of sandwich structures has been sparsely dealt with in the literature [9]. Further, unlike the flexural study of composite laminates, the non-linear dynamic analysis concerning torsional characteristics of sandwich structures has not received enough attention from researchers. Such studies are necessary for the development of structural design strategies. An attempt is made here, by extending the work given in reference [9], to analyze the non-linear torsional-free vibration and damping behavior of sandwich beam having a viscoelastic core.

In this paper, complex eigenvalue problem based on complex moduli is formulated using a new beam element developed recently by Ganapathi *et al.* [10, 11]. The formulation of the element includes warping of the cross-section. The non-linearity based on the Green's strain vector definition is incorporated into the model. The non-linear governing equations obtained here are solved using a direct iteration technique. Numerical results, to highlight the effect of amplitude of torsional vibration, shear modulus of the core layer, and thickness

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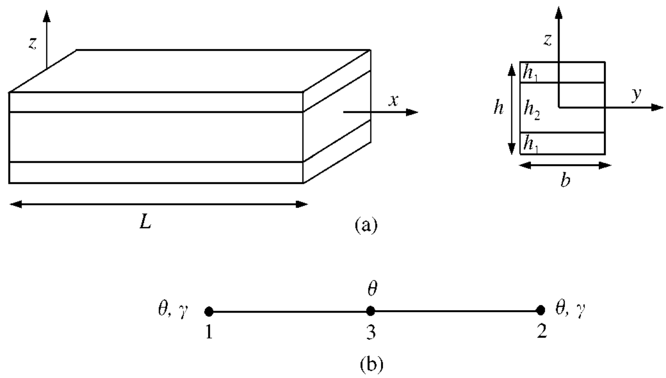

Figure 1. (a) Sandwich beam co-ordinate system. (b) Description of sandwich beam finite element.

ratios of face-to-core of the sandwich structures on the torsional frequencies and the associated system loss factors, are presented for the cantilever beams.

2. FORMULATION

A laminated composite beam of length L, width b and total thickness h, is considered with the co-ordinates x along the length, y along the width and z along the thickness directions as shown in Figure 1(a). Based on the geometrical interpretation [12], the displacements in the kth layer u^k, v^k and w^k at point (x, y, z) from the median surface are expressed in terms of warping function ϕ^k, torsional rotation θ and independent parameter γ for torsional rotation gradient in the length direction as

$$u^k(x,y,z,t) = \phi^k(y,z)\gamma(x,t), \quad v^k(x,y,z,t) = y(\cos\theta - 1) - z\sin\theta,$$

$$w^k(x,y,z,t) = z(\cos\theta - 1) + y\sin\theta, \tag{1}$$

where t is the time and θ is a function of x and t.

The torsional warping function ϕ^k used in defining the kinematics is the solution derived from three-dimensional elasticity equations for composite beam of rectangular cross-section made of different layers. The general expression for ϕ^k is in the form of a harmonic function, as outlined in references [9–11, 13], and is expressed as

$$\phi^k = \sum_{N=1,3,\ldots}^{\infty} (C_N^k \sinh(\alpha z) + D_N^k \cosh(\alpha z)) \sin(\alpha y) + yz, \tag{2}$$

where α is defined as $N\pi/b$.

The coefficients C_N^k and D_N^k, in equation (2), while defining the warping function for the rectangular cross-section, are determined by solving the boundary value problem for torsion such that the displacements are continuous at the interface of adjacent layers, and the transverse shear stress is continuous at the interface of the adjacent layers and vanishes at the top and bottom surfaces of the beam, as outlined in references [9–11, 13]. Substituting the warping function ϕ^k determined this way into equation (1) satisfies the inter-layer displacement continuity and torsional transverse shear stress continuity away from the boundaries where γ approaches $\theta_{,x}$, i.e., the Saint–Venant hypothesis. For a beam, the relevant Green's strain vector $\{\varepsilon\}$ for the kth layer can be written as

$$\{\varepsilon\}^k = \begin{Bmatrix} \varepsilon_{xx}^k \\ 2\varepsilon_{xz}^k \\ 2\varepsilon_{xy}^k \end{Bmatrix} = \begin{Bmatrix} u_{,x}^k + \frac{1}{2}(u_{,x}^{k^2} + v_{,x}^{k^2} + w_{,x}^{k^2}) \\ u_{,z}^k + w_{,x}^k + u_{,x}^k u_{,z}^k + v_{,x}^k v_{,z}^k + w_{,x}^k w_{,z}^k \\ u_{,y}^k + v_{,x}^k + u_{,x}^k u_{,y}^k + v_{,x}^k v_{,y}^k + w_{,x}^k w_{,y}^k \end{Bmatrix}. \tag{3}$$

Equation (3), after substituting the kinematics given in equation (1), can be conveniently represented as

$$\{\varepsilon\}^k = \{\varepsilon^L\}^k + \{\varepsilon^{NL}\}^k \tag{4}$$

where

$$\{\varepsilon^L\}^k = \begin{Bmatrix} \phi^k \gamma_{,x} \\ \phi_{,z}^k \gamma + y\theta_{,x} \\ \phi_{,y}^k \gamma - z\theta_{,x} \end{Bmatrix}, \quad \{\varepsilon^{NL}\}^k = \begin{Bmatrix} (\frac{1}{2})[\phi^{k^2}\gamma_{,x}^2 + (z^2 + y^2)\theta_{,x}^2] \\ \phi^k \phi_{,z}^k \gamma\gamma_{,x} \\ \phi^k \phi_{,y}^k \gamma\gamma_{,x} \end{Bmatrix}. \tag{5}$$

The superscripts L and NL denote the linear and non-linear components of strains respectively.
The Piola–Kirchhoff stress vector associated with the Green strains $\{\varepsilon\}$ is written as

$$\{\sigma\}^k = \{\sigma_{xx}\,\sigma_{xz}\,\sigma_{xy}\}^{kT}. \tag{6}$$

Here the superscript T represents the transpose of the vector. The stress–strain relation for the kth layer is written as

$$\{\sigma\}^k = \begin{bmatrix} Q_{11}^k & 0 & Q_{16}^k \\ 0 & Q_{44}^k & 0 \\ Q_{16}^k & 0 & Q_{66}^k \end{bmatrix} \{\varepsilon\}^k, \tag{7}$$

where Q_{ij}^k ($i, j = 1, 4, 6$) are the transformed stiffness coefficients of the kth layer and are complex quantities.

For a composite laminated beam of thickness h_k ($k = 1, 2, 3, \dots$), and the ply-angle ϕ_k ($k = 1, 2, 3, \cdots$), the necessary expressions for computing the stiffness coefficients, available in the literature [14], are used. Since the formulation deals with the damping model, energy dissipation under harmonic vibration due to the viscoelastic core is taken into account with complex moduli of an orthotropic material of the form as shown below

$$E_L^* = E_L^R + \mathrm{i}E_L^I, \quad E_T^* = E_T^R + \mathrm{i}E_T^I, \quad G_{LT}^* = G_{LT}^R + \mathrm{i}G_{LT}^I, \quad G_{TT}^* = G_{TT}^R + \mathrm{i}G_{TT}^I. \tag{8}$$

Here, E^* and G^* are Young's modulus and shear modulus, respectively. The subscripts L and T are the longitudinal and transverse directions, respectively, with respect to the fibres, and the superscripts R and I denote the real and imaginary parts of the complex moduli.

The material loss factors η_L, η_T under tension compression, and η_{LT}, η_{TT} under shear are defined as

$$\eta_L = E_L^I/E_L^R, \quad \eta_T = E_T^I/E_T^R, \quad \eta_{LT} = G_{LT}^I/G_{LT}^R, \quad \eta_{TT} = G_{TT}^I/G_{TT}^R. \tag{9}$$

The strain energy functional U of the system is given as

$$U(\delta)=\left(\frac{1}{2}\right)\int_0^L\int_{-b/2}^{b/2}\sum_k\int_{h_k}^{h_{k+1}}\{\sigma^k\}^{\mathrm{T}}\{\varepsilon^k\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{10}$$

The kinetic energy of the beam is written as

$$T(\delta)=\frac{1}{2}\int_0^L\int_{-b/2}^{b/2}\sum_k\int_{h_k}^{h_{k+1}}\rho^k[(\dot{u}^k)^2+(\dot{v}^k)^2+(\dot{w}^k)^2]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{11}$$

where the dot over the variable denotes the partial derivative with respect to time and ρ^k is the mass density of the kth layer.

Substituting the kinematics given in equation (1), equation (11) can be rewritten as

$$T(\delta)=\left(\frac{1}{2}\right)\int_0^L\int_{-b/2}^{b/2}\sum_k\int_{h_k}^{h_{k+1}}\rho^k[\phi^{k^2}\dot{\gamma}^2+(z^2+y^2)\dot{\theta}^2]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{12}$$

Substituting equations (10) and (12) into Lagrange's equation of motion, one obtains the governing equation for the vibration of the beam structure as

$$[M]\{\ddot{\delta}\}+[[K]_L+[\mathrm{K}(\delta)]_{NL}]\{\delta\}=\{0\}, \tag{13}$$

where $[M]$ is the consistent mass matrix, and $[K]_L$, $[K]_{NL}$ the structural linear and non-linear stiffness matrices of the beam which are of the complex forms. $\{\delta\}$ and $\{\ddot{\delta}\}$ are the vector of the degrees of freedom (d.o.f.) associated with the displacement field in a finite element discretization and its second derivative with respect to time, respectively.

The coefficients of mass and stiffness matrices involved in governing equation (13) can be rewritten as the product of the term having thickness and width co-ordinates (y and z) alone and the term containing x. In the present study, while performing the integration for the evaluation of the stiffness and mass coefficients, terms having y and z are explicitly integrated whereas the terms containing x are evaluated using the three-point Gauss integration rule.

Substituting characteristics of the time function at the point of reversal of the motion

$$\{\ddot{\delta}\}_{\max}=-\lambda^*\{\delta\}_{\max} \tag{14}$$

into equation (13), will lead to the following non-linear algebraic equation of the form

$$[[K]_L+[\mathrm{K}(\delta)]_{NL}]\{\delta\}-\lambda^*[M]\{\delta\}=[0]. \tag{15}$$

The complex eigenvalues of the form $\lambda^*=(\lambda^R+\mathrm{i}\lambda^I)=(\omega^*)^2$ where $\omega^{*^2}=\omega^{R^2}(1+\mathrm{i}\eta)$ are obtained for the above equation by using a direct iteration technique—suitably modified for the eigenvalue problems based on QR algorithm. The resonance frequencies ω and the system loss factors η are calculated from the eigenvalues [15], corresponding to different amplitudes of vibration level as

$$\omega=\omega^R=(\lambda^R)^{1/2},\quad \eta=\lambda^I/\lambda^R. \tag{16}$$

Here, a three-noded beam finite element as described in Figure 1(b) is employed, based on quadratic functions for rotation θ and linear functions for γ. Further, the element needs two nodal d.o.fs. θ and γ at both the ends of the three-noded beam element whereas the center node has only one-d.o.f. θ. These functions allow one to have the same order of

TABLE 1(a)

Convergence of torsional frequencies (Hz) of cantilever sandwich beam with number of elements (G-core $= 25 \times 10^6$ N/m^2, $b = 0{\cdot}015$ m, $h_1/h_2 = 7$, terms in warping function $N = 11$)

No. of Elements	Mode, n		
	1	2	3
2	1192·706	3762·096	14449·38
4	1192·327	3616·613	6262·816
8	1192·295	3607·77	6124·04
16	1192·291	3606·734	6110·42
32	1192·29	3606·733	6111·42

TABLE 1(b)

Convergence of torsional frequencies (Hz) of cantilever sandwich beam with number of terms in warping function (G-core $= 25 \times 10^6$ N/m^2, $b = 0{\cdot}015$ m, number of element $= 16$)

h_1/h_2	N, warp. function	5	7	9	11	13	15	17
	Mode $n = 1$	1193·838	1192·725	1192·407	1192·291	—	—	—
7	2	3611·266	3608·004	3607·074	3606·734	—	—	—
	3	6117·624	6112·423	6110·953	6110·42	—	—	—
	1	279·4161	275·2726	273·8686	273·2945	273·0273	272·8906	272·7442
1/7	2	941·7924	930·8543	927·1591	925·6505	924·9494	924·591	924·2087
	3	1866·571	1851·563	1846·513	1844·457	1843·504	1843·018	1842·502

interpolation for both $\theta_{,x}$ and γ in the definition of torsional strain, which recovers the Saint–Venant torsion ($\gamma = \theta_{,x}$). The element behaves very well for both short and long situations pertaining to torsion. It has no spurious mode and is represented by correct rigid-body modes.

3. RESULTS AND DISCUSSION

Using the above formulation, the torsional vibration and damping characteristics are investigated for the sandwich beams. Here, a three-ply symmetric sandwich beam having h_1 as thickness of face (bottom/top) layer and h_2 for core (middle), with rectangular cross-section, is considered for the analysis. The middle ply/core ply consists of soft viscoelastic materials. The material properties and the geometrical parameters as given in reference [15] are considered here.

Material properties: for top/bottom layer: $E_b = 45{\cdot}54$ GPa, $G_b = 17{\cdot}12$ GPa, $\nu_b = 0{\cdot}33$, $\rho_b = 2040$ kg/m^3, $\eta_L = \eta_T = \eta_{LT} = \eta_{TT} = 0{\cdot}0$. For core/middle layer, E_c is varied as 7·25, 72·5, 725 and 7250 MPa, $\nu_c = 0{\cdot}45$, $\rho_c = 1200$ kg/m^3, $\eta_L = \eta_T = \eta_{LT} = \eta_{TT} = 0{\cdot}5$.

Geometrical parameters: length (L) = 0·270 m, width (b) is varied as 0·0075, 0·015 and 0·030 m, total thickness of the beam (h) = 0·009 m. The ratio of thickness of face-to-core layer (h_1/h_2) is varied as $\frac{1}{7}$, 1, and 7.

Next, for the torsional vibrations study, convergence tests, for obtaining the natural frequencies ω_n and the associated loss factors η, are conducted for cantilever beams by

TABLE 2

Comparision of torsional frequencies (Hz) for cantilever sandwich beam (G-core $= 2{\cdot}5 \times 10^6$ N/m^2, $b = 0{\cdot}015$ m)

h_1/h_2	Mode, n			
	1		2	
	Present	3D FEM	Present	3D FEM
1/7	188·34	183·63	712·60	697·35
1	748·00	743·53	2302·30	2256·86
7	1147·61	1131·04	3476·20	3410·88

increasing the number of elements while keeping the number of terms in defining the warping function as $N = 11$. The results are described in Table 1(a). Convergence tests are also performed for retaining the number of terms required, in defining the warping function for the cantilever case, and results are given in Table 1(b). It is revealed from Table 1 that 16 elements idealization with $N = 11$ in the warping function series is good enough for evaluating the frequencies and the associated loss factors. However, it is seen from Table 1(b) that for the sandwich case wherein h_2/h_1 is normally high, slightly more number of terms in the warping function are necessary for predicting the converged values. This again depends very much on the variation in the material as well as the geometrical properties concerning face and core of the sandwich beam. Furthermore, an idea of the accuracy involved in the present formulation may be obtained by comparing the values of the first and third natural frequencies of torsional vibration calculated for an isotropic cantilever beam ($L = 0{\cdot}432$ m, $h = 0{\cdot}0508$ m, $b = 0{\cdot}0127$ m) 1660 and 5110 Hz, respectively, with the available experimental results [5], namely 1650 and 5097 Hz. For the sandwich case as depicted in Table 2, the present solutions for linear vibrations are in good agreement with those of three-dimensional finite element solution using a 20-noded brick element.

To solve the non-linear eigenvalue problem, an iterative procedure is used. The iteration starts from a corresponding initial mode shape obtained from linear analysis, with amplitude scaled up by a factor. This gives the initial value denoted by δ_i. Based on this initial mode shape, the non-linear stiffness matrices are formed, and an eigenvalue and its corresponding vector are evaluated. This eigenvector is then scaled up again and the iteration continues until the frequency/damping factor and the eigenvector obtained from the subsequent two iterations satisfy the required convergence criteria suggested by Bergan and Clough [16] within a tolerance of 0·01%.

Since there are no results directly concerning the non-linear torsional vibration and damping behaviour of sandwich beam with visoelastic core available in the literature, numerical experiments are conducted for analyzing the cantilever sandwich beams by considering different values for the thickness ratio (h_1/h_2). The shear modulus G of the core of the laminates is varied in such a way that one can see the behaviour of the beams made of constrained layered damping arrangement to sandwich construction. Numerical results are evaluated using eigenvalue formulation based on QR algorithm. The results concerning free vibration and damping are presented in Tables 3–5 for different thickness ratios of the beam.

It is evident from these tables that, in general, the system loss factor ratio (η_{NL}/η_L; η_L, η_{NL} are the system loss factors obtained from linear and nonlinear analysis) decreases, but the

TABLE 3

Non-linear frequency and loss factor ratios of cantilever beam ($h_1/h_2 = \frac{1}{7}$)

θ (rad)	G-core (MPa) = 2·5						25		250	
	$b(m)$ = 0·0075		0·015		0·03		0·015		0·015	
	ω_{nl}/ω_l	η_{nl}/η_l	ω_{nl}/ω_l	η_{nl}/η_l	ω_{nl}/ω_l	η_{nl}/η_l	ω_{nl}/ω_l	η_{nl}/η_l	ω_{nl}/ω_l	η_{nl}/η_l
0·20	1·0021	0·9958	1·0056	0·9889	1·0286	0·9461	1·0026	0·9949	1·0004	0·9991
0·40	1·0084	0·9835	1·0223	0·9575	1·1126	0·8178	1·0103	0·9799	1·0018	0·9966
0·60	1·0188	0·9640	1·0498	0·9103	1·2478	0·6730	1·0231	0·9562	1·0040	0·9923
0·80	1·0332	0·9383	1·0874	0·8530	1·4293	0·5451	1·0409	0·9253	1·0070	0·9865
1·00	1·0515	0·9078	1·1348	0·7910	1·6536	0·4421	1·0635	0·8891	1·0110	0·9790
1·25	1·0795	0·8651	1·2068	0·7130	1·9913	0·3447	1·0983	0·8389	1·0172	0·9677
1·50	1·1131	0·8195	1·2921	0·6389	2·3906	0·2737	1·1401	0·7862	1·0247	0·9543
1·75	1·1518	0·7727	1·3903	0·5711	—	—	1·1885	0·7332	1·0336	0·9390
2·00	1·1955	0·7263	1·5008	0·5105	—	—	1·2432	0·6817	1·0438	0·9222
ω_l (Hz) or η_l	245·81	0·0193	188·34	0·0620	137·58	0·1486	273·29	0·2891	659·75	0·4203

TABLE 4

Non-linear frequency and loss factor ratios of cantilever beam ($h_1/h_2 = 1$, $b = 0{\cdot}015$ m)

θ (rad)	G-core (MPa) = 2·5		25		250	
	ω_{nl}/ω_l	η_{nl}/η_l	ω_{nl}/ω_l	η_{nl}/η_l	ω_{nl}/ω_l	η_{nl}/η_l
0·20	1·0006	0·9987	1·0006	0·9988	1·0004	0·9993
0·40	1·0025	0·9949	1·0024	0·9953	1·0015	0·9971
0·60	1·0057	0·9887	1·0053	0·9895	1·0033	0·9936
0·80	1·0102	0·9802	1·0095	0·9815	1·0058	0·9887
1·00	1·0158	0·9696	1·0148	0·9716	1·0090	0·9824
1·25	1·0246	0·9536	1·0230	0·9565	1·0141	0·9729
1·50	1·0352	0·9351	1·0329	0·9391	1·0202	0·9617
1·75	1·0476	0·9144	1·0445	0·9195	1·0274	0·9488
2·00	1·0618	0·8921	1·0577	0·8984	1·0357	0·9345
ω_l (Hz) or η_l	748·00	0·0040	774·22	0·0367	985·51	0·1959

TABLE 5

Non-linear frequency and loss factor ratios of cantilever beam ($h_1/h_2 = 7$, $b = 0{\cdot}015$ m)

θ (rad)	G-core (MPa) = 2·5		25		250	
	ω_{nl}/ω_l	η_{nl}/η_l	ω_{nl}/ω_l	η_{nl}/η_l	ω_{nl}/ω_l	η_{nl}/η_l
0·20	1·0003	0·9994	1·0003	0·9994	1·0002	0·9996
0·40	1·0012	0·9976	1·0011	0·9977	1·0007	0·9986
0·60	1·0028	0·9945	1·0026	0·9949	1·0016	0·9968
0·80	1·0049	0·9903	1·0045	0·9910	1·0029	0·9943
1·00	1·0076	0·9850	1·0071	0·9861	1·0045	0·9912
1·25	1·0119	0·9769	1·0110	0·9785	1·0070	0·9863
1·50	1·0171	0·9672	1·0158	0·9695	1·0100	0·9804
1·75	1·0232	0·9561	1·0215	0·9591	1·0136	0·9736
2·00	1·0301	0·9437	1·0279	0·9476	1·0178	0·9659
ω_l (Hz) or η_l	1147·61	0·0045	1192·29	0·0395	1492·72	0·1524

frequency ratio (ω_{NL}/ω_L) increases with an increase in the amplitude of torsional vibration mode of the beam. This effect is more so with the increase in the width of the beam. Furthermore, it can be seen that the rate of decrease in the damping ratio and the rate of increase in the frequency ratio are less with the increase in the value of the core shear modulus. This type of trend in the damping behaviour is because of the change in the shear energy due to shear of the laminates, and it depends not only on the geometry but also on the level of vibration amplitudes. It is also inferred from these tables that the increase in the torsional frequency ratio is less when the core thickness is very less or of the order of face thickness. A similar trend is seen for decrease in the damping ratio for a low value of core thickness. In general, it can be opined that the effect of amplitudes is more on the frequency and damping ratio when the core thickness is high. It is hoped that this study will be useful for the designers/engineers while optimizing the sandwich structures for the torsional response under dynamic situations.

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