Journal of Sound and Vibration (2001) **240**(4), 667–678
doi:10.1006/jsvi.2000.3253, available online at http://www.idealibrary.com on IDEAL®

A SERIES EXPANSION APPROACH TO INTERPRETING THE SPECTRA OF THE TIMOSHENKO BEAM

S. Ekwaro-Osire, D. H. S. Maithripala and J. M. Berg

Department of Mechanical Engineering, Texas Tech University, Box 41021, *Lubbock TX* 79409-1021, *U.S.A.*

(*Received* 1 *June* 2000, *and in final form* 17 *July* 2000)

The Timoshenko beam model results in two fourth order partial differential equations in time and space. Consequently, solving the boundary value problem yields two independent sequences of natural frequencies and two corresponding sequences of mode shapes. A particular natural frequency and its corresponding mode shape describe one particular solution to the boundary value problem of the Timoshenko beam. From an eigenfunction expansion sense, all these possible solutions have to be considered in the complete series expansion of the solution. However, the question of whether these two independent sequences of natural frequencies, implies the existence of two distinct spectra of frequencies, has been a long standing topic of debate, and hitherto has not been resolved completely. The object of this paper is to provide answers to some of the issues raised by this debate. In this context, the complete solution in a series form to the Timoshenko beam is investigated, and it is shown for the first time that a particular mode shape of the solution is naturally expressed by an ordered pair of characteristic values, rather than a single characteristic value. This representation facilitates the progressive ordering of all the natural frequencies of the system and their respective mode shapes in a single set, and eliminates the remaining argument for the two spectra interpretation.

1. INTRODUCTION

The Timoshenko beam model is widely used, in the analysis of the transverse vibration of non-slender beams. It has been shown in the literature that the predictions of the Timoshenko beam model are in excellent agreement with the results obtained from the exact elasticity equations and experimental results [1–4]. In essence, the Timoshenko beam model adds rotary inertia and shear distortion effects to the Bernoulli–Euler model, which incorporates only the effects due to bending moment and lateral displacement. As a result of these terms in the Timoshenko model, a fourth order time differential term appears in the governing equations and causes a difficulty in deriving the solutions. Two main approaches exist in the literature for solving the Timoshenko beam equations. One is the Laplace transform method [5, 6], which results in an integral form solution, and the other is the method of series expansion [7]. This latter approach, also referred to as the eigenfunction expansion method, is usually preferred, since as pointed out by Anderson [8], the elements of the solution such as the mode shape and the natural frequency are thereby readily obtained.

Trail-Nash and Collar [4] carry out an exhaustive theoretical study of the uniform Timoshenko beam using the series expansion method. They point out that for natural

0022-460X/01/090667 + 12 $35.00/0

frequencies greater than a critical frequency, a change in the form of the mode shape occurs and interpret this change as a consequence of an existence of a second distinct spectra. Anderson, in an independent study [8], shows that since the Timoshenko beam equations are fourth order in both time and space, in general two sets of real frequencies and a corresponding mode shape for each frequency exist and hence the most general solution is a combination of them. Dolph [9] argued that for the hinged–hinged beam, two distinct natural frequencies correspond to the same spatial mode shape and that it constitutes the existence of a second distinct frequency spectra. However, Levinson and Cooke [10] observed, on the basis that a particular mode shape is the pair of functions describing the transverse deflection and the rotation of the cross-section, that even for the hinged–hinged beam the two frequencies from the apparent two distinct spectra will correspond to two distinct mode shapes and therefore will not imply the existence of two distinct spectra in an eigenvalue eigenfunction sense, especially since the natural frequencies form a single ordered set. Levinson and Cooke [10] further argued that for boundary conditions such as clamped–clamped or free–free, a simple uncoupling in terms of the spatial characteristic value parameters of the frequency equations is not possible and hence it might not even be relevant to consider the existence of two distinct spectra in such cases.

Downs [11] using a dynamic discretization method reported the existence of the two spectra, as well as the pure shear frequency, for a hinged–hinged beam. In this paper, a physical interpretation of the nature of the modes, particularly for those with no resultant transverse deflection, is also attempted. Abbas and Thomas [12] after a detailed analysis of the results obtained from a finite element model of the Timoshenko beam concluded that except for the hinged–hinged beam a distinct second spectrum of frequencies does not exist. However, contrary to this, Bhashyam and Prathap [13], also using a finite-element-based numerical analysis, argued that two distinct spectra exist even for boundary conditions other than hinged–hinged conditions. A letter by Prathap [14] in reply to the findings by Levinson and Cooke [10] contends that a simple ordering of the natural frequencies is insufficient and that a simultaneous ordering of their corresponding mode shapes is also required. In this context, he argues that the numerical evidence given in reference [13] shows that it is not possible to interpret the natural frequencies and their corresponding spatial eigenvalues as belonging to a single ordered set but to two independent sets, as would be in the case of two distinct spectra of natural frequencies. For the case of the hinged–hinged beam, and later more generally for other boundary conditions [13], Abbas and Thomas [12] also show that the lower set of frequencies are bounded below by the Bernoulli–Euler model and simple shear model frequencies, while the upper set of frequencies are bounded below by the pure shear model frequencies. It is also shown in references [4, 8, 12, 13] that this second set of frequencies tends to infinity as the rotary inertia and shear effects of the cross-section tend to zero. Further, Nesterenko [15] using an energy-based method argued that this second set of frequencies does not have a physical significance. Stephen [3], comparing these second set of frequencies to those values obtained from the exact elasticity equations, shows that this second set of frequencies could not be related to any particular solution obtained from the exact elasticity equations and hence argues that they do not have a physical significance.

The objective of this paper is to provide a critical analysis of the solution of Timoshenko beam equations using the classical technique of series expansion. The analysis will be performed as a whole, over the entire range of natural frequencies, without the need for considering two frequency ranges. Therefore, all the equations derived will be valid for the entire range of natural frequencies. The dependent nature of the spatial characteristic values will be investigated in section 3.2, and it will be shown for the first time that the frequency equations are a function of only one independent spatial characteristic value. Therefore, it will be shown that in fact there exist two frequency equations which yield, not only two single characteristic values for the system, but two pairs of them. The importance of considering the spatial characteristic values in pairs to represent a particular mode shape will be emphasized in section 3.3, whereby the issue of the two frequency spectra of the Timoshenko equation will be addressed. This will be considered in the context of whether or not two independent orderings of natural frequencies exist and it will in fact be shown that a single simultaneous ordering of the natural frequencies and their corresponding pairs of characteristic values is possible.

2. BEAM EQUATIONS

In deriving the beam equations, a set of assumptions are made, namely, small deflections, plane sections remain plane before and after deformation, Poisson effects on the beam are negligible, and material properties are linear elastic homogeneous and isotropic. The restriction on plane sections remaining plane before and after deformation, prompted Timoshenko to make certain corrections for the shear stresses acting on a cross-section. For a clear and concise study of the importance of this factor and for an approach of calculating this factor for various cross-sections the interested reader is referred to the paper by Cowper [16]. Under these assumptions, the Hamilton's principle for the system yields the following Timoshenko beam equations for free vibration:

$$EI\frac{\partial^2\theta}{\partial x^2}+k'GA\left(\frac{\partial y}{\partial x}-\theta\right)-\frac{I\rho}{g}\frac{\partial^2\theta}{\partial t^2}=0, \tag{1}$$

$$\frac{\rho A}{g}\frac{\partial^2 y}{\partial t^2}-k'GA\left(\frac{\partial^2 y}{\partial x^2}-\frac{\partial\theta}{\partial x}\right)=0, \tag{2}$$

where $y(x, t)$ is the total transverse deflection, $\theta(x, t)$ is the rotation of the cross-section due to bending, E is the modulus of elasticity, G is the modulus of rigidity, g is the gravitational constant, I is the area moment inertia of the cross-section, A is the cross-sectional area, ρ is the density, and k' is the shear correction factor. Furthermore, Hamilton's principle also yields the following four boundary conditions for the problem:

$$\frac{\partial\theta}{\partial x}\,\delta\theta\bigg|_0^L=0,\quad k'GA\left(\frac{\partial y}{\partial x}-\theta\right)\delta y\bigg|_0^L=0. \tag{3, 4}$$

Eliminating y or θ from equations (1) or (2), the following two uncoupled differential equations in y and θ are obtained by Huang [2]:

$$EI\frac{\partial^4 y}{\partial x^4}+\frac{\rho A}{g}\frac{\partial^2 y}{\partial t^2}-\left(\frac{\rho I}{g}+\frac{EI}{gk'}\frac{\rho}{G}\right)\frac{\partial^4 y}{\partial x^2\partial t^2}+\frac{\rho I}{g}\frac{\rho}{gk'G}\frac{\partial^4 y}{\partial^4 t}=0, \tag{5}$$

$$EI\frac{\partial^4 \theta}{\partial x^4}+\frac{\rho A}{g}\frac{\partial^2 \theta}{\partial t^2}-\left(\frac{\rho I}{g}+\frac{EI}{gk'}\frac{\rho}{G}\right)\frac{\partial^4 \theta}{\partial x^2\partial t^2}+\frac{\rho I}{g}\frac{\rho}{gk'G}\frac{\partial^4 \theta}{\partial^4 t}=0, \tag{6}$$

with coupled boundary conditions (3) and (4).

3. SOLUTION OF THE BEAM EQUATIONS

An analysis of the procedure of solving the Timoshenko beam equations given by equations (5) and (6) subjected to the boundary conditions (3) and (4) will be carried out in this section. The derivation of the characteristic equation and the general solution for the spatial variables will be presented in section 3.1, along with a description of the ideas that had led to the claim of the existence of two distinct frequency spectra. In section 3.2, the inherent-dependent relationship between the spatial characteristic values will be demonstrated for the first time while in section 3.3 the issue, on the interpretation of the existence or the non-existence of a second spectra would be dealt with, in an eigenvalue eigenfunction sense.

3.1. THE SPATIAL SOLUTION

Equations (5) and (6) are fourth order homogeneous linear PDE and since a solution harmonic in time is been sought, product solutions of the form

$$y(x,t) = A_y \mathrm{e}^{\mathrm{i}\omega t} Y(x), \quad \theta(x,t) = B_\theta \mathrm{e}^{\mathrm{i}\omega t} \Theta(x), \tag{7, 8}$$

are assumed. Substituting equations (7) and (8) into equations (5) and (6), respectively, the following two ODE for $Y(x)$ and $\Theta(x)$ are obtained:

$$EI\frac{\mathrm{d}^4 Y}{\mathrm{d}x^4} + \omega^2\left(\frac{\rho I}{g} + \frac{EI}{gk'}\frac{\rho}{G}\right)\frac{\mathrm{d}^2 Y}{\mathrm{d}x^2} - \omega^2\left(\frac{\rho A}{g} - \omega^2\frac{\rho I}{g}\frac{\rho}{gk'G}\right)Y = 0, \tag{9}$$

$$EI\frac{\mathrm{d}^4 \Theta}{\mathrm{d}x^4} + \omega^2\left(\frac{\rho I}{g} + \frac{EI}{gk'}\frac{\rho}{G}\right)\frac{\mathrm{d}^2 \Theta}{\mathrm{d}x^2} - \omega^2\left(\frac{\rho A}{g} - \omega^2\frac{\rho I}{g}\frac{\rho}{gk'G}\right)\Theta = 0. \tag{10}$$

This set of ODE can be transformed into the following non-dimensional form by substituting $\zeta = x/L$ and

$$\eta(\zeta) = \left\{\begin{matrix}\dfrac{\mathrm{Y}(\zeta)}{\mathrm{L}} \\ \Theta(\zeta)\end{matrix}\right\}, \quad \phi^2 = \omega^2\frac{\rho A L^4}{EIg}, \tag{11, 12}$$

into equations (9) and (10), yielding

$$\frac{\mathrm{d}^4\eta}{\mathrm{d}\zeta^4} + \phi^2\left(\frac{I}{AL^2} + \frac{I}{AL^2}\frac{E}{k'G}\right)\frac{\mathrm{d}^2\eta}{\mathrm{d}\zeta^2} - \phi^2\left[1 - \phi^2\left(\frac{I}{AL^2}\frac{I}{AL^2}\frac{E}{k'G}\right)\right]\eta = 0. \tag{13}$$

Here ϕ is a parameter proportional to the natural frequency, ω, of the system and is usually referred to as the natural frequency parameter of the system. Furthermore, using $\alpha = \mathrm{I}/AL^2$ and $\beta = (I/AL^2)\,E/k'G$, equation (13) reduces to

$$\frac{\mathrm{d}^4\eta}{\mathrm{d}\zeta^4} + \phi^2(\alpha+\beta)\frac{\mathrm{d}^2\eta}{\mathrm{d}\zeta^2} - \phi^2(1 - \phi^2\alpha\beta)\eta = 0. \tag{14}$$

Trail-Nash and Collar [4] pointed out that with $\alpha = 0$ equation (14) reduces to the simple shear model equation and with $\alpha = 0$ and $\beta = 0$ corresponds to the simple Bernoulli–Euller model equation.

Since equation (14) is a fourth order ODE, it has four exponential solutions of the form

$$\eta(\zeta) = \begin{Bmatrix} c \\ d \end{Bmatrix} e^{\lambda\zeta}, \tag{15}$$

where it has been shown in references [2, 8] that the relationship between the two arbitrary constants c and d can be deduced from equation (2) as

$$d = \frac{1}{\lambda L}(\phi^2\beta + \lambda^2)c. \tag{16}$$

Substituting equation (15) into equation (14) the characteristic equation of the system is obtained as

$$\lambda^4 + \phi^2(\alpha + \beta)\lambda^2 - \phi^2(1 - \alpha\beta\phi^2) = 0, \tag{17}$$

which is fourth order in both the spatial characteristic value λ and the natural frequency parameter ϕ. Hence, for a given particular natural frequency parameter ϕ_n, there exist four roots for λ, which can be expressed in the form $\lambda^2 = -p_n^2$ and $\lambda^2 = q_n^2$ and, thus, the general solution for equation (14) can be expressed as

$$\eta_n(\zeta) = \begin{Bmatrix} c_{n_1} \\ d_{n_1} \end{Bmatrix} \sin p_n\zeta + \begin{Bmatrix} c_{n_2} \\ d_{n_2} \end{Bmatrix} \cos p_n\zeta + \begin{Bmatrix} c_{n_3} \\ d_{n_3} \end{Bmatrix} \sinh q_n\zeta + \begin{Bmatrix} c_{n_4} \\ d_{n_4} \end{Bmatrix} \cosh q_n\zeta, \tag{18}$$

where c_{n_i} and d_{n_i} are arbitrary constants. From equation (16) it would be clear that only four of the arbitrary constants appearing in equation (18) are independent. The relationship between them is to be deduced from equation (16) and is given in Appendix A.

For a particular natural frequency parameter ϕ_n, the four solutions $\pm ip_n$ and $\pm q_n$ of the characteristic equation (17) are given by

$$2p_n^2 = \phi_n^2(\alpha + \beta) + [\phi_n^4(\alpha - \beta)^2 + 4\phi_n^2]^{1/2}, \tag{19}$$

$$2q_n^2 = -\phi_n^2(\alpha + \beta) + [\phi_n^4(\alpha - \beta)^2 + 4\phi_n^2]^{1/2}, \tag{20}$$

and it can be seen from equations (19) and (20) that

$$q_n^2 - p_n^2 = -\phi_n^2(\alpha + \beta), \quad p_n^2 q_n^2 = \phi_n^2(1 - \alpha\beta\phi_n^2). \tag{21, 22}$$

Further, it is evident from equation (19) that for all ϕ_n, p_n^2 is greater than zero. Hence, equation (22) implies that q_n^2 is positive for ϕ_n satisfying $\alpha\beta\phi_n^2 < 1$, and q_n^2 is negative for ϕ_n satisfying $\alpha\beta\phi_n^2 > 1$. Therefore, for conditions satisfying $\alpha\beta\phi_n^2 > 1$, q_n^2 can be expressed as $q_n^2 = -r_n^2$, where r_n is real and thus it can be deduced that, in this range of frequencies, equation (18) will contain only trigonometric terms. This change of form in equation (18) is what led Trail–Nash and Collar [4], and others as well, to speculate on the existence of a second spectrum where they interpreted it as a change in the mode shape of the solution. The frequency corresponding to this apparent mode change is given by

$$\alpha\beta\phi_c^2 = 1. \tag{23}$$

However, in contrast to the convention in the cited papers, and others, we feel that there is no necessity to consider two forms of mode shapes for the two frequency ranges. Rather it is sufficient to consider equation (18) as the sole form of the mode shape corresponding to all frequencies, where, when q_n is purely imaginary in equation (18), the second form of the mode shape, which contains only trigonometric terms, naturally results.

3.2. THE DEPENDENCY OF THE SPATIAL CHARACTERISTIC VALUES

The dependent nature of the spatial characteristic values will be presented in this section for the first time, where from equations (21) and (22) it can be inferred that the spatial characteristic values p_n and q_n are not independent, and that the dependencies can be expressed as

$$p_n^2 q_n^2 = \frac{p_n^2 - q_n^2}{\alpha + \beta}\left[1 - \alpha\beta\left(\frac{p_n^2 - q_n^2}{\alpha + \beta}\right)\right]. \tag{24}$$

For a particular value of p_n, equation (24) can be expressed as

$$\alpha\beta q_n^4 + [(\alpha^2 + \beta^2)p_n^2 + \alpha + \beta]q_n^2 + [\alpha\beta p_n^2 - (\alpha + \beta)]p_n^2 = 0, \tag{25}$$

which is a quadratic equation in q_n^2, depending only on p_n, and hence two solutions $q_{n_1}^2$ and $q_{n_2}^2$ will exist for q_n^2.

From the discriminant of equation (25) given by

$$\Delta = (\alpha^2 - \beta^2)^2 p_n^4 + 2(\alpha + \beta)^3 p_n^2 + (\alpha + \beta)^2 > 0, \tag{26}$$

it is evident that both the solutions $q_{n_1}^2$ and $q_{n_2}^2$ of equation (25) are real and that one of the two roots $q_{n_2}^2$, is less than zero for any value of p_n. However, it should be recalled that for the p_n that correspond to $\alpha\beta\phi_n^2 < 1$, the value of q_n^2 should be greater than zero. Owing to this contradiction, the solution of equation (25) that yields a negative value for q_n^2 for any p_n can be excluded from consideration. Thus, the value of $q_{p_n}^2$ for a particular p_n^2 can be uniquely determined by

$$2\alpha\beta q_{p_n}^2 = -[(\alpha^2 + \beta^2)p_n^2 + \alpha + \beta] + \sqrt{\Delta}. \tag{27}$$

In a similar fashion it also follows that given a purely imaginary q_n, there exists an unique $p_{q_n}^2$, that can be calculated using

$$2\alpha\beta p_{q_n}^2 = [-(\alpha^2 + \beta^2)q_n^2 + \alpha + \beta] + \sqrt{\Delta'}, \tag{28}$$

where

$$\Delta' = (\alpha^2 - \beta^2)^2 q_n^4 - 2(\alpha + \beta)^3 q_n^2 + (\alpha + \beta)^2 > 0. \tag{29}$$

Further note from equation (25), that $q_{p_n}^2$ will be less than zero for

$$p_n^2 > \frac{\alpha + \beta}{\alpha\beta} \overset{\text{def}}{=} p_c^2, \tag{30}$$

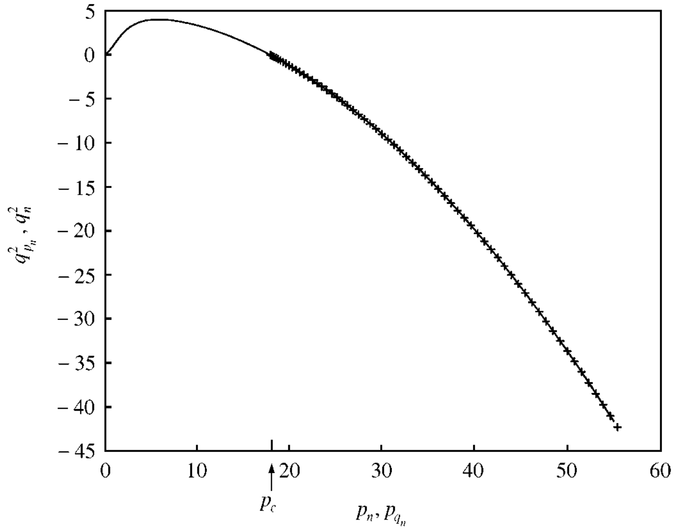

Figure 1. Dependency of the spatial characteristic values (cf. equations (28) and (27)): (——), $q_{p_n}^2$ versus p_n; (+), q_n^2 verses p_{q_n}.

and will be greater than zero for $p_n^2 < p_c^2$, where p_c is referred to as the critical spatial characteristic value of the beam. Following a similar argument it can also be shown that for positive q_n^2, p_n^2 will be less than p_c^2 and that for negative q_n^2, $p_{q_n}^2$ will be greater than p_c^2. Thus, it can be seen that $p_n^2 < p_c^2$ will imply $\alpha\beta\phi_n^2 < 1$ and that $p_n^2 > p_c^2$ will imply $\alpha\beta\phi_n^2 > 1$. By substituting equation (27) into equation (28) it can also be shown that if $q_n = q_{p_n}$ then $p_{q_n} = p_n$, provided that q_n is purely imaginary. This result, for the sake of clarity, is also shown in Figure 1, where for a particular p_n or p_{q_n}, that is greater than the critical characteristic parameter p_c, the corresponding q_{p_n} and q_n coincide.

Thus, as a result of the existence of these dual characteristic values, it will be incomplete to consider only p_n and q_n in isolation as the spatial characteristic values of the beam. Rather it will be preferable, to consider the ordered pairs $\langle p_n, q_{p_n}\rangle$ and $\langle p_{q_n}, q_n\rangle$ as the spatial characteristic values of the beam. In what follows, we shall see that, in fact, this gives rise to the desired ordering of the natural frequencies along with their corresponding mode shapes.

Recall that the solution to the governing ODE (14) for $\eta(\zeta)$ is given by equation (18). Four boundary conditions in $\eta(\zeta)$ are needed to determine the four independent arbitrary constants in equation (18). Substituting equation (18) into the four boundary conditions given by equations (3) and (4) results in four linear homogeneous equations in the four unknowns. The determinant of the coefficient matrix of the afore referred system of equations need to be zero, in order to obtain a non-trivial solution for the four independent arbitrary constants. Hence, setting the determinant to zero yields the frequency equation of the system, which from equation (18) is seen to contain both trigonometric terms in p_n and hyperbolic terms in q_n. However, as discussed in the previous paragraph, these p_n and q_n are not independent and thus it is more appropriate to consider two frequency equations; one in terms of p_n and q_{p_n},

$$f_1(p_n) = f(p_n, q_{p_n} \sin p_n, \cos p_n, \sinh q_{p_n}, \cosh q_{p_n}) = 0, \tag{31}$$

and the other in terms of q_n and p_{q_n},

$$f_2(q_n) = f(p_{q_n}, q_n, \sin p_{q_n}, \cos p_{q_n}, \sinh q_n, \cosh q_n) = 0. \tag{32}$$

The p_n and q_{p_n} appearing in equation (31) are related by equation (27), while p_{q_n} and q_n appearing in equation (32) are related by equation (28). Generally, an infinite sequence of real values of p_n satisfying equation (31) exists and for each of these p_n, the corresponding q_{p_n} will be given by equation (27). However, as q_n appears in the hyperbolic terms of the frequency equation (32), the sequence of values $\{q_n\}_{n=0}^{\infty}$ satisfying the frequency equation (32) will be purely imaginary and the corresponding p_{q_n} will be given by equation (28). These solutions to the two frequency equations in p_n and r_n where $q_n = \mathrm{i}r_n$ can be found using numerical non-linear equation solving methods. Therefore, the two frequency equations will yield two sequences of pairs of characteristic values $\{\langle p_n, q_{p_n}\rangle\}_{n=0}^{\infty}$ and $\{\langle p_{q_n}, q_n\rangle\}_{n=0}^{\infty}$. Note that the sequence of values in $\{p_{q_n}\}_{n=0}^{\infty}$ is all real while the first few values of q_{p_n} in $\{q_{p_n}\}_{n=0}^{\infty}$ corresponding to $p_n < p_c$ are real and the rest of the values, corresponding to $p_n > p_c$, are purely imaginary.

Note also that for a given set of boundary conditions, the critical spatial characteristic value p_c, given by equation (30), is most likely not in $\{p_n\}_{n=0}^{\infty}$ or $\{p_{q_n}\}_{n=0}^{\infty}$. If p_c is a value in the first sequence then it should be a solution of the first frequency equation (31) of the system and the corresponding q_{p_n} will be zero, while if it is a value in the second sequence then it should result as a consequence of $q_n = 0$ being a solution of the second frequency equation (32). The existence of this critical spatial characteristic value pair $\langle p_c, 0\rangle$, in either of the two ordered pairs of characteristic value sequences, will imply the existence of a mode of vibration, usually referred to as the pure shear mode of vibration. This mode of vibration exists only for special boundary conditions. Notably, the hinged–hinged beam admits the solution $q_n = \mathrm{i}r_n = 0$ in its second frequency equation given by

$$f_2(q_n) = \sin p_{q_n} \sin r_n = 0 \tag{33}$$

and hence gives rise to the pure shear mode of vibration. The natural frequency parameter, ϕ_c, corresponding to this mode is given by $\alpha\beta\phi_c^2 = 1$. This pure shear mode of vibration for the hinged–hinged beam was noted and discussed in references [4, 11, 12].

From the above discussion, it is clear that

$$\eta_{p_n}(\zeta) = C_{n_1} \sin p_n\zeta + C_{n_2} \cos p_n\zeta + C_{n_3} \sinh q_{p_n}\zeta + C_{n_4} \cosh q_{p_n}\zeta, \tag{34}$$

$$\eta_{q_n}(\zeta) = C'_{n_1} \sin p_{q_n}\zeta + C'_{n_2} \cos p_{q_n}\zeta + C'_{n_3} \sinh q_n\zeta + C'_{n_4} \cosh q_n\zeta \tag{35}$$

are both solutions of the ODE (14). Two distinct mode shapes, namely $\eta_{p_n}(\zeta)$ and $\eta_{q_n}(\zeta)$, are now seen to correspond to the two ordered pairs of characteristic values $\langle p_n, q_{p_n}\rangle$ and $\langle p_{q_n}, q_n\rangle$. Note that since q_n can be expressed as $q_n = \mathrm{i}r_n$, the mode shape (35) contains trigonometric terms only, where the last two hyperbolic terms of q_n in equation (35) are actually trigonometric terms in r_n. However, this form of representation will be maintained in order to highlight the fact that it is a pair of characteristic values that correspond to a particular mode shape and not p_n or q_n in an individual sense.

3.3. ON THE TWO FREQUENCY SPECTRA

In order to derive the complete solution to the Timoshenko beam equation it is necessary to determine the natural frequency parameter, ϕ_n, that corresponds to the mode shape $\eta_{p_n}(\zeta)$. Substituting $\lambda^2 = -p_n^2$ into equation (17), the following fourth order equation in ϕ_n is obtained:

$$\alpha\beta\phi_n^4 - (1 + (\alpha + \beta)p_n^2)\phi_n^2 + p_n^4 = 0. \tag{36}$$

It can be seen that since the characteristic values q_n are actually $\mathrm{i}r_n$, where r_n is real, for a particular q_n, the corresponding ϕ_n^2 will also be given by equation (36). Since equation (36) is quadratic in ϕ_n^2, two solutions exists for ϕ_n^2, and will be given by

$$2\alpha\beta\phi_{p_n}^2 = [1 + p_n^2(\alpha+\beta)] - [1 + p_n^4(\alpha-\beta)^2 + 2p_n^2(\alpha+\beta)]^{1/2}, \tag{37}$$

$$2\alpha\beta\phi_{q_n}^2 = [1 + p_n^2(\alpha+\beta)] + [1 + p_n^4(\alpha-\beta)^2 + 2p_n^2(\alpha+\beta)]^{1/2}. \tag{38}$$

Further, it can also be deduced from equation (36) that

$$\phi_{p_n}^2\phi_{q_n}^2 = \frac{\mathrm{p}_n^4}{\alpha\beta}. \tag{39}$$

Examining equations (37)–(39) it can be seen that both $\phi_{q_n}^2$ and $\phi_{p_n}^2$ are greater than zero and thus four solutions for ϕ_n, namely $\pm\phi_{p_n}$ and $\pm\phi_{q_n}$ exist. However, it is clear from equation (38) that $\alpha\beta\phi_{q_n}^2 > 1$ for all values of p_n. Therefore, the smaller frequency parameter ϕ_{p_n} will correspond to the spatial characteristic value pair $\langle p_n, q_{p_n}\rangle$. Since $q_n = \mathrm{i}r_n$ and the corresponding p_{q_n} was shown to be greater than p_c, which implies that $\alpha\beta\phi_n^2 > 1$, the larger frequency parameter ϕ_{q_n} will correspond to the spatial characteristic value pair $\langle p_{q_n}, q_n\rangle$, where ϕ_{q_n} is calculated from equation (39), with p_n replaced by r_n. Thus, the mode shape $\eta_{p_n}(\zeta)$ will correspond to ϕ_{p_n} and the mode shape $\eta_{q_n}(\zeta)$ will correspond to ϕ_{q_n}. This correspondence can be expressed as an ordered triple $\langle p_n, q_{p_n}, \phi_{p_n}\rangle$ and $\langle p_{q_n}, q_n, \phi_{q_n}\rangle$, where each triple represents a particular solution, or mode of vibration, of the Timoshenko beam.

It now follows that both

$$\eta_{p_n}(\zeta, t) = \eta_{p_n}(\zeta)(A'_{p_n}\mathrm{e}^{\mathrm{i}\omega_{pn}t} + A'_{q_n}\mathrm{e}^{-\mathrm{i}\omega_{pn}t}), \tag{40}$$

$$\eta_{q_n}(\zeta, t) = \eta_{q_n}(\zeta)(B'_{p_n}\mathrm{e}^{\mathrm{i}\omega_{qn}t} + B'_{q_n}\mathrm{e}^{-\mathrm{i}\omega_{qn}t}), \tag{41}$$

are solutions of the Timoshenko beam equations given by equations (1) and (2), where the corresponding natural frequencies, ω_{p_n} and ω_{q_n}, of the system are calculated by substituting the respective natural frequency parameters ϕ_{p_n} and ϕ_{q_n} in equation (12), and A'_{p_n}, A'_{q_n}, $B'_{p_n}$and B'_{q_n} are four arbitrary constants. Therefore, the two sequences of the ordered triple of values $\{\langle p_n, q_{p_n}, \phi_{p_n}\rangle\}_{n=0}^{\infty}$ and $\{\langle p_{q_n}, q_n, \phi_{q_n}\rangle\}_{n=0}^{\infty}$ will give all the possible solutions to the boundary value problem of the Timoshenko beam and thus the complete solution will be the series expansion given by

$$\eta(x, t) = \sum_{n=0}^{\infty}\left[\eta_{p_n}\left(\frac{x}{L}\right)(A_{n1}\sin\omega_{p_n}t + A_{n2}\cos\omega_{p_n}t) + \eta_{q_n}\left(\frac{x}{L}\right)(B_{n1}\sin\omega_{q_n}t + B_{n2}\cos\omega_{q_n}t)\right], \tag{42}$$

where it should be recalled that η_{p_n} and η_{q_n} correspond to the ordered pairs $\langle p_n, q_{p_n}\rangle$ and $\langle p_{q_n}, q_n\rangle$ respectively. A_{n_1}, A_{n_2}, B_{n_1} and B_{n_2} appearing in equation (42) are arbitrary constants. Four initial conditions on $\eta(x, t)$ are required to determine them and thereby completely solve the initial value boundary value problem of the Timoshenko beam.

Considering the two modes of vibration, corresponding to the triples, $\langle p_n, q_{p_n}, \phi_{p_n}\rangle$ and $\langle p_{q_n}, q_n, \phi_{q_n}\rangle$, it follows from equation (21) that

$$q_{p_n}^2 - p_n^2 = -\phi_{p_n}^2(\alpha+\beta), \quad q_n^2 - p_{q_n}^2 = -\phi_{q_n}^2(\alpha+\beta). \tag{43, 44}$$

TABLE 1

Progression of the ordered triple values for the hinged–hinged beam

p_m	q_m	ϕ_m
π	$0{\cdot}576\pi$	5·74
2π	$0{\cdot}638\pi$	13·31
3π	$0{\cdot}602\pi$	20·64
4π	$0{\cdot}516\pi$	27·86
5π	$0{\cdot}364\pi$	35·03
$5{\cdot}81\pi$	0	40·83†
6π	$0{\cdot}184\pi i$	42·16
7π	$0{\cdot}483\pi i$	49·29
8π	$0{\cdot}681\pi i$	56·40
9π	$0{\cdot}852\pi i$	63·50
$9{\cdot}94\pi$	πi	70·17†
10π	$1{\cdot}010\pi i$	70·60
⋮	⋮	⋮
17π	$1{\cdot}985\pi i$	120·23
$17{\cdot}11\pi$	$2\pi i$	121·03†
18π	$2{\cdot}117\pi i$	127·31
⋮	⋮	⋮
24π	$2{\cdot}895\pi i$	169·82
$24{\cdot}8\pi$	$3\pi i$	175·64†
25π	$3{\cdot}023\pi i$	176·90
⋮	⋮	⋮

†Values are from the sequence $\{\langle p_{q_n}, q_n, \phi_{q_n}\rangle\}_{n=0}^{\infty}$ and the rest are from the sequence $\{\langle p_n, q_{p_n}, \phi_{p_n}\rangle\}_{n=0}^{\infty}$.

It was pointed out in section 3.2 with the aid of Figure 1 that, if $p_n = p_{q_m}$, then $q_{p_n} = q_m$ and hence it is clear from equations (43) and (44) that $\phi_{p_n} = \phi_{q_m}$ for $p_n = p_{q_m}$. Therefore, for $p_n > p_c$, $p_n \leqslant p_{q_m}$ implies $q_{p_n} \leqslant q_m$ and $\phi_{p_n} \leqslant \phi_{q_m}$ for two adjacent values of p_n and p_{q_m}, giving rise to a simultaneous progressive ordering of the ordered triple of values. This is shown in Table 1 for the case of the hinged–hinged beam where the values marked by an "†" are those from the sequence $\{\langle p_{q_n}, q_n, \phi_{q_n}\rangle\}_{n=0}^{\infty}$ and the rest are from the sequence $\{\langle p_n, q_{p_n}, \phi_{p_n}\rangle\}_{n=0}^{\infty}$. Thus, it is concluded that the two sequences of the ordered triple of values, which give all the modes of vibration, can be combined as $\{\langle p_m, q_m, \phi_m\rangle\}_{m=0}^{\infty}$ to give a single progressively ordered set.

Levinson and Cooke [10] argue, for the case of the hinged–hinged boundary conditions only, that since all mode shapes are considered in forming the general series expansion (42), it might not be relevant to consider ω_{p_n} and ω_{q_n} as components of two distinct spectra, especially since they can be combined to form a single ordered set. Prathap [14] counters that it is not sufficient to obtain only a ordering of the natural frequencies but that a simultaneous ordering of their corresponding mode shapes was required as well. He further argues that this would be possible only if the two natural frequencies are considered to be from two distinct spectra. However, as pointed out at the end of the previous paragraph, a complete, single, progressive ordering of all the natural frequencies and their corresponding mode shapes into one single set results, when the ordered pairs of characteristic values $\langle p_n, q_{p_n}\rangle$ and $\langle p_{q_n}, q_n\rangle$ are considered collectively to be representative of the mode shapes η_{p_n} and η_{q_n}. It should be noted that the latter result is obtained in this paper irrespective of the boundary conditions and hence is general. Thus, we have obtained the ordering required by Prathap [14], and have thereby answered the last argument for considering the spectrum of solutions to have two distinct components.

4. NUMERICAL EXAMPLE

How a single progressive ordering of the mode shapes and their corresponding natural frequencies are facilitated by the natural representation of the mode shapes by a pair of characteristic values will be demonstrated in this section with the aid of the box beam used in reference [4]. The parameters of the beam result in $\alpha = 0{\cdot}19695$ and $\beta = 0{\cdot}00305$ which gives

$$\alpha + \beta = 0{\cdot}20000, \quad \alpha\beta = 0{\cdot}00060. \tag{45, 46}$$

Substituting equations (45) and (46) into equations (27) and (28), Figure 1 can be obtained. Figure 1 shows the dependency between the characteristic values p_n and q_{p_n}, and p_{q_n} and q_n. The solid line gives the relationship between $q_{p_n}^2$ and p_n given by equation (27) and the line marked with " + " gives the relationship between q_n^2 and p_{q_n} given by equation (28). This clearly shows that for $q_n^2 < 0$, if $p_n = p_{q_n}$ then $q_{p_n} = q_n$ and thus a simultaneous progressive ordering of the ordered pairs of characteristic values or in other words a simultaneous progressive ordering of the mode shapes is made possible. It should be noted here that these conditions are achieved irrespective of the boundary conditions and hence are general.

Next, the simultaneous ordering of the mode shapes and their corresponding natural frequencies, which was presented in section 3.3, will be demonstrated for the hinged–hinged boundary conditions.

4.1. HINGED–HINGED BOUNDARY CONDITION

The two frequency equations (31) and (32) for the hinged–hinged boundary conditions as given in reference [4] are

$$f_1(p_n) = \sin p_n \sinh q_{p_n} = 0, \quad f_2(r_n) = \sin p_{q_n} \sin r_n = 0. \tag{47, 48}$$

$p_n = n\pi$ for $n = 0, 1, 2, 3, \ldots$ is a solution of equation (47) and the corresponding q_{p_n} will be given by equation (27). Similarly, $r_n = n\pi$ for $n = 0, 1, 2, 3, \ldots$ is a solution of equation (48) and the corresponding p_{q_n} will be given by equation (28). The corresponding natural frequencies ϕ_{p_n} and ϕ_{q_n} are calculated from equations (37) and (38) respectively. Table 1 gives the solution pairs $\langle p_n, q_{p_n}\rangle$ of $f_1(p_n) = 0$ and $\langle p_{q_n}, q_n\rangle$ of $f_2(r_n) = 0$ and their corresponding natural frequencies. The values marked by an "†" are those from the sequence $\{\langle p_{q_n}, q_n, \phi_{q_n}\rangle\}_{n=0}^{\infty}$ and the rest are from the sequence $\{\langle p_n, q_{p_n}, \phi_{p_n}\rangle\}_{n=0}^{\infty}$. Thus, as concluded in section 3.3 it is seen that the two sequences of the ordered triple of values can be combined as $\{\langle p_m, q_m, \phi_m\rangle\}_{m=0}^{\infty}$ to give a single progressively ordered set.

5. CONCLUSION

The series expansion solution to the Timoshenko beam was investigated in this paper with the objective of providing answers to open questions concerning the nature of the spectrum of the Timoshenko beam model. In this context it was shown in section 3.2 that a particular mode shape of the solution can be expressed by an ordered pair of characteristic values, and that two independent sequences of such ordered pairs exists for a particular given beam conditions. Further, it was shown in section 3.3 that for each of these ordered pairs of characteristic values, there exists a corresponding unique natural frequency. Thus, if these pairs of characteristic values and the corresponding natural frequency were to be considered as an ordered triple, it would completely represent one particular solution of the

Timoshenko beam model. It was also shown that a progressive ordering of all the ordered triples into one single set is possible. This implies that there exists a single progressive ordering of all the natural frequencies and their corresponding modeshapes of the Timoshenko beam model and puts to rest the last remaining criticism of the single spectrum interpretation of the Timoshenko beam model structure.

REFERENCES

1. S. M. HAN, H. B. BENAROYA and T. WEI 1999 *Journal of Sound and Vibration* **225**, 935–988. Dynamics of transversely vibrating beams using four engineering theories.
2. T. C. HUANG 1961 *Journal of Applied Mechanics* **83**, 579–584. The effects of rotary inertia and of shear deformation on the frequency and normal mode equations of uniform beams with simple end conditions.
3. N. G. STEPHEN 1982 *Journal of Sound and Vibration* **80**, 578–582. The second frequency spectrum of Timoshenko beams.
4. R. W. TRAIL-NASH and A. R. COLLAR 1953 *Quarterly Journal of Mechanics and Applied Mathematics* **6**, 186–222. The effects of shear flexibility and rotary inertia on the bending vibration of beams.
5. B. A. BOLEY and C. C. CHAO 1955 *Journal of Applied Mechanics* **75**, 579–586. Some solutions of the Timoshenko beam equations.
6. N. ORTNER and P. WAGNER 1996 *Journal of Elasticity* **42**, 217–241. Solution of the initial-boundary value problem for the simply supported semi-infinite Timoshenko beam.
7. H. REISMANN and P. S. PAWLIK 1974 *Elastokinetics*. St. Paul, MN: West Publishing Co.
8. R. A. ANDERSON 1953 *Journal of Applied Mechanics* **75**, 504–510. Flexural vibrations in uniform beams according to the Timoshenko theory.
9. C. L. DOLPH 1954 *Quarterly of Applied Mathematics* **12**, 175–187. On the Timoshenko theory of transverse beam vibrations.
10. M. LEVINSON and D. W. COOKE 1982 *Journal of Sound and Vibration* **84**, 319–326. On the two frequency spectra of Timoshenko beams.
11. B. DOWNS 1976 *Journal of Applied Mechanics* **43**, 671–674. Transverse vibration of a uniform simply supported Timoshenko beam without transverse deflection.
12. B. A. H. ABBAS and J. THOMAS 1977 *Journal of Sound and Vibration* **51**, 123–137. The second frequency spectrum of Timoshenko beams.
13. G. R. BHASHYAM and G. PRATHAP 1981 *Journal of Sound and Vibration* **76**, 407–420. The second frequency spectrum of Timoshenko beams.
14. G. PRATHAP 1983 *Journal of Sound and Vibration* **90**, 443–446. The two frequency specrta of Timoshenko beams—a re-assessment.
15. V. V. NESTERENKO 1993 *Journal of Applied Mathematics and Mechanics* **57**, 669–677. A theory for transverse vibrations of the Timoshenko beam.
16. G. R. COWPER 1966 *Journal of Applied Mechanics* **88**, 335–340. The shear coefficient in Timoshenko's beam theory.

APPENDIX A: COEFFICIENTS

$$d_{n_1} = \frac{1}{p_n L}(\phi_n^2 \beta - p_n^2) c_{n_2}, \quad d_{n_2} = \frac{-1}{p_n L}(\phi_n^2 \beta - p_n^2) c_{n_1}, \qquad \text{(A.1, A.2)}$$

$$d_{n_3} = \frac{1}{q_n L}(\phi_n^2 \beta + q_n^2) c_{n_4}, \quad d_{n_4} = \frac{1}{q_n L}(\phi_n^2 \beta + q_n^2) c_{n_3}. \qquad \text{(A.3, A.4)}$$