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FREE VIBRATION ANALYSIS OF RECTANGULAR PLATES ON ELASTIC INTERMEDIATE SUPPORTS

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A procedure using the finite strip element method in combination with a spring system is proposed to treat the free vibration analysis of plates on elastic intermediate supports. Results indicate that the spring system can successfully simulate elastic intermediate supports such as point supports, line supports, local uniformly distributed supports and mixed edge supports. From the results, it is also evident that support stiffness and support areas have significant influence on the free vibration response of plates on line supports and local uniformly distributed supports.

1. INTRODUCTION

Rectangular plates on intermediate supports find use in many engineering structures and other areas of practical interest, such as slabs on columns, printed circuit boards or solar panels supported at a few points. With its potential applications, the vibration of point supported plates and plates with complex boundary conditions have received considerable attention from researchers. Venkateswara Rao *et al.* [1, 2] analyzed the vibration of rectangular plates with mixed boundary conditions and with point supports. Raju and Amba-Rao [3] considered the free vibration of a square plate symmetrically supported at four points on the diagonals. Narita [4–6] presented a series solution for the vibration analysis of rectangular plates with complex mixed conditions, point supports and cantilever plates with point constraints. Kersterns *et al.* [7] treated rectangular plates with point supports, while Bhat [8] used the characteristic orthogonal polynomial in the Rayleigh–Ritz method to analyze the vibration of rectangular plates with point and line supports. Gorman [9–11] presented the solutions to rectangular plates with symmetrically distributed point supports and uniform elastic edge supports. Bapat *et al.* [12–17] discuss the vibration characteristics of rectangular plates having various types of supports such as a single point support, arbitrary multiple point supports within the plate and at the edges. Kim and Dickinson [18] investigated the flexural vibration of rectangular plates with point supports in detail. Lee and Lee [19] investigated rectangular plates on elastic point supports and discussed the effect of the support stiffness. The most general study in this particular area is that by Fan and Cheung [20], in which they used the spline strip element method to analyze plates with complex boundary conditions and point supports.

In this paper, the finite strip element method combined with a spring system is employed to treat the free vibration analysis of plates on elastic intermediate supports. The elastic

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intermediate supports are modelled by a spring system which can simulate point supports, line supports, local uniformly distributed (or patch) supports and mixed complex boundary conditions. Results are compared with those from the literature. The effects of elastic support stiffness and support area on the vibration response are discussed. Finally, this method is applied to analyze the free vibration of a highway bridge with column supports.

2. FINITE STRIP ELEMENT AND GOVERNING EQUATION

Considering a finite strip element, the deflection of the element can be expressed as

$$w(x, y, t) = \sum_{m=1}^{M} [N]_m \{\delta\}_m. \tag{1}$$

For the low order strip element (LO2) [21, 22], the following expressions hold:

$$\{\delta\}_m = \{w_{im}(t), \theta_{im}(t), w_{jm}(t), \theta_{jm}(t)\}^{\mathrm{T}}, \qquad [N]_m = [N_{1m}, N_{2m}, N_{3m}, N_{4m}], \tag{2}$$

$$N_{1m} = (1 - 3\xi^2 + 2\xi^3)\, Y_m(y), \qquad N_{2m} = e\xi(1 - 2\xi + \xi^2)\, Y_m(y),$$

$$N_{3m} = (3\xi^2 + 2\xi^3)\, Y_m(y), \qquad N_{4m} = e(-\xi^2 + \xi^3)\, Y_m(y),$$

$$\xi = \frac{x - x_i}{x_j - x_i}, \qquad e = x_j - x_i.$$

Here, x_i and x_j denote the plate edges and $Y_m(y)$ are beam eigenfunctions which satisfy the end conditions of the beam. For a single span $Y_m(y)$ has the following form [21]:

$$Y_m(y) = A_1 \sin\left(\frac{\alpha_m y}{b}\right) + A_2 \cos\left(\frac{\alpha_m y}{b}\right) + A_3 \sinh\left(\frac{\alpha_m y}{b}\right) + A_4 \cosh\left(\frac{\alpha_m y}{b}\right). \tag{3}$$

In the above equation, A_i and α_m are determined from the boundary conditions at the discontinuous ends of the strip and b is the length of the element. For example, if the plate strip is simply supported at both ends, the function takes the simpler form

$$Y_m(y) = A_1 \sin\left(\frac{\alpha_m y}{b}\right), \qquad \alpha_m = m\pi, \quad m = 1, 2, \ldots\ . \tag{4}$$

From the displacement function, the curvatures and bending moments can be easily obtained. The strain vector of curvatures and the stress vector of moments are then given by

$$\{\varepsilon\} = \sum_{m=1}^{M} [B]_m \{\delta\}_m, \qquad \{\varepsilon\} = \{\chi_x, \chi_y, \chi_{xy}\}^{\mathrm{T}}, \tag{5}$$

$$\{\sigma\} = \sum_{m=1}^{M} [D]\,[B]_m \{\delta\}_m, \qquad \{\sigma\} = \{M_x, M_y, M_{xy}\}^{\mathrm{T}}. \tag{6}$$

Here,

$$[B]_m = \left\{ -\frac{\partial^2}{\partial x^2}, -\frac{\partial^2}{\partial y^2}, 2\frac{\partial^2}{\partial x\,\partial y} \right\}^{\mathrm{T}} [N]_m, \qquad [D] = \begin{bmatrix} D_x & D_1 & 0 \\ D_1 & D_y & 0 \\ 0 & 0 & D_{xy} \end{bmatrix},$$

$$D_x = \frac{E_x h^3}{12(1-\mu_x \mu_y)}, \qquad D_y = \frac{E_y h^3}{12(1-\mu_x \mu_y)},$$

$$D_{xy} = \frac{G_{xy} h^3}{12}, \qquad D_1 = \mu_y D_x = \mu_x D_y.$$

In the above equation $[D]$ is the constitutive matrix, h the plate thickness and E_x, E_y, μ_x, μ_y, G_{xy} are the elastic constants of the orthotropic plate.

In order to apply the Lagrangian equation to the plate system, the total energy of the plate is required. This will include the strain energy of the plate, the strain energy of the elastic foundation, the potential energy of the load and the kinetic energy of the mass system. The strain energy of the plate strip can be expressed as

$$U_p = \int_e \frac{1}{2} \{\varepsilon\}^{\mathrm{T}} \{\sigma\}\, \mathrm{d}A = \frac{1}{2} \sum_{m=1}^{M} \sum_{n=1}^{M} \{\delta\}_m^{\mathrm{T}} [K_p]_{mn} \{\delta\}_n, \quad [K_p]_{mn} = \int_e [B]_m^{\mathrm{T}} [D]\, [B]_n\, \mathrm{d}A. \quad (7)$$

Three types of spring systems can be considered:

k_f: spring stiffness for the displacement (w)
k_x: spring stiffness for the rotation about the y-axis ($\theta_x = \partial w/\partial x$)
k_y: spring stiffness for the rotation about the x-axis ($\theta_y = \partial w/\partial y$).

The strain energy of the elastic foundation, consisting of these springs, can then be expressed as

$$U_f = \int_e \frac{1}{2} k_f w^2\, \mathrm{d}A = \frac{1}{2} \sum_{m=1}^{M} \sum_{n=1}^{M} \{\delta\}_m^{\mathrm{T}} [K_f]_{mn} \{\delta\}_n, \qquad [K_f]_{mn} = \int_e [N]_m^{\mathrm{T}} k_f [N]_n\, \mathrm{d}A, \quad (8)$$

$$U_x = \int_e \frac{1}{2} k_x \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}A = \frac{1}{2} \sum_{m=1}^{M} \sum_{n=1}^{M} \{\delta\}_m^{\mathrm{T}} [K_x]_{mn} \{\delta\}_n, \qquad [K_x]_{mn} = \int_e [G_x]_m^{\mathrm{T}} k_x [G_x]_n\, \mathrm{d}A, \quad (9)$$

$$U_y = \int_e \frac{1}{2} k_y \left(\frac{\partial w}{\partial y}\right)^2 \mathrm{d}A = \frac{1}{2} \sum_{m=1}^{M} \sum_{n=1}^{M} \{\delta\}_m^{\mathrm{T}} [K_y]_{mn} \{\delta\}_n, \qquad [K_y]_{mn} = \int_e [G_y]_m^{\mathrm{T}} k_y [G_y]_n\, \mathrm{d}A. \quad (10)$$

Here,

$$[G_x] = \frac{\partial}{\partial x} [N], \qquad [G_y] = \frac{\partial}{\partial y} [N].$$

Considering only transverse loads, the potential energy can be expressed as

$$V = \int_e q(x, y, t) w\, \mathrm{d}A = \sum_{m=1}^{M} \{\delta\}_m^{\mathrm{T}} \{Q\}_m, \qquad \{Q\}_m = \int_e [N]_m^{\mathrm{T}} q(x, y, t)\, \mathrm{d}A. \quad (11)$$

In the above equation, q is the distributed load and Q_m is the load vector for the harmonic m. The kinetic energy of the strip can be expressed as

$$T_e = \int_e \frac{1}{2} m\dot{w}^2 \,\mathrm{d}A = \frac{1}{2} \sum_{m=1}^{M} \sum_{n=1}^{M} \{\dot{\delta}\}_m^{\mathrm{T}} [M]_{mn} \{\dot{\delta}\}_n, \qquad [M]_{mn} = \int_e [N]_m^{\mathrm{T}}\, m [N]_n \,\mathrm{d}A, \quad (12)$$

where ρ is the plate density. The total energy of the structure is obtained by adding all the above contributions and is given by

$$\prod = \sum_e \prod = \sum (U_p + U_f + U_x + U_y - V), \qquad T = \sum T_e. \quad (13, 14)$$

Using the Lagrangian equation on the entire plate system, the governing (dynamic) equation of the structure can be formulated as

$$[M]\{\ddot{\delta}\} + ([K_p] + [K_f] + [K_x] + [K_y])\{\delta\} = \{Q\}. \quad (15)$$

In the above equation, $[K_p]$, $[K_f]$, $[K_x]$, $[K_y]$ are the stiffness matrices of plate and spring systems, respectively, and $[M]$ is the mass matrix.

When free vibration is treated, the load is set to zero and the deflection can be expressed as

$$w(x, y, t) = \sum_{m=1}^{M} [N]_m \{\bar{\delta}\}_m \sin(\omega t + \varphi), \qquad \{\bar{\delta}\}_m = \{\bar{w}_{im}, \bar{\theta}_{im}, \bar{w}_{jm}, \bar{\theta}_{jm}\}^{\mathrm{T}}.$$

Then the dynamic equation (15) reduces to the eigenvalue problem:

$$\{([K_p] + [K_f] + [K_x] + [K_y]) - \omega^2 [M]\}\{\bar{\delta}\} = \{0\}, \quad (16)$$

where ω is the natural frequency in radians and φ the phase.

3. MODELS OF INTERMEDIATE SUPPORTS

Different types of intermediate supports can be simulated by appropriate modelling of the spring system. Elastic intermediate supports can be placed within the interior of the plate or at its edges. When line intermediate supports are placed at the edges of the plate, mixed complex boundary conditions can be simulated.

3.1. LOCAL UNIFORMLY DISTRIBUTED ELASTIC SUPPORTS

In order to treat localized uniform elastic supports, the spring system is modelled as having uniform stiffness within the support area. For example, if the elastic support area is a rectangular patch within $x_1 \leqslant x \leqslant x_2$, $y_1 \leqslant y \leqslant y_2$, then the springs modelling this type of support can be expressed as

$$\begin{aligned} k_f(x, y) &= k_f H[r_1^2 - (x - c_1)^2] H[r_2^2 - (y - c_2)^2], \\ k_x(x, y) &= k_x H[r_1^2 - (x - c_1)^2] H[r_2^2 - (y - c_2)^2], \\ k_y(x, y) &= k_y H[r_1^2 - (x - c_1)^2] H[r_2^2 - (y - c_2)^2], \end{aligned} \quad (17)$$

where $H(x)$ is the Heaviside unit function given by

$$H[x] = \begin{cases} 1, & x \geqslant 0, \\ 0, & x < 0, \end{cases}$$

$$r_1 = (x_2 - x_1)/2, \qquad c_1 = (x_2 + x_1)/2,$$

$$r_2 = (y_2 - y_1)/2, \qquad c_2 = (y_2 + y_1)/2.$$

3.2. ELASTIC LINE SUPPORTS AND POINT SUPPORTS

Line supports and point supports are common in several types of plate structures. Supports parallel to both edges of the rectangular plate can be treated, with uniformly distributed stiffness along the length of the supports, parallel to the x- and y-axes. They can be represented as follows:

1. Line supports parallel to the x direction with $x_1 \leqslant x \leqslant x_2$ at $y = y_0$:

$$\begin{aligned} k_f(x, y) &= k_f H[r_1^2 - (x - c_1)^2]\delta(y - y_0), \\ k_x(x, y) &= k_x H[r_1^2 - (x - c_1)^2]\delta(y - y_0), \\ k_y(x, y) &= k_y H[r_1^2 - (x - c_1)^2]\delta(y - y_0). \end{aligned} \tag{18}$$

2. Similarly, the line supports parallel to the y direction with $y_1 \leqslant y \leqslant y_2$ at $x = x_0$:

$$\begin{aligned} k_f(x, y) &= k_f H[r_2^2 - (y - c_2)^2]\delta(x - x_0), \\ k_x(x, y) &= k_x H[r_2^2 - (y - c_2)^2]\delta(x - x_0), \\ k_y(x, y) &= k_y H[r_2^2 - (y - c_2)^2]\delta(x - x_0). \end{aligned} \tag{19}$$

3. Point support at (x_0, y_0):

$$\begin{aligned} k_f(x, y) &= k_f \delta(x - x_0)\delta(y - y_0), \\ k_x(x, y) &= k_x \delta(x - x_0)\delta(y - y_0), \\ k_y(x, y) &= k_y \delta(x - x_0)\delta(y - y_0), \end{aligned} \tag{20}$$

where $\delta(x)$ is the Dirac Delta function.

3.3. STIFFNESS OF COLUMN (PATCH) SUPPORTS

When plates supported by elastic columns are considered, it is usual to treat the columns as clamped at one end and flexible at the point of attachment to the plate. For a column support, shown in Figure 1, with height H, area of cross-section $A = b_1 \times h_1$, Young's modulus E_c, the axial stiffness k_f and rotational stiffness about the y- and x-axis k_x and k_y are given by

$$k_f = E_c A/H, \qquad k_x = 4EI_y/H, \qquad k_y = 4EI_x/H,$$

where I_y and I_x are second moments of area about the y- and x-axis respectively.

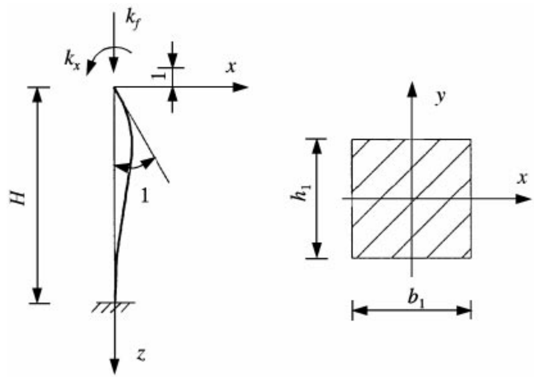

Figure 1. Column stiffness and section.

When the column is treated as a distributed (or patch) support, it is assumed to have only a mean axial stiffness and no rotational stiffness. This type of support is relevant in certain bridge decks where neoprene (synthetic rubber-based material) pads are used between the deck and the piers. If the point support model is adopted, the column will have at its end an axial stiffness and rotational stiffness about the two axes. That is:

Distributed (*or patch*) *support*:

$$k_f = E_c/H, \quad k_x = 0, \quad k_y = 0. \tag{21}$$

Each point support:

$$k_f = E_c A/H, \quad k_x = (E_c A/H)(b_1^2/3), \quad k_y = (E_c A/H)(h_1^2/3). \tag{22}$$

The use of elastic point supports with rotational stiffness k_x and k_y is illustrated in the example treated in section 6. For a column support, $k_f = E_c A/H$ is termed the total stiffness for the distributed support or the point support in the ensuing discussion.

4. VALIDATION OF PROCEDURE

Plates with mixed boundary conditions or inner point supports have been treated by many researchers. In order to validate the procedure, a square plate with mixed boundary conditions or a centre point support, as shown in Figure 2 is considered. The plate structures in Figures 2(a)–2(c) have a mix of simple and fixed boundary conditions at the edges, while the plate structure in Figure 2(d) is simply supported along all four edges and has a central point support as well. The spring stiffness is set to the high value of 10^{15} kN/m^2 to simulate a rigid line boundary condition, 10^{15} kN/m to simulate a rigid point boundary condition, and the Poisson ratio is set to 1/3. The results for the first three frequencies are shown in Table 1, with the frequency parameters defined as $\lambda = \omega B^2\sqrt{\rho h/D}$. Convergence of the results has been ensured and the converged results in Table 1 have been obtained by using 10 finite strip elements and 25 terms of the series parallel to the mixed boundary edge.

From the results in Table 1, it can be seen that the present results compare well with those from others. In the case of the model with the elastic point support at the centre, the present results cannot be compared directly with those in reference [19], as unsymmetrical modes are absent in the results of Lee and Lee [19]. Hence, only the symmetric modes can be

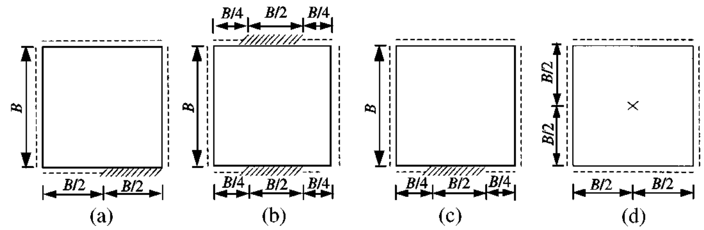

Figure 2. Square plates with mixed boundary conditions and centre point support.

TABLE 1

Natural frequency parameters of square plates with mixed boundary conditions and centre point support

Plates	References	Frequency parameters		
		λ_1	λ_2	λ_3
Figure 2(a)	Venkateswara Rao *et al.* [1]	22·96		
	Narita [4]	22·63	50·04	55·95
	Fan and Cheung [20]	22·73	50·15	56·23
	Present	22·81	50·26	56·36
Figure 2(b)	Venkateswara Rao *et al.* [1]	28·62		
	Narita [4]	22·44	53·49	67·85
	Fan and Cheung [20]	28·65	54·00	68·58
	Present	28·67	54·09	68·54
Figure 2(c)	Fan and Cheung [20]	23·54	51·43	58·36
	Present	23·55	51·43	58·33
Figure 2(d)	Venkateswara Rao *et al.* [1]		52·62	
	Lee and Lee [19]		53·09	
	Kim and Dickinson [18]	49·35†	53·17	78·96
	Fan and Cheung [20]	49·35†	52·78	78·96
	Present	49·35†	52·75	78·96

†Two kinds of mode shapes.

compared. The first mode in the present case being unsymmetrical, the frequency of the second mode (which is symmetric) is compared with the frequency of the first mode in reference [19], and the results are seen to match well. Hence, it may be concluded that the spring system can successfully simulate the complex mixed boundary conditions and the inner point supports–either rigid or elastic.

5. PLATES ON LINE AND LOCALLY DISTRIBUTED ELASTIC SUPPORTS

Though line and locally distributed elastic supports are also common types of support in engineering structures, they have not received much attention from researchers. In the present work, plates on elastic line supports and locally distributed rectangular supports are considered. The support stiffness and support length or area are taken as parameters in the study. At first, a rectangular plate with the two shorter edges simply supported and having two line supports along the middle line parallel to the shorter edges, as shown in Figure 3(a), is considered. A square plate simply supported along two opposite edges and having a central patch support, as shown in Figure 3(b), is treated next. The other sides of these two plate structures are free. Material properties for both these cases are the same and are given by: $E = 31$ GPa, $\mu = 0{\cdot}25$, $\rho = 2500$ kg/m^3, where E, μ, and ρ are Young's modulus, the Poisson ratio and the mass density respectively. The plate dimensions are: $B = 10$ m, $L = 20$ m, $h = 0{\cdot}3$ m for the line-supported plate, and $B = L = 10$ m, $h = 0{\cdot}3$ m for the square plate. The plate is divided into 10 finite strip elements and 25 terms of the series, parallel to the free edges, are taken into account. Rotational stiffness is not considered in these examples.

The first three natural frequency parameters are presented in Tables 2 and 3 for the plate with elastic line supports and for the plate on the elastic support at the plate centre respectively. Unfortunately, the authors have not come across any similar work in the literature for comparing results. In order to investigate the effect of support (length or) area on the natural frequencies, the total stiffness of the support is keep constant. From Table 2, it can be seen that the support stiffness does affect the natural frequency. When the support stiffness is small, different support lengths yield different frequencies, but when the stiffness and support length increase, the plate behaviour resembles that of a continuous plate. For the square plate with a central support, it can be seen from Table 3 that both support

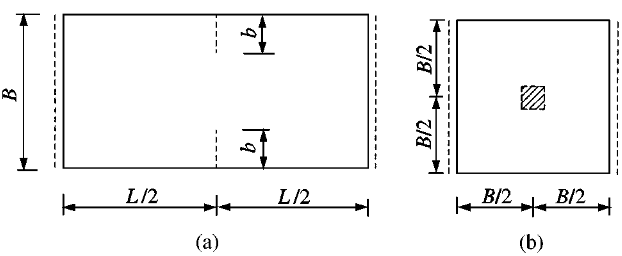

Figure 3. Plates with line and local distributed supports.

TABLE 2

Natural frequency parameters of plate on line elastic supports with different support stiffness and support dimensions

Support dimension b/B	Support stiffness k_f (kN/m^2)	Frequency parameters		
		λ_1	λ_2	λ_3
0·1	$1{\cdot}0 \times 10^7$	5·46199	9·71072	10·47248
	$1{\cdot}0 \times 10^8$	9·71072	10·37520	16·49109
	$1{\cdot}0 \times 10^9$	9·71072	14·66484	16·49108
	$1{\cdot}0 \times 10^{10}$	9·71072	16·49108	16·89209
0·2	$1{\cdot}0 \times 10^7$	7·13920	9·71072	11·94123
	$1{\cdot}0 \times 10^8$	9·71072	13·53014	16·49109
	$1{\cdot}0 \times 10^9$	9·71072	16·49109	21·80026
	$1{\cdot}0 \times 10^{10}$	9·71072	16·49109	29·24047
0·5	$1{\cdot}0 \times 10^7$	9·71072	10·49049	12·85587
	$1{\cdot}0 \times 10^8$	9·71072	16·49109	19·53419
	$1{\cdot}0 \times 10^9$	9·71072	16·49109	37·17847
	$1{\cdot}0 \times 10^{10}$	9·71072	16·49109	37·17847

TABLE 3

Natural frequency parameters of plate on local distributed elastic support at the centre point with different support stiffnesses and support areas

Total stiffness k_f (kN/m)	Distributed area A (m^2)	Frequency parameter		
		λ_1	λ_2	λ_3
10^7	Point	10·86777	16·49106	37·76607
	0·5 × 0·5	10·86767	16·49336	37·76220
	1·0 × 1·0	10·86551	16·50018	37·74962
	2·0 × 2·0	10·48820	16·52611	37·69785
10^8	Point	16·01589	16·49106	39·12869
	0·5 × 0·5	16·07790	16·51379	39·15430
	1·0 × 1·0	16·18525	16·58025	39·22887
	2·0 × 2·0	16·32791	16·82842	39.49974
10^9	Point	16·49112	21·01836	39·12869
	0·5 × 0·5	16·70649	21·29933	39·37427
	1·0 × 1·0	17·27542	21·73815	40·02445
	2·0 × 2·0	18·97154	22·47696	41·99695
10^{10}	Point	16·49112	21·98679	39·12869
	0·5 × 0·5	17·96738	22·69499	40·86395
	1·0 × 1·0	20·20092	23·25170	43·56065
	2·0 × 2·0	23·47683	24·42372	47·91951

stiffness and support area have an effect on the natural frequencies, with the stiffness having a greater influence. When the stiffness is relatively small, the support area affects the frequencies slightly, but when it is greater that 10^9, the natural frequencies are quite different for different support areas. This means that for this range of support stiffness, locally distributed supports cannot be replaced by point supports, but the true support area should be taken into account.

6. EFFECT OF (CENTRAL) COLUMN STIFFNESS

In order to investigate the effects of different types of support models and support areas, the square plate with four simply supported edges and a column support at the centre is considered, as shown in Figure 2(d). As mentioned in section 4, this plate structure had attracted the attention of many researchers who treated the column as either a rigid point support or an elastic point support. Herein the column is modelled as having four different types of support conditions and the effect of this on the free vibration characteristics is investigated. The four types of column support are: rigid point support, elastic point support without rotational stiffness (type A), elastic point support with rotational stiffness (type B) and uniformly distributed support (type C). In the numerical analysis, the dimensions and material properties of the plate are as follows: $B = 10$ m, $h = 0{\cdot}3$ m, $E = 31$ GPa, $\mu = 1/3$, $\rho = 2500$ kg/m^3. The property of column support is given by $E_c/H = 5{\cdot}0 \times 10^9$ N/m^3 which is typical for a concrete column and three different column sections are considered.

The first five natural frequency parameters are shown in Table 4, obtained by using 10 elements and 25 terms of the series, which ensured convergence. Generally, a rectangular plate has two axes of symmetry parallel to the edges. In the following discussion, symmetry and asymmetry of the modes of vibration are referred with respect to either set of axes. From this table, it can be seen that the support (model) type and column area have significant effects on the natural frequencies and mode shapes. When the column section is small ($A = 0{\cdot}5 \times 0{\cdot}5\,\mathrm{m}^2$), elastic point support models yield a symmetric–symmetric fundamental mode shape, while the second and third mode shapes are symmetric–asymmetric. The distributed support model, on the other hand, gives results which are quite different. Here, the first and second mode shapes are symmetric–asymmetric and the third mode is symmetric–symmetric. For larger areas of cross-section of the column ($A = 0{\cdot}8 \times 0{\cdot}8$ and $1{\cdot}0 \times 1{\cdot}0\,\mathrm{m}^2$), the first and second mode shapes are symmetric–asymmetric when the central support is modelled as a point support without rotational stiffness or as a distributed support. At these larger column areas when the support is modelled as a point support having rotational stiffness, the fundamental mode is symmetric–symmetric while the second and third modes are symmetric–asymmetric. Table 4 also shows that the rotational stiffness and column sectional area (and hence axial stiffness) have significant effects on the natural frequencies and associated mode shapes. When the column sectional area increases, the values of frequency increase, with the frequencies obtained from distributed elastic support model increasing more rapidly than those obtained from the other models. This means that the distributed model is more rigid than the others. An interesting feature is that both the models with a point support (either with or without rotational stiffness) give the same fourth and fifth frequencies and mode shapes. It seems that the rotational stiffness has no effect on these two mode shapes: asymmetric–asymmetric (fourth mode) and symmetric–symmetric (fifth

TABLE 4

Natural frequency parameters of a square plates with four simply supported edges and supported by a column at the centre point

Column section	Support model	Frequency parameters λ_1	λ_2	λ_3	λ_4	λ_5
0·5 × 0·5	A	46·872†	49·348‡	49·351‡	78·959§	98·711†
	B	46·872†	50·865‡	50·955‡	78·959§	98·711†
	C	49·842‡	49·853‡	50·926†	78·962§	102·715†
0·8 × 0·8	A	49·348‡	49·351‡	50·171†	78·959§	98·711†
	B	50·180†	53·171‡	53·225‡	78·959§	98·711†
	C	52·046‡	52·146‡	58·253†	79·002§	108·265†
1·0 × 1·0	A	49·348‡	49·351‡	51·033†	78·959§	98·711†
	B	51·033†	53·908‡	54·087‡	78·959§	98·711†
	C	54·666‡	55·002‡	61·600†	79·116§	112·618†
Rigid point support	Fan [20]	49·35‡	49·35‡	52·78†	78·96§	98·71†
	Kim [18]	49·348‡	49·348‡	53·170†	78·959§	98·696†
	Present	49·348‡	49·351‡	52·667†	78·959§	98·711†

Note: *Support models*:
A—point support without rotational stiffness; B—point support with rotational stiffness; C—local distributed support.
Mode shapes: (axes of symmetry parallel to the plate edges):
†Symmetric–symmetric
‡Symmetric–asymmetric (asymmetric–symmetric).
§Asymmetric–asymmetric.

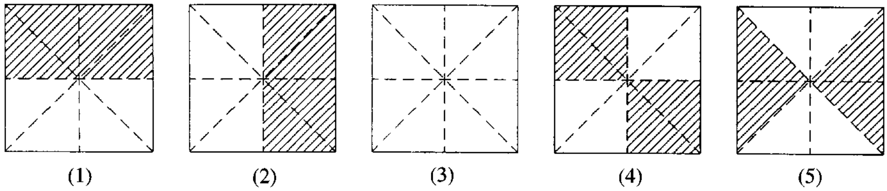

Figure 4. Mode shapes of a simply supported square plate with a rigid support at the centre point.

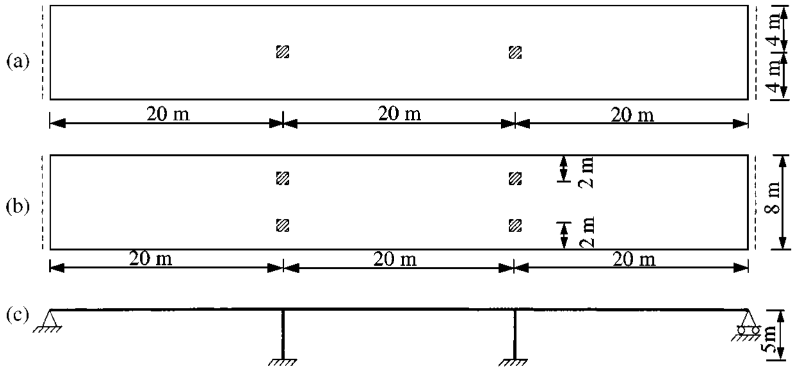

Figure 5. Highway bridges supported by columns.

mode), with respect to axes parallel to the plate edges. These shapes are shown in Figure 4 (fourth and fifth mode plates)

In the above discussion, symmetric and asymmetric mode shapes were with respect to the axes of symmetry parallel to the plate edges. But, in a square plate, there are four axes of symmetry: two are parallel to the plate edges and the other two are the diagonals. Figure 4 shows the mode shapes obtained for a plate with a rigid point support, and these modes are also common to the other support models, though in a different order. In Figure 4, the first two mode shapes are symmetric–asymmetric with respect to the axes of symmetry parallel to the plate edges. In the elastic support models, the rotational stiffness will affect these mode shapes, but axial stiffness will not. The third mode shape is symmetric with respect to all four axes of symmetry, and hence the axial stiffness has some effect on it, but not the rotational stiffness. The fourth mode shape is asymmetric–asymmetric with respect to the axes of symmetry parallel to the plate edges and symmetric–symmetric with respect to the diagonals. The fifth mode shape is exactly the opposite of the fourth. For these two mode shapes, with zero deflection and slope at the centre, the axial and rotational stiffnesses have no effect when point support models are used. But when a uniformly distributed support model is used, both axial stiffness and support area affect the frequencies and mode shapes.

7. FREE VIBRATION OF A HIGHWAY BRIDGE

A plate supported by columns is a structural form used in highway bridges. In the present work, two kinds of support forms are considered. In the first case, the bridge deck is supported by two single columns as shown in Figure 5(a) and in the other case there are two columns at each support location as shown in Figure 5(b). In both cases, the bridge deck (plate) and the supports (columns) are assumed to have the same properties ($E = 31$ GPa, $\mu = 0{\cdot}25$, $\rho = 2500\,\mathrm{kg/m^3}$). The dimensions of the structure are: $B = 8\,\mathrm{m}$, $L = 3 \times 20\,\mathrm{m}$, bridge thickness $h = 0{\cdot}3\,\mathrm{m}$, the height of columns is $H = 5\,\mathrm{m}$. Two column sections are considered: $0{\cdot}5 \times 0{\cdot}5$ and $1{\cdot}0 \times 1{\cdot}0\,\mathrm{m}^2$. In the numerical analysis, the bridges are divided into eight finite strips and 25 terms of the series are taken into account, to ensure convergence of results. In this problem the rotational stiffness of columns is not considered and only the axial stiffness is used. The stiffness of the point and patch (distributed) support stiffness are calculated by $k_f = EA/H$ and $k_f = E/H$ respectively.

The numerical results are shown in Table 5. It can be seen from this table that when the columns are modelled as point supports, the fundamental frequency is unaffected by different support forms (i.e., one or two columns at each support location) and stiffness. When the areas of the supports are taken into consideration, the fundamental frequencies are significantly affected by the column stiffness. The reason for this is that the distributed supports can restrict the deformation of plate in the supported areas and relatively restrict the rotation (deformation). On the other hand, point supports are able to restrict only

TABLE 5

Natural frequency parameters of bridge deck on elastic point and patch supports

Support cases	Column section	Total stiffness	Frequency parameters λ_1	λ_2	λ_3
Figure 5(a)	Point	$1{\cdot}55 \times 10^9$	1·53538	1·78457	1·97127
	Point	$6{\cdot}20 \times 10^9$	1·53538	1·78491	1·97372
	$0{\cdot}5 \times 0{\cdot}5$	$1{\cdot}55 \times 10^9$	1·64250	2·22460	3·83637
	$1{\cdot}0 \times 1{\cdot}0$	$6{\cdot}20 \times 10^9$	2·47137	4·64142	5·47462
Figure 5(b)	Point	$1{\cdot}55 \times 10^9$	1·53538	1·97622	2·89580
	Point	$6{\cdot}20 \times 10^9$	1·53538	1·97751	2·90086
	$0{\cdot}5 \times 0{\cdot}5$	$1{\cdot}55 \times 10^9$	1·73382	5·58618	6·40759
	$1{\cdot}0 \times 1{\cdot}0$	$6{\cdot}20 \times 10^9$	2·79143	5·78172	8·01763

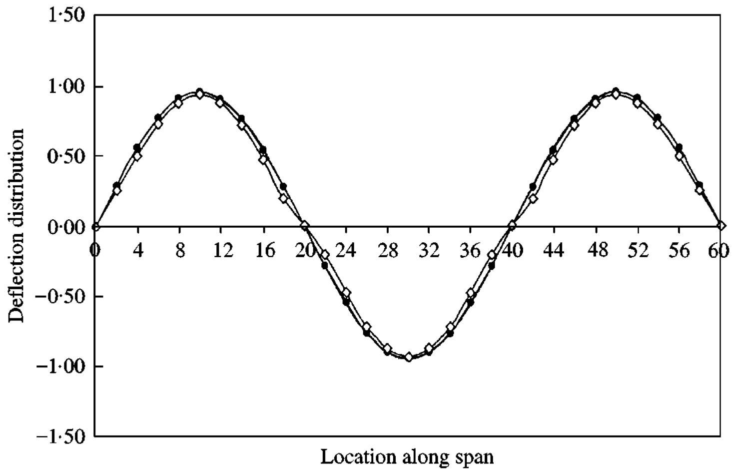

Figure 6. Deflection distribution of the first mode shape along bridge span at the supports line (Figure 5(a)): ——, $A = 0{\cdot}0$; —●—, $A = 0{\cdot}5 \times 0{\cdot}5$; —◇—, $A = 1{\cdot}0 \times 1{\cdot}0$.

the transverse deformation at that point, while the rotational deformation is free. Figure 6 shows the deflection distributions of first mode shape along the support line for the structure in Figure 5(a). It can be seen that there is a restriction in the deflection. From these results it is evident that for practical analysis, the concrete column support stiffness is just about the value of 10^9 kN/m^3 and for such a situation, the column supports cannot be modelled as point support for treating free vibration.

8. CONCLUSIONS

A procedure using the finite strip element method and a spring system has been proposed to treat the free vibration of plates resting on intermediate elastic supports. The spring system can be used to model point supports, line supports, locally distributed (patch) supports and complex boundary conditions. A comparison of the present results with those in the literature shows that this spring system can satisfactorily simulate the inner supports and mixed complex boundary conditions. Numerical analysis shows that the support stiffness and distributed support area have significant effects on the natural frequencies and associated mode shapes.

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