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A FAMILY OF EXACT TRANSIENT SOLUTIONS FOR ACOUSTIC WAVE PROPAGATION IN INHOMOGENEOUS, NON-UNIFORM AREA DUCTS

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This paper presents a family of exact solutions for quasi-one-dimensional, transient acoustic wave propagation in ducts with mean temperature and area variations in the absence of mean flow. These solutions are obtained using a transformation of the spatial and acoustic variables in a manner suggested by the WKB approximation. Exact travelling wave-type solutions are obtained for a class of temperature and area profiles. These solutions differ from the classical travelling wave solution, however, in that the acoustic pressure and velocity are not algebraically related by the local value of the acoustic impedance, $\bar{\rho}(x)\bar{c}(x)$. Although these solutions resemble the approximate, "high frequency", WKB form of solution of the wave equation, they have the interesting property that they are exact, regardless of the scale of the acoustic disturbance relative to that of the inhomogeneity.

1. INTRODUCTION

This paper presents a family of exact solutions for acoustic wave propagation in ducts with mean temperature and cross-sectional area variations. Such solutions are of interest because of the wide variety of applications involving acoustic wave propagation in inhomogeneous or non-constant area ducts, e.g., in horn loudspeakers, automobile exhaust systems, or combustors.

Attention is restricted in this paper to quasi-one-dimensional acoustic wave propagation. Even in this greatly simplified situation, however, there does not currently exist an exact, general solution for the acoustic field. As such, theoretical investigations have primarily sought either exact solutions for ducts with prescribed temperature or area variations or approximate solutions. For example, the Kinsler *et al.* acoustic text [1] or the paper by Eisenberg and Kao [2] present several exact solutions for harmonic wave propagation in constant temperature ducts with certain area profiles. In a similar manner, Sujith *et al.* [3] and Kumar and Sujith [4] present exact solutions for inhomogeneous, constant area ducts with specified temperature profiles. In addition, a few solutions have been obtained incorporating mean flow [2] and dissipative [5] affects. Finally, Dokumaci [6] and Cummings [7] have developed approximate solutions using WKB-type methods that are

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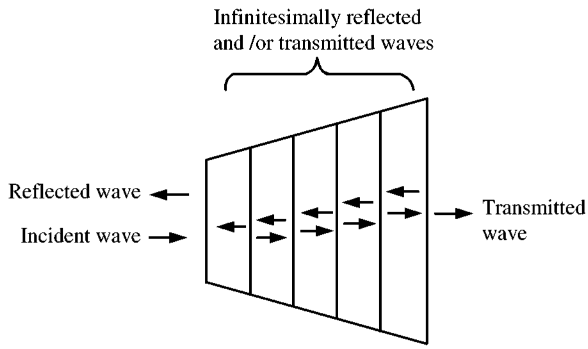

Figure 1. Simplified description of wave propagation in an inhomogeneous, non-constant area duct.

valid when the scale of the acoustic wavelength is small relative to that over which the area variation or inhomogeneity occurs [7, 8].

Essentially all of the available exact solutions of the wave equation in non-constant temperature or area ducts assume harmonic wave propagation, however. Relatively few solutions are known for general transient wave propagation. Of course, a formal transient solution of these frequency domain solutions could be obtained by determining their inverse Fourier transform. However, little insight into the characteristics of the wave field can be obtained from the resultant solutions, which generally contain integrals over an infinity of frequencies.

This paper presents a family of exact, explicit solutions for transient wave propagation in non-constant temperature and area ducts. It is organized in the following manner: the following section gives a brief background on wave propagation in inhomogeneous ducts and describes approximate WKB-type solutions of the wave equation. Then, using transformations suggested by the time domain analog of the approximate WKB solutions of the wave equation, the solutions section derives a family of exact travelling wave-type solutions for the acoustic pressure and velocity.

2. BACKGROUND

The acoustic field in a homogeneous, constant area duct can be described by the superposition of rightward and leftward propagating plane waves, $f(t - x/c)$ and $g(t + x/c)$ respectively. The functions f and g are arbitrary and are determined by the initial or boundary conditions. If these waves impinge on a discontinuous area or gas property change, they are partially reflected and partially transmitted. If the area or gas property change smoothly, the acoustic field can be considerably more complex. A convenient way of thinking about the resultant acoustic field is to consider it as the result of a large number of reflections and transmissions in a duct composed of a series of small discontinuities attached to each other; see Figure 1. As shown in the figure, a wave going through a smooth change in conditions undergoes a series of infinitesimal reflections and transmissions so that its final shape is distorted. Thus, a description of the acoustic field as a superposition of two plane waves independently propagating in opposite directions is, in general, not adequate, because a wave propagating to the left continuously excites “reflected” rightward propagating waves and *vice versa*, i.e., the two waves propagating in opposite directions are coupled [6].

Such an approximate description of the acoustic field is quite accurate, however, if the gas properties or duct area changes occur over scales that are long relative to that of the disturbance. In this case, the duct area or gas properties "look" uniform to the wave, so that reflections are negligible. Neglecting these reflections altogether is the essence of "high-frequency" ray or WKB-type approximations [8]. In a one-dimensional context, these approximations describe the wave field as independently propagating waves whose local amplitudes vary in order to conserve the wave's energy flux. For example, since the energy flux in a travelling wave is given by $I(x,t) = p'(x,t)u'(x,t)A(x)$, the approximate WKB solution shows that the amplitude of a rightward travelling wave of the form $f(t - x/\bar{c})$ is progressively rescaled in the following manner:

$$p'(x,t) \propto \frac{f(t - x/\bar{c})}{A^{1/2}(x)}, \qquad u'(x,t) \propto \frac{1}{\bar{\rho}\bar{c}}\frac{f(t - x/\bar{c})}{A^{1/2}(x)}. \tag{1}$$

Equation (1) can be generalized to situations where the mean density, $\bar{\rho}(x)$, and/or speed of sound, $\bar{c}(x)$, of the medium also change slowly relative to the scale of the acoustic disturbance. Assuming that the acoustic pressure and velocity are related by the acoustic impedance, $\bar{\rho}(x)\bar{c}(x)$ (as they are in plane travelling waves), the disturbance evolves as

$$p'(x,t) \propto \left(\frac{\bar{\rho}(x)\bar{c}(x)}{A(x)}\right)^{1/2} f\left(t - \int_0^x \frac{\mathrm{d}\xi}{\bar{c}(\xi)}\right), \qquad u'(x,t) \propto \left(\frac{\bar{\rho}(x)\bar{c}(x)}{A(x)}\right)^{1/2} \frac{f(t - \int_0^x \mathrm{d}\xi/\bar{c}(\xi))}{\bar{\rho}(x)\bar{c}(x)}. \tag{2}$$

Note that for a perfect gas:

$$(\rho(x)c(x))^{1/2} = \left(\frac{\gamma\bar{p}^2}{R\bar{T}}\right)^{1/4}. \tag{3}$$

Assuming that there is no ambient flow, the mean pressure, $\bar{p}$, is constant. Assuming for simplicity that γ and R are constants as well, leads to the following modified form of equation (2) that relates the amplitude of the acoustic pressure or velocity to the local area or temperature:

$$p'(x,t) \propto \frac{f(t - \int_0^x \mathrm{d}\xi/\bar{c}(\xi))}{A(x)^{1/2}\bar{T}^{1/4}(x)}, \qquad u'(x,t) \propto \bar{T}^{1/4}(x)\frac{f(t - \int_0^x \mathrm{d}\xi/\bar{c}(\xi))}{A^{1/2}(x)}. \tag{4}$$

It should be emphasized that equations (1, 2) and (4) are approximate solutions that are valid only when the length scale of the acoustic disturbance is small relative to that of the gas property or area change. It will be shown in the next section, however, that exact solutions of the form shown in equation (4) exist for a family of temperature and area profiles.

3. A FAMILY OF EXACT SOLUTIONS

For completeness, we begin with the derivation of the wave equation for a variable area duct with a mean temperature gradient and negligible mean flow. Assuming a perfect, inviscid and non-heat conducting gas, the quasi-one-dimensional continuity, momentum, energy and state equations can be written as [9]

$$\text{continuity:} \quad A\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u A)}{\partial x} = 0, \tag{5}$$

$$\text{momentum:} \quad \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right] = -\frac{\partial p}{\partial x}, \tag{6}$$

$$\text{energy:} \quad A\frac{\partial p}{\partial t} + Au\frac{\partial p}{\partial x} + \gamma p\frac{\partial (Au)}{\partial x} = 0. \tag{7}$$

These quasi-one-dimensional equations approximately describe wave propagation in ducts where area variations occur over length scales that are long relative to that of the acoustic disturbance. Writing each dependent variable as the sum of a steady and time-dependent component, i.e., $u = u'(x, t)$, $p = \bar{p}(x) + p'(x, t)$, $\rho = \bar{\rho}(x) + \rho'(x, t)$, and substituting these expressions into equations (5–7) yields a system of equations for the steady and unsteady variables. The resulting linearized acoustic momentum and energy equations are

$$\bar{\rho}\frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial x}, \qquad \frac{A}{\gamma\bar{p}}\frac{\partial p'}{\partial t} + \frac{\partial (u'A)}{\partial x} = 0. \tag{8, 9}$$

Utilizing the perfect gas relation,

$$\frac{1}{\bar{\rho}}\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}x} + \frac{1}{\bar{T}}\frac{\mathrm{d}\bar{T}}{\mathrm{d}x} = 0,$$

and combining equations (8) and (9), yields the following wave equations for the acoustic velocity and pressure:

$$\frac{\partial^2 p'}{\partial x^2} + \left[\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x} + \frac{1}{\bar{T}}\frac{\mathrm{d}\bar{T}}{\mathrm{d}x}\right]\frac{\partial p'}{\partial x} - \frac{1}{\bar{c}^2}\frac{\partial^2 p'}{\partial t^2} = 0, \tag{10}$$

$$\frac{\partial^2 u'}{\partial x^2} + \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x}\frac{\partial u'}{\partial x} + u'\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x}\right] - \frac{1}{\bar{c}^2}\frac{\partial^2 u'}{\partial t^2} = 0, \tag{11}$$

where $\bar{c} = (\gamma R\bar{T})^{1/2}$.

The ensuing analysis determines the conditions under which equations (10) and (11) admit exact travelling wave solutions.

3.1. EXACT TRAVELLING WAVE SOLUTIONS FOR THE ACOUSTIC PRESSURE

First, we determine the conditions under which the wave equation admits exact travelling wave solutions for the acoustic pressure. Motivated by the discussion in the background section and the form of equation (4), we introduce the transformations

$$\tilde{x} = \int_0^x \frac{\mathrm{d}\xi}{\bar{c}(\xi)}, \qquad p'(x, t) = \Phi_p(A(x), \bar{T}(x))\,\tilde{p}'(\tilde{x}, t). \tag{12, 13}$$

Note that the transformed spatial variable, $\tilde{x}$, is the transit time for a disturbance to propagate through a distance x. The acoustic pressure is scaled by a function of the local area and temperature, $\Phi_p(A(x), T(x)) = \Phi_p(x)$. Note that the "high-frequency" solutions in Equation (4) suggest defining this function as $\Phi_p(x) = A^{-1/2}(x)T^{-1/4}(x)$. While the analysis below will show that these area and temperature profiles are the only ones that admit travelling wave solutions, the more general form in equation (13) will be retained initially.

Substituting equations (12) and (13) into the wave equation for the pressure, equation (10), yields the transformed wave equation

$$\left[\frac{\partial^2\tilde{p}'}{\partial\tilde{x}^2}-\frac{\partial^2\tilde{p}'}{\partial t^2}\right]\frac{\Phi_p}{\gamma R\bar{T}}+\frac{\partial\tilde{p}'}{\partial\tilde{x}}\frac{1}{c}\left[2\frac{\mathrm{d}\Phi_p}{\mathrm{d}x}+\frac{1}{2\bar{T}}\frac{\mathrm{d}\bar{T}}{\mathrm{d}x}\Phi_p+\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x}\Phi_p\right]$$
$$+\tilde{p}'\left[\frac{\mathrm{d}^2\Phi_p}{\mathrm{d}x^2}+\left[\frac{1}{\bar{T}}\frac{\mathrm{d}\bar{T}}{\mathrm{d}x}+\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x}\right]\frac{\mathrm{d}\Phi_p}{\mathrm{d}x}\right]=0. \tag{14}$$

This equation reduces to the classical wave equation, $\partial^2\tilde{p}'/\partial\tilde{x}^2-\partial^2\tilde{p}'/\partial t^2=0$ (that admits travelling wave solutions of the form $\tilde{p}'=f(t-\tilde{x})+g(t+\tilde{x})$) when the following relations hold:

$$\frac{\mathrm{d}^2\Phi_p}{\mathrm{d}x^2}+\frac{1}{A(x)\bar{T}(x)}\frac{\mathrm{d}(A(x)\bar{T}(x))}{\mathrm{d}x}\frac{\mathrm{d}\Phi_p}{\mathrm{d}x}=0, \tag{15}$$

$$2\frac{\mathrm{d}\Phi_p}{\mathrm{d}x}+\left(\frac{1}{A(x)}\frac{\mathrm{d}A(x)}{\mathrm{d}x}+\frac{1}{2T(x)}\frac{\mathrm{d}\bar{T}(x)}{\mathrm{d}x}\right)\Phi_p=0. \tag{16}$$

Integrating equation (16) yields

$$\Phi_p(x)=\frac{constant}{A^{1/2}(x)\bar{T}^{1/4}(x)}. \tag{17}$$

Equation (17) explicitly shows that the only solutions for the area and temperature profiles are those previously given in equation (4) for the "high-frequency" approximation. However, not every temperature and area profile admits travelling wave solutions because they must also satisfy equation (15). Integrating this equation once and substituting into equation (17) yields the relation $(\mathrm{d}\Phi_p/\mathrm{d}x)\,A(x)\bar{T}(x)=constant$ which can be written as

$$\frac{\mathrm{d}(A^{-1/2}(x)\bar{T}^{-1/4}(x))}{\mathrm{d}x}A(x)\bar{T}(x)=constant. \tag{18}$$

Because equation (18) is a single differential equation for two quantities, $A(x)$ and $T(x)$, there exists an infinite number of temperature and area profiles that satisfy it. For example, travelling wave solutions of the wave equation exist for any arbitrary temperature profile, provided the area profile satisfies equation (18), i.e., solving equation (18) for $A(x)$, given an arbitrary $T(x)$, yields

$$A(x)=\frac{(C_2-C_1\int_0^x \mathrm{d}x'/\sqrt{T(x')})^2}{\sqrt{T(x)}}, \tag{19}$$

where C_1 and C_2 are arbitrary constants. In the same manner, solving equation (18) for $T(x)$, given an arbitrary $A(x)$, yields:

$$T(x)=\frac{(C_2+C_1\int_0^x A(x')\,\mathrm{d}x')^{4/3}}{A^2(x)}. \tag{20}$$

In the special case where the area is constant, the solution of equation (18) for $T(x)$ is given by

$$\frac{\bar{T}(x)}{T_0}=(1+a_px)^{n_p}, \tag{21}$$

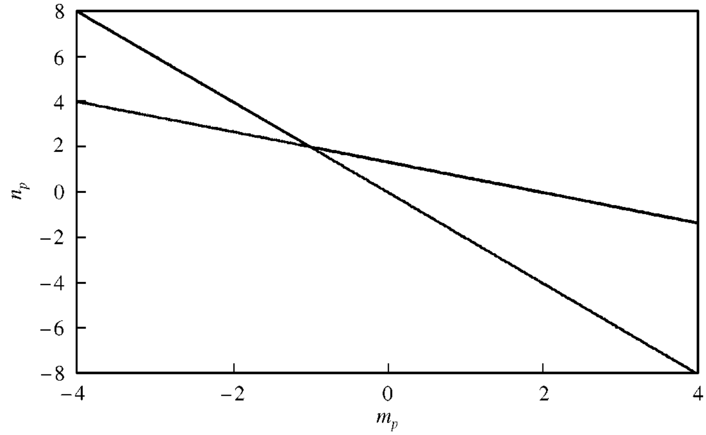

Figure 2. Two possible solutions for the relationship between the temperature (n_p) and area (m_p) variation exponents given by equation (21).

where $n_p = 4/3$ and a_p is an arbitrary constant. In a constant temperature duct, the analogous solution for the area is

$$\frac{A(x)}{A_0} = (1 + a_p x)^{m_p}, \tag{22}$$

where $m_p = 0$ or 2. Note that the latter case corresponds to wave propagation in a conical duct and thus represents a solution equivalent to spherical wave propagation. Thus, equation (22) shows that only uniform and conical ducts admit non-dispersive travelling wave solutions for the acoustic pressure.

A similar family of exponential profiles exists in the more general case where both temperature and area vary simultaneously. It can be verified by direct substitution that equations (21) and (22) exactly satisfy equation (18) when n_p and m_p are related by either of the relationships

$$n_p = -\,2m_p \quad \text{or} \quad n_p = -\,(2m_p - 4)/3. \tag{23}$$

Figure 2 plots the relationship between the exponents m_p and n_p in equation (23). Note that two possible solutions exist for m_p given an arbitrary n_p (and *vice versa*). Note also that n_p and m_p are inversely related in both solutions, i.e., if the temperature increases with x, then the area must decrease, and *vice versa*. This trend can be understood by energy conservation considerations and equation (17).

Thus, for the temperature and area profiles given by any of equations (19–23), the exact solution of the wave equation (10) is described by

$$p'(x,t) = \frac{f(t - \int_0^x \mathrm{d}\xi/\bar{c}(\xi)) + g(t + \int_0^x \mathrm{d}\xi/\bar{c}(\xi))}{A^{1/2}(x)\,\bar{T}^{1/4}(x)}. \tag{24}$$

Substituting equation (24) into the linearized momentum equation (8) yields the following solution for the corresponding acoustic velocity field:

$$u'(x,t) = \frac{1}{\bar{\rho}(x)\bar{c}(x)}\left[\frac{(f(\zeta) - g(\eta))}{A^{1/2}(x)\bar{T}^{1/4}(x)} - \bar{c}(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{A^{1/2}(x)T^{1/4}(x)}\right)\int (f(\zeta)\,\mathrm{d}\zeta + g(\eta)\,\mathrm{d}\eta)\right], \tag{25}$$

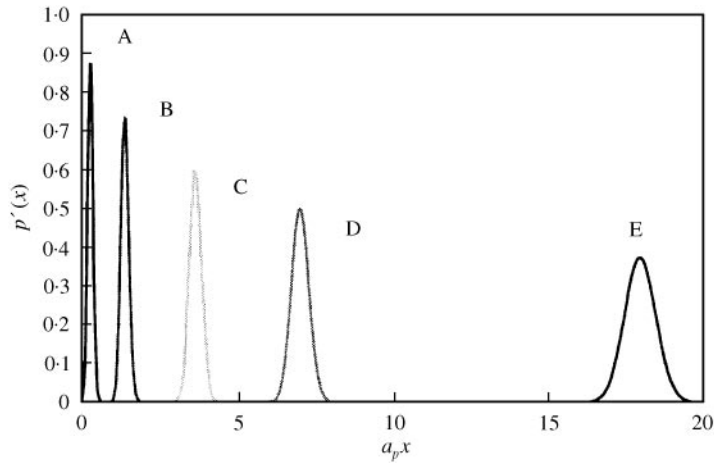

Figure 3. Evolution of a Gaussian pulse through a non-uniform temperature, constant area duct. Curves denoted A–E denote waveform at times $t = 0{\cdot}25/a_p/c_0$, $1/a_p/c_0$, $2/a_p/c_0$, $3/a_p/c_0$, and $5/a_p/c_0$ respectively ($a_p c_0 \tau = 0{\cdot}1$, $m_p = 0$, $n_p = 4/3$).

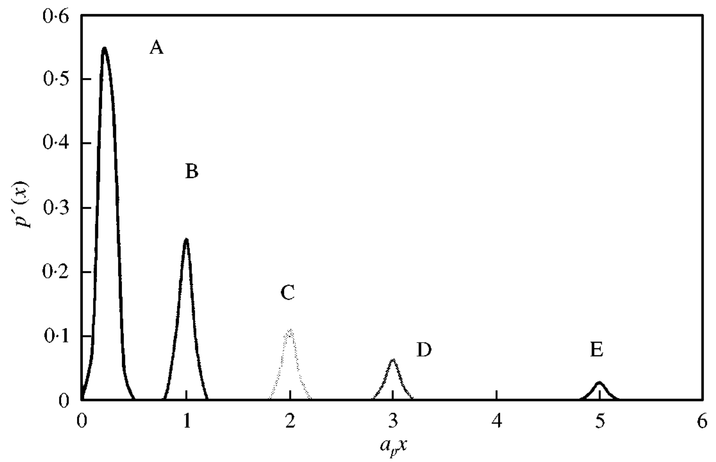

Figure 4. Evolution of a Gaussian pulse through a non-uniform area, constant temperature duct. Curves denoted A–E denote waveform at times $t = 0{\cdot}25/a_p/c_0$, $1/a_p/c_0$, $2/a_p/c_0$, $3/a_p/c_0$, and $5/a_p/c_0$ respectively ($a_p c_0 \tau = 0{\cdot}1$, $m_p = 2$, $n_p = 0$).

where

$$\zeta = t - \int_0^x \frac{d\xi}{\bar{c}(\xi)} \quad \text{and} \quad \eta = t + \int_0^x \frac{d\xi}{\bar{c}(\xi)}. \tag{26}$$

It is important to note that equation (25) does not have the same form as the "high-frequency" travelling wave solution in equation (4) because the pressure and velocity in each wave are not related algebraically by the local acoustic impedance, $\bar{\rho}(x)\bar{c}(x)$.

The results of sample calculations illustrating the evolution of a pulse through non-uniform temperature and area ducts are shown in Figures 3 and 4. These figures plot the evolution of a rightward propagating pulse whose initial shape is described by $f(t) = \exp(-(t/\tau)^2)$ at $x = 0$ through a constant area, varying temperature (solution given by equations (21) and (24)) and constant temperature, varying area duct (solution given by equations (22) and (24)). Figure 3 illustrates that temperature variations exert two effects upon the wave. First, the wave shape changes because different parts of the wave propagate at different speeds. Second, the wave amplitude decreases because of the change in impedance. In contrast, Figure 4 shows that in a constant temperature, varying area duct, the wave shape remains the same, although the amplitude steadily decreases in increasing direction of propagation because of the increasing duct area.

It will be shown next that an analogous family of exact travelling wave solutions also exists for the acoustic velocity.

3.2. EXACT TRAVELLING WAVE SOLUTIONS FOR THE ACOUSTIC VELOCITY

An additional family of exact solutions can be obtained by transforming the acoustic velocity in an analogous manner as done in equation (13):

$$u'(x, t) = \Phi_u(A(x), T(x))\, \tilde{u}'(\tilde{x}, t). \tag{27}$$

Substituting equations (12) and (27) into the wave equation for acoustic velocity, i.e. equation (11), yields

$$\frac{\Phi_u}{\bar{c}^2}\left[\frac{\partial^2 \tilde{u}'}{\partial \tilde{x}^2} - \frac{\partial^2 \tilde{u}'}{\partial t^2}\right] + \frac{1}{\bar{c}}\frac{\partial \tilde{u}'}{\partial \tilde{x}}\left[2\frac{\mathrm{d}\Phi_u}{\mathrm{d}x} + \left[\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x} - \frac{1}{2\bar{T}}\frac{\mathrm{d}\bar{T}}{\mathrm{d}x}\right]\Phi_u\right]$$
$$+ \tilde{u}'\left[\frac{\mathrm{d}^2\Phi_u}{\mathrm{d}x^2} + \frac{\mathrm{d}}{\mathrm{d}x}\left[\Phi_u \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x}\right]\right] = 0. \tag{28}$$

This equation reduces to the classical wave equation, $\partial^2\tilde{u}'/\partial\tilde{x}^2 - \partial^2\tilde{u}'/\partial t^2 = 0$, when the following relations hold:

$$\Phi_u(x) = \frac{constant * \bar{T}^{1/4}(x)}{A^{1/2}(x)}, \qquad \frac{\mathrm{d}\Phi_u}{\mathrm{d}x} + \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x}\Phi_u = constant. \tag{29, 30}$$

In analogy to equation (18), there exists an infinite number of temperature and area profiles satisfying equation (30). Solving equation (30) for $A(x)$, given an arbitrary $T(x)$, yields

$$A(x) = \frac{1}{\sqrt{T(x)}}\frac{1}{(C_2 - C_1\int_0^x (\mathrm{d}x'/\sqrt{T(x')}))^2}, \tag{31}$$

where C_1 and C_2 are arbitrary constants. In the same manner, solving for $T(x)$, given an arbitrary $A(x)$, yields

$$T(x) = \frac{(C_2 + C_1\int_0^x A(x')\,\mathrm{d}x')^4}{A^2(x)}. \tag{32}$$

In the special case where the area is constant, the unique solution of equation (30) for $T(x)$ is given by

$$\frac{\bar{T}(x)}{T_0} = (1 + a_u x)^{n_u}, \tag{33}$$

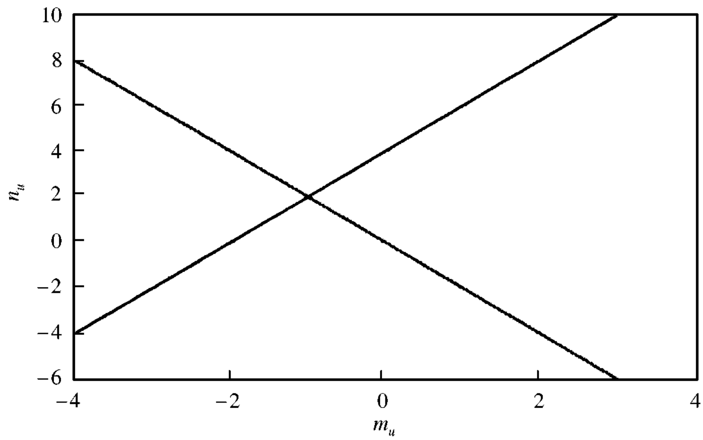

Figure 5. Two possible solutions for the relationship between the temperature (n_u) and area (m_u) variation exponents given by equation (31).

where $n_u = 0$ or 4 and a_u is an arbitrary constant. In a constant temperature duct, the analogous solution for the area is

$$\frac{A(x)}{A_o} = (1 + a_u x)^{m_u}, \tag{34}$$

where $m_u = 0$ or -2.

Analogously to equation (23), equations (33) or (34) more generally satisfy equation (30) when the exponents m_u and n_u are related by either of the expressions

$$n_u = -2m_u \quad \text{or} \quad n_u = 2m_u + 4. \tag{35}$$

The relationship between the exponents m_u and n_u in equation (31) is presented graphically in Figure 5. Similar to the relationships illustrated in Figure 3, two possible solutions exist for m_u given an arbitrary n_u (and *vice versa*). In contrast to Figure 3, however, note that these two solutions exhibit opposite trends. That is, for the $n_u = -2m_u$ solution, an increase in temperature is accompanied by a decrease in area, while the opposite trend is exhibited by the $n_u = 2m_u + 4$ solution.

For the temperature and area profiles given by equations (31–35), the acoustic velocity is given by

$$u'(x, t) = \frac{\bar{T}^{1/4}(x)}{A^{1/2}(x)} f\left(t - \int_0^x \frac{\mathrm{d}\xi}{\bar{c}(\xi)}\right) - g\left(t + \int_0^x \frac{\mathrm{d}\xi}{\bar{c}(\xi)}\right). \tag{36}$$

Substituting equation (36) into the acoustic energy equation (9) yields the following solution for the corresponding acoustic pressure field:

$$p' = \bar{\rho}(x)\bar{c}(x)\left[\frac{\bar{T}^{1/4}}{A^{1/2}}(f(\zeta) + g(\eta)) - \frac{\bar{c}}{A}\frac{\mathrm{d}}{\mathrm{d}x}(A^{1/2}\bar{T}^{1/4})\int(f(\zeta)\,\mathrm{d}\zeta - g(\eta)\,\mathrm{d}\eta)\right], \tag{37}$$

where ζ and η are given in equation (26). Again, note that the pressure and velocity in each wave are not related algebraically by the acoustic impedance, $\bar{\rho}(x)\bar{c}(x)$.

4. CONCLUSIONS

This paper presents a family of exact travelling wave-type solutions in ducts with mean temperature and area variations. These solutions resemble the approximate WKB solutions of the wave equation in that they are essentially travelling waves whose shapes are distorted by sound speed variations (because different parts of the wave travel at different speeds) and whose amplitudes are scaled by the local duct area and acoustic impedance. These solutions have the interesting property that they resemble the WKB solutions, but exactly satisfy the wave equation regardless of the scale of the acoustic disturbance relative to the scale of the inhomogeneity. These solutions differ from the approximate WKB solutions, however, in that the acoustic pressure and velocity in the travelling waves are not algebraically related by the local value of the acoustic impedance, $\bar{\rho}(x)\bar{c}(x)$.

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APPENDIX A: NOMENCLATURE

A	duct cross-sectional area
c	speed of sound
f	travelling wave propagating in $+x$ direction
g	travelling wave propagating in $-x$ direction
m	area variation exponent, see equations (22) and (34)
n	temperature variation exponent, see equations (21) and (33)
p	pressure
R	gas constant
t	time
T	temperature
u	velocity
x	spatial co-ordinate
γ	ratio of specific heats
η	retarded time
ρ	density
Φ	temperature and area variation function, see equations (13) and (27)

ξ	integration variable for axial distance
ζ	retarded time

Subscripts and superscripts

$(\ )'$	fluctuating quantity
$(\bar{\ })$	mean quantity