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A SEMI-ANALYTICAL SOLUTION OF THE FLEXURAL VIBRATION OF ORTHOTROPIC PLATES OF VARIABLE THICKNESS

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A semi-analytical method based on the finite strip technique in conjunction with the transition matrix is applied to investigate the flexural vibration of orthotropic rectangular plates of variable thickness in one direction. The effect of orthotropy, aspect ratio, and taper ratio on the natural frequencies of the vibration of such plates is investigated. The efficiency of the proposed technique is illustrated by comparing the results with those available in the literature for several examples. The results show excellent agreement and stable convergence for all plate configurations considered in this paper.

1. INTRODUCTION

In many engineering applications, the designer has to use variable thickness plates to suit some design requirements, for example, to save material, and hence the cost requirement, or to change the natural frequency of the plate away from the driving frequency. During last decade, many papers have been presented in the literature dealing with the flexural vibration of variable thickness isotropic plates (for example, references [1, 2, 3 4]). It is not intended in this paper to review the publications on such plates. For the work between 1973 and 1985, the reader can consult the outstanding review papers by Leissa [5–7]. Cheung and Zhou [1] have summarized the review of recent research work in this subject. Composite plates are usually used when the weight and strength of the plate material are of great concern. However, only few research works has been directed towards understanding the effect of the orthotropy in the flexural vibration of such plates.

Malhotra *et al.* [8] used Rayleigh–Ritz method to study the effect of the fiber orientation, boundary conditions and degree of thickness variation on the fundamental frequency of orthotropic square plates of varying thickness (parabolic variation). They considered only four boundary condition configurations. Liu and Chen [9] reported only in graphical form, the fundamental frequency parameter versus the thickness ratio of orthotropic rectangular plates with variable thickness. Bert and Malik [10] analyzed the free vibration of tapered plates using the differential quadrature method. They considered only particular types of thin rectangular plate configurations, which are the ones having two opposite edges simply supported and the other two edges (of varying thickness) are quite general. Lal *et al.* [11] studied the transverse vibration of non-uniform orthotropic rectangular plates using the

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quintic splines interpolation technique when two edges of the plates are simply supported while the other two ends (the varying direction) are restricted to three different combinations of clamped, simply supported, and free edges. In this paper, the flexural vibration of orthotropic plates of generally varying thickness and of general boundary conditions is investigated using the finite strip transition matrix technique developed by Farag [12] and Farag and Ashour [13]. Up to the author knowledge, some of the plate configurations reported in this paper for orthotropic plates of variable thickness are new in the literature.

2. FORMULATION OF THE PROBLEM

Within the classical deformation theory, equation of motion of a thin, orthotropic, and homogenous rectangular plate is given by

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -\left[\bar{p} - \rho h(y)\frac{\partial^2 W}{\partial t^2}\right], \quad (1)$$

where ρ is the density per unit area of the plate, $h(y)$ is the variable plate thickness, and $\bar{p}$ is the dynamic load function. The bending and the twisting moments in terms of displacements are given by

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = -\begin{bmatrix} D_x & v_y D_x & 0 \\ v_x D_y & D_y & 0 \\ 0 & 0 & -2D_t \end{bmatrix}\begin{bmatrix} W_{xx} \\ W_{yy} \\ W_{xy} \end{bmatrix}. \quad (2)$$

Here v, D and E are Poisson's ratio, plate flexural rigidity and modulus of elasticity, respectively, with subscript corresponding to orthotropically oriented co-ordinate axes and defined by

$$D_x = D_{0x}\frac{h^3}{h_0^3}, \quad D_y = D_{0y}\frac{h^3}{h_0^3}, \quad D_t = D_{0t}\frac{h^3}{h_0^3}, \quad (3)$$

where h_0 is the plate thickness at $y = 0$,

$$D_{0x} = \frac{E_x h_0^3}{12(1 - v_x v_y)}, \quad D_{0y} = \frac{E_y h_0^3}{12(1 - v_x v_y)}, \quad 2D_{ot} = H_{oxy} - v_y D_{ox}, \quad (4)$$

$2H_{0xy}$ is the effective torsional rigidity, and $v_y D_{0x} = v_x D_{0y}$. Thus, the equation of motion for a special orthotropic plate (the principal axes of orthotropy are coincident with the edges of the plate) with general variable thickness in the y direction can be written as

$$D_{0x}\frac{h^3(y)}{h_0^3}W_{xxxx} + 2H_{0xy}\frac{h_3(y)}{h_0^3}W_{xxyy} + 2H_{0xy}\frac{1}{h_0^3}\frac{\partial h^3(y)}{\partial y}W_{xxy}$$

$$+ v_x D_{0y}\frac{1}{h_0^3}\frac{\partial^2 h^3(y)}{\partial y^2}W_{xx} + \frac{D_{0y}}{h_0^3}\left(h^3(y)W_{yyyyy} + 2\frac{\partial h^3(y)}{\partial y}W_{yyy} + \frac{\partial^2 h^3(y)}{\partial y^2}W_{yy}\right)$$

$$= \bar{P} - \bar{m}_0\frac{h(y)}{h_0}W_{tt}, \quad (5)$$

where $\bar{m}_0 = \rho h_0$. For time-harmonic-dependent solution, the displacement $W(x, y, t)$ can be assumed as

$$W(x, y, t) = w(x, y)\mathrm{e}^{\mathrm{j}\omega t},$$

where ω is the harmonic frequency (rad/s), and $w(x, y)$ is the flexural displacement at any point (x, y). By defining the following non-dimensional quantities:

$$\xi = \frac{x}{a}, \quad \eta = \frac{y}{b}, \tag{6}$$

and a taper ratio α

$$\alpha = \frac{h_b - h_0}{h_0}, \tag{7}$$

where h_b is the thickness at $\eta = 1$. One can obtain the following non-dimensional equation of motion for the free vibration of a thin orthotropic rectangle plate of variable thickness in the y direction:

$$\begin{aligned} &\frac{D_{0x}}{D_{0y}} h^3(\eta) w_{\xi\xi\xi\xi} + 2\frac{H_{0xy}}{D_{0y}}\beta^2 \frac{\partial h^3(\eta)}{\partial \eta} w_{\xi\xi\eta} + 2\frac{H_{0xy}}{D_{0y}}\beta^2 h^3(\eta) w_{\xi\xi\eta\eta} \\ &\quad + v_x \beta^2 \frac{\partial^2 h^3(\eta)}{\partial \eta^2} w_{\xi\xi} + \beta^4 \left\{ h^3(\eta) w_{\eta\eta\eta\eta} + 2\frac{\partial h^3(\eta)}{\partial \eta} w_{\eta\eta\eta} + \frac{\partial^2 h^3(\eta)}{\partial \eta^2} w_{\eta\eta} \right\}, \\ &\quad - \lambda^2 \frac{h(\eta)}{h_0} h_0^3 w = 0, \end{aligned} \tag{8}$$

where $\lambda^2 = \omega^2 (\bar{m}_0 a^4)/(D_{0y})$, and $\beta = a/b$ is the aspect ratio of the plate.

The solution of the above equation may assume as

$$w(\xi, \eta) = \sum_{m=1}^{N} X_m(\xi) Y_m(\eta), \tag{9}$$

where $X_m(\xi)$ are functions constructed to satisfy identically the boundary conditions at the edges $\xi = 0$ and 1. In this paper, the classical beam functions $X_m(\xi)$ are used. By multiplying equation (8) by $X_n(\xi)$ and integrating from 0 to 1, one obtains the differential equations in $Y(\eta)$ as

$$\begin{aligned} \sum_{m=1}^{N} \Big[&\psi_1 Y_m C_{nm} + 2\psi_2 h_1(\eta) \beta^2 Y'_m B_{nm} + 2\psi_2 \beta^2 Y''_m B_{nm} + v_x \beta^2 h_2(\eta) Y_m B_{nm} \\ &+ \beta^4 \left\{ Y_m^{iv} + 2h_1(\eta) Y''_m + h_2(\eta) Y''_m \right\} A_{nm} - \lambda^2 h_3(\eta) Y_m A_{nm} \Big] = 0, \quad n = 1, 2, \ldots, N, \end{aligned} \tag{10}$$

where the following parameters are defined as

$$\psi_1 = \frac{D_{0x}}{D_{0y}}, \quad \psi_2 = \frac{H_{0xy}}{D_{0y}}, \tag{11}$$

$$h_1(\eta) = \frac{1}{h^3}\frac{\partial h^3}{\partial \eta}, \quad h_2(\eta) = \frac{1}{h^3}\frac{\partial^2 h^3}{\partial \eta^2}, \quad h_3(\eta) = \frac{h_0^2}{h^2}, \tag{12}$$

and

$$A_{nm}=\int_0^1 X_n(\xi)X_m(\xi)\,\mathrm{d}\xi,\quad B_{nm}=\int_0^1 \frac{\mathrm{d}^2X_n(\xi)}{\mathrm{d}\xi^2}X_m(\xi)\,\mathrm{d}\xi, \tag{13, 14}$$

$$C_{nm}=\int_0^1 \frac{\mathrm{d}^4X_n(\xi)}{\mathrm{d}\xi^4}X_m(\xi)\,\mathrm{d}\xi. \tag{15}$$

It is to be noted that the orthogonality conditions yield $A_{nm}=C_{nm}=0$ for $m\neq n$. Thus, the system of differential equations (10) may be simplified as

$$\begin{aligned}Y_n''''(\eta)=&-2h_1(\eta)Y_n'''-h_2(\eta)Y_n''\\&-\frac{1}{\beta^2}\sum_{m=1}^{N}\left\{2\psi_2Y_m''+2\psi_2h_1(\eta)Y_m'+v_xh_2(\eta)Y^m\right\}\frac{B_{nm}}{A_{nn}}\\&-\frac{1}{\beta^4}\left[\psi_1\frac{C_{nn}}{A_{nn}}-\lambda^2h_3(\eta)\right]Y_n(\eta),\quad n=1,2,3,\ldots,N.\end{aligned} \tag{16}$$

By transforming the fourth order differential equations (16) into $4N$ first order ordinary differential equations, the following equation can be obtained at any nodal line i of the divided plate:

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\{F\}_i=[S]_i\{F\}_i,\quad i=1,2,3,\ldots,Q, \tag{17}$$

where Q is the number of strips in this plate,

$$\{F\}_i=\left\{Y_1\,\frac{\mathrm{d}Y_1}{\mathrm{d}\eta}\,\frac{\mathrm{d}^2Y_1}{\mathrm{d}\eta^2}\,\frac{\mathrm{d}^3Y_1}{\mathrm{d}\eta^3}\;Y_2\,\frac{\mathrm{d}Y_2}{\mathrm{d}\eta}\,\frac{\mathrm{d}^2Y_2}{\mathrm{d}\eta^2}\,\frac{\mathrm{d}^3Y_2}{\mathrm{d}\eta^3}\cdots Y_N\,\frac{\mathrm{d}Y_N}{\mathrm{d}\eta}\,\frac{\mathrm{d}^2Y_N}{\mathrm{d}\eta^2}\,\frac{\mathrm{d}^3Y_N}{\mathrm{d}\eta^3}\right\}_i^{\mathrm{T}} \tag{18}$$

and $[\mathbf{S}_i]$ is a $4N\times 4N$ matrix. Applying the transition matrix technique [14], the solution of the above-coupled system of equation can be obtained as

$$\{F\}_i=[\mathbf{T}]_i\{F\}_{i-1}, \tag{19}$$

where $[T]_i$ is called the transition matrix of the strip i while $\{F\}_i$ and $\{F\}_{i-1}$ are the nodal vectors of the strip boundaries i and $i-1$. The solution of the $4N$ first order differential equations of motion (17) is carried out using $2N$-number of initial vectors $\{F\}_0$ at $\eta=0$. Relation (19) is applied across the divided plate until the final end at $\eta=1$ is reached. Therefore, $2N$-number of solutions S_K, where $k=1,2,\ldots,2N$, can be obtained. The true solutions S can be obtained as a linear combination of these solutions as

$$S=\sum_{k=1}^{2N}C_kS_k, \tag{20}$$

where C_k are arbitrary constants. These constants can be determined by satisfying $2N$-number of boundary conditions at $\eta=1$ [13].

3. BOUNDARY CONDITIONS

In this paper, the following boundary conditions have been considered to illustrate the technique used.

(a) *simply supported edges along* ($y = 0, 6$):

$$W = 0, \tag{21a}$$

$$\frac{\partial^2 W}{\partial y^2} + v_x \frac{\partial^2 W}{\partial x^2} = 0, \tag{21b}$$

(b) *clamped edges along* ($y = 0, 6$):

$$W = 0, \tag{22a}$$

$$\frac{\partial W}{\partial y} = 0, \tag{22b}$$

(c) *free edge along* ($y = 0, 6$):

$$\frac{\partial^2 W}{\partial y^2} + v_x \frac{\partial^2 W}{\partial x^2} = 0, \tag{23a}$$

$$D_y \frac{\partial^3 W}{\partial y^3} + (2H_{xy} - v_x D_y) \frac{\partial^3 W}{\partial x^2 \partial y} = 0. \tag{23b}$$

Substituting from equation (9) into equations (21)–(23), multiplying both sides by $Y_n(\zeta)\,d\zeta$ and integrating them from 0 to 1, reduces the above boundary conditions to the following forms:

$$Y_n = 0, \tag{24a}$$

$$\frac{\partial^2 Y_n}{\partial \eta^2} = 0 \tag{24b}$$

for simply supported edges at ($\eta = 0, 1$),

$$Y_n = 0, \tag{25a}$$

$$\frac{\partial Y_n}{\partial \eta} = 0 \tag{25b}$$

for clamped edges at ($\eta = 0, 1$), and

$$\frac{\partial^3 Y_n}{\partial \eta^3} + \sum_{m=1}^{N} \frac{1}{\beta^2} \left(\frac{2H_{xy}}{D_y} - v_x \right) \frac{B_{nm}}{A_{nn}} \frac{\partial Y_m}{\partial \eta} = 0, \tag{26a}$$

$$\frac{\partial^2 Y_n}{\partial \eta^2} + \frac{v_x}{\beta^2} \sum_{m=1}^{N} \frac{B_{nm}}{A_{nn}} Y_m = 0 \tag{26b}$$

for free edges at ($\eta = 0, 1$).

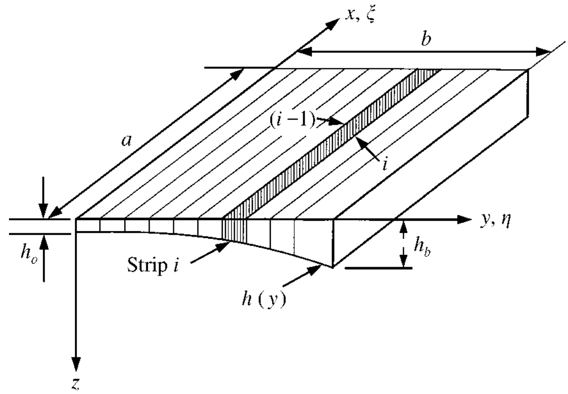

Figure 1. Geometry of generally variable thickness rectangular plate.

TABLE 1

Material properties of the special orthotropic (unidirectional composite) used in the investigation

Composites	$\psi_1 = D_x/D_y$	$\psi_2 = H_{xy}/D_y$	ν_{12}
5-ply maple plywood	3·117	0·648	0·12
E-glass/epoxy	0·4086	0·4808	0·23

TABLE 2

Comparison of the present semi-analytical results with the analytical (exact), the approximate and the other semi-analytical results for uniform orthotropic (5-ply maple plywood) SCSF plate

	λ_1	λ_2	λ_3	λ_4	λ_5
Present method	26·0606	97·6745	161·725	212·0271	254·687
Analytical, Hearmon [15]	26·06	97·68	161·72	212·04	254·68
Lal *et al.* [11]	26·057	97·671	161·722	212·016	254·675
Bert and Malik [10]	26·0571	97·6685	161·714	211·999	254·681

4. NUMERICAL RESULTS AND DISCUSSION

The finite strip transition matrix method is used to investigate the flexural vibration of an orthotropic plate of generally variable thickness in the y direction as shown in Figure 1. A linearly tapered plate is used to illustrate the above technique with the following non-dimensional variable thickness:

$$h(\eta) = 1 + \alpha\eta, \tag{27}$$

where α is the taper ratio. The material properties of the plates used in this investigation are shown in Table 1. In order to validate the developed technique, first, the first five-frequency parameters for uniform orthotropic (5-ply maple plywood) SCSF plate are calculated and compared with those available from other semi-analytical, approximate, and exact methods

TABLE 3

Convergence and comparison of the frequency parameter for E-glass/epoxy plates compared with the results available from the literature for three different plates configuration SSSS, SCSC, CCCC

	N	β	α	λ_1	λ_2	λ_3	λ_4
SSSS							
	3	0·5	0	8·3249	15·1947	27·1391	27·2707
				(8·3249)	(15·1947)	(27·1391)	(27·2707)
	3	0·5	0·2	9·1445	16·6940	29·9491	29·6943
	3	0·5	0·4	9·9424	18·1593	31·9945	32·5456
				(9·9476)	(18·1814)	(32.3980)	(32·5219)
	3	0·5	0·6	10·7231	19·5976	34·1149	35·0772
	3	1	0	15·1947	33·2996	44·4188	60·7787
				(15·1946)	(33·2996)	(44·4188)	(60·7787)
	3	1	0·2	16·6972	36·5779	48·7871	66·7760
	3	1	0·4	18·1712	39·7694	53·0316	72·6372
				(18·1814)	(39·7790)	(52·7135)	(72·7779)
	3	1	0·6	19·6223	42·8923	57·1780	78·3904
	4	2	0	44·4188	60·7787	90·3014	133·1983
				(44·4188)	(60·7786)	(90·3014)	(133·1984)
	4	2	0·2	48·7635	66·7889	99·2404	146·3117
	4	2	0·4	52·9447	72·6846	108·0254	159·0777
				(52·7135)	(72·7779)	(107·8828)	(158·9576)
	4	2	0·6	56·9965	78·4891	116·6871	171·5692
CSCS							
	2	2	0	47·2241	69·7198	164·346	182·9858
	4	2	0	47·0645	69·5645	106·3063	156·2531
	6	2	0	47·0397	69·5313	106·1997	156·1833
	8	2	0	47·0341	69·5218	106·1717	156·1622
				(47·0311)	(69·5152)	(106·1533)	(156·1433)
	2	2	0·4	56·3363	83·4075	196·1701	218·4978
	4	2	0·4	56·1419	83·2206	127·1029	195·7527
	6	2	0·4	56·1115	83·1804	126·9759	186·3151
	8	2	0·4	56·1045	83·1689	126·9425	186·2896
				(55·9489)	(83·0122)	(126·6412)	(186·1889)
CCCC							
	2	1	0·4	34·7652	60·606	80·4346	102·509
	3	1	0·4	34·7214	60·606	80·2629	102·509
	4	1	0·4	34·7214	60·573	80·2629	102·313
	6	1	0·4	34·7148	60·5655	80·2337	102·268
	8	1	0·4	34·7131	60·5631	80·2263	102·254
	2	1	0	29·1484	50·8655	67·4648	85·9024
	3	1	0	29·1118	50·8655	67·3212	85·9024
	4	1	0	29·1118	50·838	85·7384	67·3212
	6	1	0	29·1062	50·8318	67·2967	85·7003
	8	1	0	29·1048	50·8298	67·2906	85·6885

Note: Values in parentheses are from reference [10].

as shown in Table 2. The symbol SCSF means that the edges $x = 0$, $y = 0$, $x = a$, $y = b$ are simply supported, clamped, simply supported and free respectively. It can be seen that results obtained by the finite strip transition matrix method agree excellently with other methods.

Second, a convergence study is carried out to investigate the effect of the series harmonic number 'N' on the frequency parameter of tapered plates for wide range of taper ratio α and

TABLE 4

The frequency parameter for E-glass/epoxy plates for different plate configurations, compared with some of the available results in the literature

N	*CSCC*	β	α	λ_1	λ_2	λ_3	λ_4	λ_5
6		0·5	0	22·0267	33·9806	41·094	46·6472	51·5958
6		0·5	0·2	24·179	37·3436	44·6013	51·1544	62·6478
6		0·5	0·4	26·2934	40·6112	47·6064	55·656	61·6383
6		0·5	0·6	28·3752	43·8032	50·3062	60·073	66·4415
6		1	0	23·982	47·8048	56·4625	76·8934	85·1897
6		1	0·2	26·4329	52·1664	62·2796	84·4648	93·9788
6		1	0·4	28·826	56·3561	67·9203	91·8485	102·1856
6		1	0·6	31·173	60·4152	73·4216	99·0817	109·9381
6		2	0	67·297	85·7005	118·6251	165·9129	204·982
		2	0·2	75·2916	94·8047	130·1436	181·2802	247·607
6		2	0·4	83·0712	103·6955	141·3646	196·1258	247·454
6		2	0·6	90·6792	112·4155	152·353	210·5717	286·146
6		2	0·6	90·6788	112·415	152·353	210·571	286·146
	CCCS							
6		0·5	0	22·0267	33·9806	41·094	46·6472	51·5958
6		0·5	0·2	24·2384	37·3014	45·1158	51·5403	56·6106
6		0·5	0·4	26·4167	40·5273	48·6001	56·4399	61·4557
6		0.5	0.6	28·568	43·6786	51·7428	61·2575	66·1667
6		1	0	23·982	47·8048	56·4625	76·8934	85·1897
6		1	0·2	26·259	52·7912	61·7284	84·4813	93·9789
6		1	0·4	28·4784	57·5917	66·8174	91·8856	102·1857
6		1	0·6	30·652	62·2408	71·7663	99·145	109·9382
6		2	0	67·2967	85·7003	118·625	165·913	204·982
6		2	0·2	72·5086	93·4853	130·472	183·086	223·5954
6		2	0·4	77·5094	101·061	142·011	199·701	241·497
6		2	0·6	82·3446	108·47	153·3	215·861	258·8783
	CSCF							
6		0·5	0	14·5066	16·8728	23·3296	35·2085	42·5063
6		0·5	0·2	16·4386	18·7576	25·7255	38·7055	44·3243
6		0·5	0·4	18·0895	20·8109	28·1136	42·1232	47·5445
6		0·5	0·6	19·5379	22·9716	30·4982	45·4781	50·2869
6		1	0	15·1625	26·5776	40·5576	52·6382	58·4732
6		1	0·2	17·2421	29·2282	46·0619	58·2636	64·0476
6		1	0·4	19·3044	31·8802	51·1972	64·0413	69·4891
6		1	0·6	21·3451	34·5344	55·9968	69·9732	74·8239
6		2	0	17·5575	44·0155	70·9812	82·3486	133·005
6		2	0·2	19·7424	49·7656	76·8423	93·3904	149·395
6		2	0·4	21·9888	55·5382	82·6037	104·241	163·31
6		2	0·6	24·2725	61·3094	88·2785	114·902	177·216
	CFCS							
6		0·5	0	14·5138	16·8731	23·3178	35·1882	42·5096
6		0·5	0·2	15·2441	18·4744	25·5743	41·3828	38·6384
6		0·5	0·4	15·8633	20·0887	27·8196	42·0101	42·6639
6		0·5	0·6	16·4348	21·6793	30·0541	43·8026	54·32
6		1	0	15·1723	26·5484	40·5685	52·6236	58·4215
6		1	0·2	16·1159	29·2085	42·8415	57·6114	64·3855
6		1	0·4	17·0661	31·8678	44·9208	62·608	70·2191
6		1	0·6	18·0301	34·5258	46·8984	67·5743	75·9529
6		2	0	17·551	44·0213	70·8774	82·3646	133·017

TABLE 4
(*continued*)

6	2	0·2	18·9586	47·1254	79·081	87·6504	140·922
6	2	0·4	20·4592	50·3114	87·1813	92·8946	148·392
6	2	0·6	22·0431	53·5844	95·1093	98·2533	155·63
CFCC							
6	0·5	0	38·6266	50·2405	77·5545	99·5664	130·737
6	0·5	0·2	41·3862	46·9212	87·3617	109·099	132·079
6	0·5	0·4	42·6641	46·1572	82·3953	97·4484	132·322
6	0·5	0·6	43·8022	49·7904	84·1692	102·172	131·522
6	1	0	31·5902	56·2977	69·1337	93·8894	129·167
6	1	0·2	34·96	61·4433	76·4441	102·351	134·957
6	1	0·4	38·2943	66·5301	83·552	108·895	139·634
6	1	0·6	41·6012	71·5548	88·8597	117·457	143·845
6	2	0	23·3965	47·6538	84·9473	95·787	135·085
6	2	0·2	26·2704	51·2704	90·1557	107·956	131·036
6	2	0·4	29·4692	54·9898	95·3312	119·946	143·699
6	2	0·6	32·7168	58·8146	100·521	131·804	158·532
CCCF							
6	0·5	0	14·5835	17·6151	25·328	38·6266	39·7189
6	0·5	0·2	16·6633	19·6022	27·9068	42·4469	44·7761
6	0·5	0·4	18·5521	21·6743	30·4514	46·134	48·5241
6	0·5	0·6	20·2657	23·8307	32·9712	49·7393	51·7217
6	1	0	31·6279	56·3201	78·7791	127·787	129·161
6	1	0·2	34·5356	62·3979	75·4781	140·65	139·376
6	1	0·4	37·4076	68·4896	81·5777	153·694	150·894
6	1	0·6	40·2523	74·6237	87·5223	166·649	162·042
6	2	0	47·6605	84·9685	154·209	197·812	264·171
6	2	0·2	53·6444	96·6436	154·058	224·374	293·409
6	2	0·4	59·6656	109·217	138·821	245·62	296·296
6	2	0·6	65·6941	115·597	148·89	266·776	277·392
CSCS							
6	0·5	0	15·4821	20·4268	30·8892	40·8561	45·6785
	0·5	0	(15·4820)	(20·4255)	(30·8822)	(40·8560)	(45·6770)
6	0·5	0·2	61·971	22·4661	33·9359	44·5183	50·3253
6	0·5	0·4	18·3624	24·4974	36·9114	47·5764	54·9996
	0·5	0·4	(18·4633)	(24·3672)	(36·8152)	(48·7007)	(54·4635)
6	0·5	0·6	19·6813	26·5175	39·8306	50·294	60·524
6	1	0	20·4265	45·6783	47·0409	69·5317	83·7139
	1	0	(20·4255)	(45·6770)	(47·0311)	(69·5152)	(83·7103)
6	1	0·2	22·449	50·1365	51·6727	76·4203	91·6789
6	1	0·4	24·4372	54·4114	56·1828	83·1986	98·9829
	1	0·4	(24·3672)	(54·4635)	(55·9489)	(83·0122)	(99·7828)
6	1	0·6	26·3981	58·5439	60·5963	89·8866	105·807
6	2	0	47·0397	69·5306	106·1980	156·1800	163·9156
			(47·0311)	(69·5152)	(106·1533)	(156·1433)	(163·8662)
6	2	0·2	51·6524	76·414	116·692	180·019	171·497
6	2	0·4	56·1114	83·1979	126·974	186·311	195·65
			(55·9490)	(83·0122)	(126·6412)	(186·1889)	(191·2562)
6	2	0·6	60·4488	89·8512	137·086	200·729	210·903
CCCC							
6	0·5	0	16·4657	23·9828	37·4507	41·3914	47·8053
6	0·5	0·2	18·0652	26·3455	41·1254	45·2475	52·5834
6	0·5	0·4	19·5859	28·6478	44·6808	48·6591	57·3256

TABLE 4
(*continued*)

6	0·5	0·6	21·0447	30·9017	48·1419	51·7711	61·9893
6	1	0	29·1064	50·8319	67·3011	85·704	87·1507
6	1	0·2	31·9593	55·7976	73·8896	94·1193	95·5666
6	1	0·4	34·715	60·5657	80·2394	102·2728	103·4832
6	1	0·6	37·3938	65·1778	86·4003	110·2187	111·0132
6	2	0	93·6451	108·115	136·337	179·653	237·872
6	2	0·2	102·8156	118·711	149·699	197·232	261·175
6	2	0·4	111·636	128·939	162·599	214·148	287·303
6	2	0·6	120·199	138·373	174·837	229·666	316·044
CFCF							
6	0·5	0	14·2711	15·1675	26·5629	39·3428	39·778
6	0·5	0·2	15·2238	17·0464	29·2083	41·3788	43·7034
6	0·5	0·4	15·8609	19·041	31·8364	42·6634	47·5402
6	0·5	0·6	16·4344	20·9289	34·4515	43·8021	53·9613
6	1	0	14·2488	17·5542	34·0095	39·275	44·0184
6	1	0·2	15·5557	19·442	37·4094	42·4288	49·032
6	1	0·4	16·7007	21·5251	40·8084	44·7588	54·6021
6	1	0·6	17·7723	23·71	44·207	46·8221	60·2208
6	2	0	14·2249	24·7217	39·1686	55·7197	76·8423
6	2	0·2	15·6595	27·295	42·9832	61·4977	83·9412
6	2	0·4	17·1235	30·0453	46·6517	67·6225	90·1953
6	2	0·6	18·6171	32·9284	50·2449	73·9864	95·9987
SSSS							
3	0·5	0	8·3249	15·195	27·1391	27·2741	33·2999
3	0·5	0·2	9·1445	16·6943	29·6943	29·9531	36·627
3	0·5	0·4	9·9424	18·1597	31·9945	32·5500	39·9373
3	0·5	0·6	10·7231	19·5981	34·1150	35·0822	43·2134
3	1	0·0	15·1947	33·2996	44·4188	60·7787	64·5298
3	1	0·2	16·6972	36·5779	48·7871	66·776	70·7633
3	1	0·4	18·1712	39·7694	53·0316	72·6372	76·6326
3	1	0·6	19·6223	42·8923	57·1780	78·3904	82·2243
3	2	0·0	44·4188	60·7787	90·3014	133·1984	162·7172
3	2	0·2	48·7635	66·7889	99·2404	146·3118	178·7062
3	2	0·4	52·9447	72·6846	108·025	159·0779	194·2184
3	2	0·6	56·9965	78·4891	116·687	171·5697	209·3531
SCSS							
3	0·5	0·0	9·0179	17·2423	30·6918	34·5671	46·9576
3	0·5	0·2	9·9401	18·9163	30·2661	33·6539	51·6615
3	0·5	0·4	10·8355	20·5411	32·8314	36·5138	56·2569
3	0·5	0·6	11·709	22·1269	35·2274	39·2925	60·7665
3	1	0·0	19·646	36·0717	54·2508	66·3707	68·9691
			(19·646)	(36·0717)	(54·2508)	(66·3707)	(68·9691)
3	1	0·2	21·3536	39·7605	59·2587	73·2597	75·6654
3	1	0·4	23·0108	43·3418	64·089	79·8097	82·1646
3	1	0·6	24·6283	46·836	68·7782	86·0886	88·5077
3	2	0·0	65·4179	78·5837	104·408	144·2872	240·4265
3	2	0·2	70·3469	85·4257	114·544	159·0421	263·1849
3	2	0·4	75·0261	92·0874	124·454	173·3632	285·1972
3	2	0·6	79·573	98·5482	134·146	187·3282	256·6865
SFSS							
3	0·5	0·0	10·3839	18·6145	25·5822	31·844	37·1821
3	0·5	0·2	11·3797	20·4358	26·8626	32·1108	34·9829
3	0·5	0·4	12·3687	22·2195	27·921	34·8879	38·0329

TABLE 4
(*continued*)

3		0·5	0·6	13·3513	23·9743	28·8802	37·5544	41·0134
3		1	0·0	7·7812	22·2652	26·759	41·5357	56·0705
				(7·8694)	(22·4295)	(26·8733)	(41·8213)	(56·1945)
3		1	0·2	8·4326	24·6105	28·4569	45·5189	61·8397
3		1	0·4	9·109	26·9432	30·1003	49·475	64·3581
3		1	0·6	9·8089	29·2686	31·7153	53·4052	67·0546
3		2	0·0	11·1559	31·1248	62·8826	68·7382	89·0613
3		2	0·2	12·3683	33·7305	67·3221	76·852	98·4427
3		2	0·4	13·659	36·4361	71·7819	84·8929	107·774
3		2	0·6	15·0166	39·2357	76·2847	92·8902	117·0759
	SSSC							
3		0·5	0·0	9·0179	17·2423	30·6918	34·5671	46·9576
3		0·5	0·2	9·8683	18·9582	29·9161	33·747	51·516
3		0·5	0·4	10·6934	20·625	32·1439	36·7005	55·9649
3		0·5	0·6	11·4988	22·2529	34·2195	39·5739	60·3263
3		1	0·0	19·646	36·0717	54·2508	66·3707	68·9691
		1	0	(19·6460)	(36·07175)	(54·2509)	(66·3707)	(68·969)
3		1	0·2	21·8069	39·4733	59·8827	72·3644	75·833
3		1	0·4	23·9162	42·7734	65·3381	78·0417	82·5001
3		1	0·6	25·984	45·9952	70·6542	83·4806	89·0115
3		2	0·0	65·418	78·5837	104·408	144·2872	203·9998
3		2	0·2	73·3146	87·2278	114·853	157·8935	225·5005
3		2	0·4	80·9941	95·6683	125·045	171·094	233·6955
3		2	0·6	88·4993	103·947	135·039	183·9812	266·5942
	SSSF							
3		0·5	0·0	10·3839	18·6145	25·5822	31·844	37·1821
3		0·5	0·2	11·4592	20·4705	28·9114	32·4575	34·967
3		0·5	0·4	12·5318	22·287	31·6292	35·9453	37·9992
3		0·5	0·6	13·6039	24·0717	33·9325	39·5911	40·959
3		1	0·0	7·7812	22·2652	26·759	41·5357	56·0705
3		1	0·2	8·709	24·3543	30·3237	45·8368	61·3554
3		1	0·4	9·6507	26·4245	33·7934	50·2173	66·507
3		1	0·6	10·5995	28·4777	37·175	54·4155	71·549
3		2	0·0	11·1559	31·1248	62·8826	68·7385	89·0613
3		2	0·2	12·2564	34·8361	71·0017	74·2654	97·4179
3		2	0·4	13·4146	38·6027	79·0734	79·6869	105·6991
3		2	0·6	14·6095	42·398	85·0154	87·087	113·9124
	SCSC							
3		0·5	0·0	19·646	9·9853	27·4626	34·4626	36·0718
3		0·5	0·2	21·5714	10·9633	30·5783	37·8356	39·6373
3		0·5	0·4	23·4307	11·9069	33·0635	41·085	43·1295
3		0·5	0·6	25·2378	12·8232	35·4039	44·2366	46·5579
3		1	0·0	25·6218	39·941	65·418	68·8223	78·5839
				(25·6218)	(39·941)	(65·418)	(68·8223)	(78·5839)
3		1	0·2	28·1324	43·8532	71·8187	75·5164	86·2855
3		1	0·4	30·556	47·6275	77·9808	81·8938	93·7229
3		1	0·6	32·9108	51·2927	83·9546	88·0227	100·951
3		2	0·0	92·2352	102·513	124·0537	159·7595	210·3085
3		2	0·2	11·4697	112·775	111·6573	175·7465	231·3504
3		2	0·4	111·628	124·063	122·8341	158·1808	254·4964
3		2	0.6	122·802	111·666	110·5597	142·3717	229·0558

Note: Values in parentheses are from reference [10].

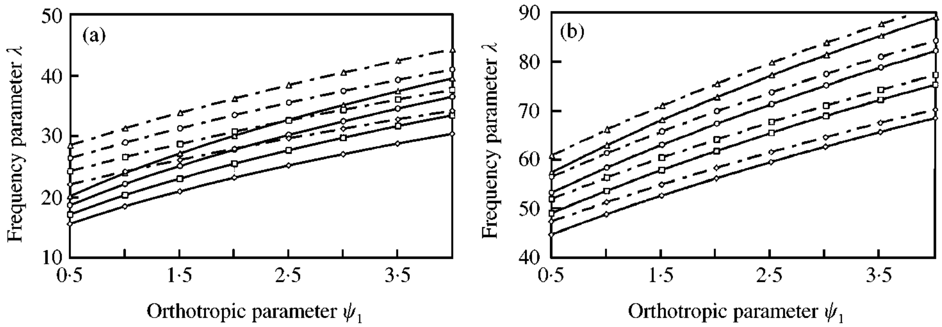

Figure 2. The fundamental frequency parameter versus the orthotropic parameter ψ_1 for SSSS plate, (a) for $\beta = 1$ and (b) for $\beta = 2$. The solid lines (——) for $\psi_2 = 0{\cdot}5$ and the dotted lines (·····) for $\psi_2 = 3$. Key: (◇) $\alpha = 0{\cdot}0$; (□) $\alpha = 0{\cdot}2$; (○) $\alpha = 0{\cdot}4$; and (△) $\alpha = 0{\cdot}6$.

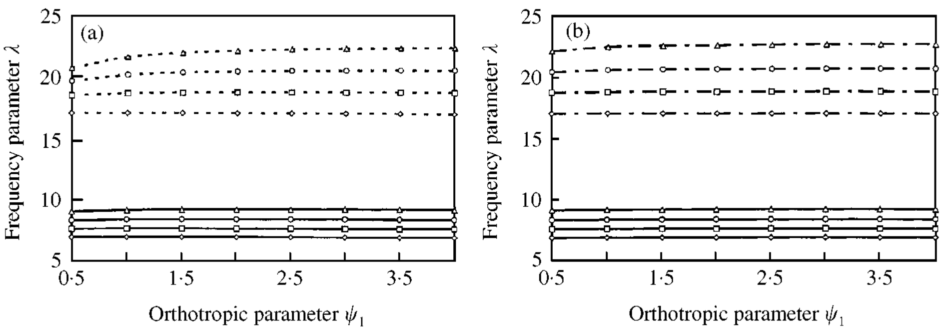

Figure 3. The fundamental frequency parameter versus the orthotropic parameter ψ_1 for SFSF plate, (a) for $\beta = 1$ and (b) $\beta = 2$. The solid lines (——) for $\psi_2 = 0{\cdot}5$ and the dotted lines (·····) for $\psi_2 = 3$. Key: (◇) $\alpha = 0{\cdot}0$; (□) $\alpha = 0{\cdot}2$; (○) $\alpha = 0{\cdot}4$; and (△) $\alpha = 0{\cdot}6$.

for different aspect ratio β. The first five-frequency parameters for three different E-glass/epoxy plate configurations (SSSS, SCSC, CCCC) are calculated and compared with those results available from the literature as shown in Table 3. The results show that the technique used in this paper has excellent convergence and agree well with results available from literature. It seems that if two edges of the plate are simply supported, the x direction, the method converges using only one term in the series solution. For other cases, the method converges using only few terms (four to six terms) in the series solution.

Third, the first five-frequency parameters of E-glass/epoxy tapered plate for different aspect ratio, taper ratio, and plate configurations are calculated and tabulated in Table 4. Note that in this case, a large number of plate configurations exists, therefore, so our results are limited to those cases of simply supported and clamped at the two edges in the x direction and all possible combinations in the y direction. In the cases where two edges in the x direction are simply supported, three terms of the series solution are used. For other cases, six terms are used. The results are compared with those available in the literature. Excellent agreement can be noticed with maximum difference of less than 1%.

Finally, the effect of the orthotropic parameters ψ_1 and ψ_2 on the fundamental frequency is investigated for four plate configurations. Figures 2–5 show the fundamental frequency parameter versus the orthotropic parameter ψ_1 for SSSS, SFSF, CFCF, and CCCC plate configurations, respectively, for different taper ratio α. In each figure, two cases of the orthotropic parameter ψ_2 have been considered, i.e., $\psi_2 = 0{\cdot}5$ (solid lines) and $\psi_2 = 3$

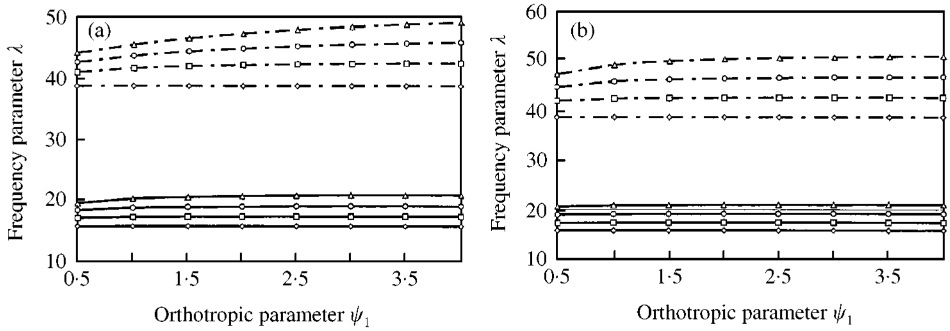

Figure 4. The fundamental frequency parameter versus the orthotropic parameter ψ_1 for CFCF plate, (a) for $\beta = 1$ and (b) for $\beta = 2$. The solid lines (——) for $\psi_2 = 0{\cdot}5$ and the dotted lines (·····) for $\psi_2 = 3$. Key: (◇) $\alpha = 0{\cdot}0$; (□) $\alpha = 0{\cdot}2$; (○) $\alpha = 0{\cdot}4$; and (△) $\alpha = 0{\cdot}6$.

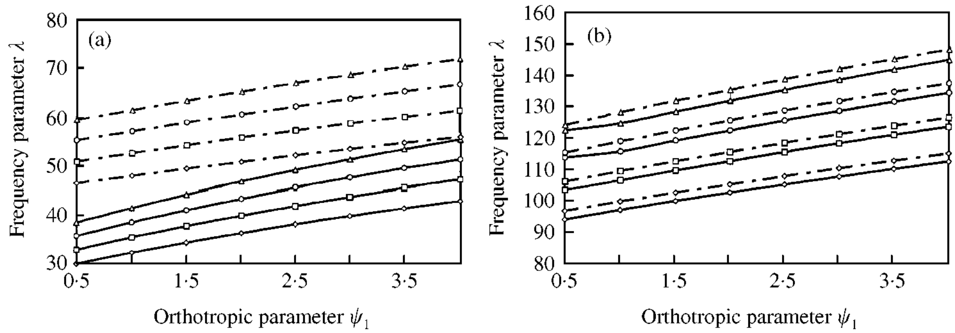

Figure 5. The fundamental frequency parameter versus the orthotropic parameter ψ_1 for CCCC plate, (a) for $\beta = 1$ and (b) for $\beta = 2$. The solid lines (——) for $\psi_2 = 0{\cdot}5$ and the dotted lines (·····) for $\psi_2 =$ Key: (◇) $\alpha = 0{\cdot}0$; (□) $\alpha = 0{\cdot}2$; (○) $\alpha = 0{\cdot}4$; and (△) $\alpha = 0{\cdot}6$.

(dotted lines). Also, two cases of the aspect ratio β have been considered for each plate configuration (a) $\beta = 1$ and (b) $\beta = 2$. The results of this investigation show that for SSSS and CCCC plate configurations, the fundamental frequency increases with the increase of the orthotropic parameter ψ_1 while the effect of the orthotropic parameter ψ_2 is much less for higher aspect ratio as seen from the comparison of Figures 2(a), 5(a) and 2(b), 5(b). For the SFSF and CFCF plate configurations, the fundamental frequency does not change significantly with the orthotropic parameter ψ_1 while it is mainly controlled by the orthotropic parameter ψ_2, as is shown in Figures 3(a), 4(a) and 3(b), 4(b).

5. CONCLUSION

The finite strip transition matrix method is applied to calculate the frequency parameters of variable thickness orthotropic plates in the y direction for different combinations of boundary conditions. The method is illustrated for linearly tapered E-glass/epoxy plate for different taper ratio, aspect ratio, and different combinations of boundary conditions. Also, the effect of the orthotropic parameters on the fundamental frequency is investigated for four plate configurations with different aspect and taper ratios. Numerical results obtained from this technique show the accuracy and the validity of the method.

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APPENDIX A: NOMENCLATURE

a, b	length and width of the plate
D_x, D_y	plate flexural rigidity in x and y directions respectively
E_x, E_y	Young's modulus in x and y directions respectively
$\{F\}_i$	nodal vector at the boundary i
$h(y)$	plate thickness as function of y
h_o, h_b	plate thickness at $y = 0$ and b respectively
H_{xy}	effective torsional rigidity of plate
$\bar{m}_0$	$= \rho\, h_0$, mass density per unit area of the plate
M_x, M_y, M_{xy}	bending moment in the x and y directions and twisting moment respectively
N	number of terms used in series solution
$\bar{p}$	dynamic load function
Q	number of strips
$[\mathbf{T}]_i$	transition matrix of the strip i
$W(x, y, t)$	time-dependent flexural displacement at any point (x, y)
$\mathrm{w}(x, y)$	flexural displacement at any point (x, y)
x, y	real space co-ordinate
X_m	classical beam functions in the x direction
Y_n	series solution in the y direction
α	taper ratio

β	$= a/b$, plate aspect ratio
λ^2	$= \omega^2(\bar{m}_0 a^4/D_{0y})$, eigenvalue parameter
v_x,v_y	Poisson's ratio in x and y directions respectively
ρ	density per unit area of the plate
ω	angular frequency
ξ, η	$= x/a$ and y/b respectively
ψ_1, ψ_2	$= H_{0xy}/D_{0y}$ and D_{0x}/D_{0y}, respectively, rigidity ratios

APPENDIX B: FLOW CHART SHOWING THE SEQUENCE OF THE COMPUTATION LOGIC