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INDUCED DAMPING BY A *NEARLY* CONTINUOUS DISTRIBUTION OF *NEARLY* UNDAMPED OSCILLATORS: LINEAR ANALYSIS

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The induced damping in a master oscillator contributed by a set of satellite oscillators is obtained in terms of a summation over a discrete distribution of the set. (The distribution is with respect to the resonance frequencies of the satellite oscillators in the set. The distribution is cast in an ascending order and is assumed to be centered about the resonance frequency of the master oscillator in isolation). If the modal overlap parameters are less than unity, significant undulations are present in the induced damping; the less the modal overlap parameters are compared with unity, the more prominent are the undulations. The undulations are largely suppressed when the local modal overlap parameters exceed unity. Moreover, appropriately averaging the undulations yields values for the induced damping that coincide with those obtained when the modal overlap parameters exceed unity. Further, it transpires that these common values are independent of the individual modal overlap parameters. When the summation is replaced by an integration, the first order results are undulations-free and the values, so obtained, again, coincide with those pertaining to modal overlap parameters that exceed unity. Without defining the excursions in the undulations, the transition from a discrete-to-a continuous distribution, that is implied by a summation-to-an integration, must assume that the modal overlap parameters exceed unity. In particular, without careful and meaningful qualifications, it may be misleading to assume, *a priori*, that the modal overlap parameters are equal to zero.

1. INTRODUCTION

A central theme to a few publications is that a set of satellite oscillators attached to a master oscillator contributes to the damping of the master oscillator even if all the loss factors of the satellite oscillators in the set, are equal to zero [1–6] (cf. Figure 1). The satellite oscillators are numerable and their resonance frequencies are distributed on both sides of the resonance frequency of the master oscillator. The conclusion that a set of lossless satellite oscillators can, nonetheless, contribute to the damping of a master oscillator, has been supported by analyses that *a priori* assume the satellite oscillators to posses zero loss factors [4, 5]. In contrast to the analysis presented here, some of the analyses; e.g., the analysis in reference [5], are conducted in the time domain [3]. With respect to the material presented in this paper, whether the analysis is conducted in the time domain or in the frequency domain, is considered to be merely a matter of choice, notwithstanding that in the case of a closely packed distribution of resonance frequencies of the satellite oscillators, the relaxation times involved may be quite long. In this case, to relate the results in the time domain to those in the frequency domain may require a long time to capture the entire data [3].

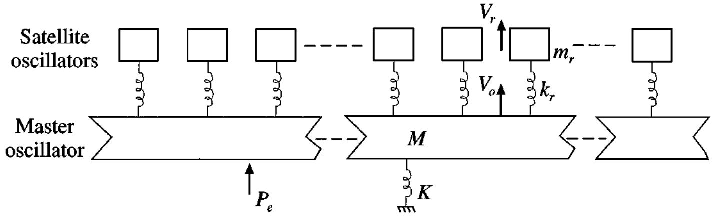

Figure 1. Master oscillator coupled to a set of resonance frequency distributed satellite oscillators. The master oscillator is defined by the mass M and the stiffness K and the rth satellite oscillator by mass m_r and stiffness k_r. Only the master oscillator is driven, by an external drive $P_e(\omega)$, generating the response $V_o(\omega)$ in the master oscillator and the response $V_r(\omega)$ in the rth satellite oscillator.

In this present paper, a fundamental initial oversight that besets, all these publications is brought to the attention of the reader. Admittedly, once that oversight is addressed, the remaining analyses and arguments in all these publications are largely validated; e.g., validated is the argument that the specifically defined measure of the damping that is provided to the host master oscillator by a distribution of nearly lossless oscillators, is independent of the loss factors that account for the dampings in these oscillators. The loss factors, in turn, are defined in terms of associated modal overlap parameters. The independence of the induced damping in the master oscillator of these loss factors, or equivalently of the modal overlap parameters, does not, however, imply that individual loss factors that are equal to zero are admissible without strict qualifications. An analysis of *nearly* continuous distribution of *nearly* undamped satellite oscillators may bring insights into the manner by which these qualifications can be stated. In part, these insights are obscured in an analysis that is subjected *a priori* to limiting asymptotic conditions of negligibility and continuity; notwithstanding that such impositions may require careful consideration if apparent singular behaviors of the mechanical system are to be avoided [4, 5]. Here it is argued that the determination of the induced loss factor is readily achieved by inserting integration for the summation over satellite oscillators. However, such an insertion demands the obedience of an auxiliary relationship. This relationship is commensurate with a satisfied modal overlap condition. A satisfied condition of modal overlap states that if the loss factors of the satellite oscillators are made to approach infinitesimal values, the number of these satellite oscillators must approach infinity at least at the same rate. This condition, then, precludes setting the loss factors of the satellite oscillators equal to zero *a priori* and in that sense, the imposition of the descriptive adjective "nearly" is meaningful. Conversely, it is asserted that *ad hoc* replacement of a summation by integration renders the condition of modal overlap implicitly satisfied. Were the condition of modal overlap not satisfied, the true values of the induced damping undulate, as a function of frequency, within the distribution range of the resonance frequency of the satellite oscillators. The undulations are suppressed by the transition from a summation-to-an integration, i.e., a discrete-to-a continuous format of evaluation. The excursions of these undulations must then be stated in a suitable and a definitive manner. The significance of these statements is that whether or not the condition of modal overlap is satisfied, collectively the satellite oscillators possess enough damping and, when appropriate, high enough peaks in their responses to explain the physical presence of the induced loss factors [4]. Then, inventing exotic mechanisms to account for the induced damping becomes moot [5].

2. THE FORMALISM OF A DISCRETE AND OF A CONTINUOUS DISTRIBUTION OF RESONANCE FREQUENCIES

Consider the mechanical system comprising a master oscillator, with mass M and stiffness K, that is coupled to a set of satellite oscillators. This mechanical system is sketched in Figure 1 [1–6]. The rth satellite oscillator is defined by a mass m_r and a stiffness k_r. The damping is assumed to be associated with the stiffness elements

$$K = K_o(1 + \mathrm{i}\eta_o), \qquad K_o/M = \omega_o^2, \tag{1a}$$

$$k_r = k_{or}(1 + \mathrm{i}\eta_r), \qquad k_{or}/m_r = \omega_r^2, \tag{1b}$$

where η_o and η_r are loss factors. Each of these loss factors characterizes an individual oscillator and quantifies the dissipation that the oscillator, in isolation, can handle when externally driven. The linear equations of motion of the master oscillator *in situ* and of a typical satellite oscillator *in situ* are

$$[(\mathrm{i}\omega M) + (K/\mathrm{i}\omega)]\, V_o(\omega) + \sum_1^R (k_r/\mathrm{i}\omega)[V_o(\omega) - V_r(\omega)] = P_e(\omega), \tag{2}$$

$$(\mathrm{i}\omega\, m_r)\, V_r(\omega) + (k_r/\mathrm{i}\omega)[V_r(\omega) - V_o(\omega)] = 0, \tag{3}$$

respectively, where $V_o(\omega)$ and $V_r(\omega)$ are the responses of the mass M of the master oscillator and of the mass m_r of the rth satellite oscillator, respectively, R is the number of satellite oscillators that are coupled with the master oscillator and $P_e(\omega)$ is the drive that is assumed applied externally to the master oscillator; the satellite oscillators are not driven externally [7] (cf. Figure 1). The satellite oscillators, in equations (2) and (3), are assumed uncoupled from each other; the satellite oscillators are coupled with the master oscillator only. The summation in equation (2) is over a set of oscillators, where, again, R is the number of satellite oscillators in the set. From equations (2) and (3), one obtains

$$Z_o(\omega)\, V_o(\omega) = P_e(\omega), \qquad V_r(\omega) = B(y, x_r, \eta_r)\, V_o(\omega), \tag{4}$$

where

$$Z_o(\omega) = (\mathrm{i}\omega M)\{1 - (y)^{-2}[1 - s(y)] - \mathrm{i}(y)^{-2}[\eta_o + \eta_s(y)]\}, \tag{5}$$

$$[s(y) - \mathrm{i}\eta_s(y)] = (y)^2 \sum_1^R A(y, \bar{m}_r, x_r, \eta_r), \qquad A(y, \bar{m}_r, x_r, \eta_r) = \bar{m}_r B(y, x_r, \eta_r), \tag{6}$$

$$B(y, x_r, \eta_r) = (x_r)^2[(1 + \eta_r^2)(x_r)^2 - (y)^2 - \mathrm{i}\eta_r(y)^2]\{[(x_r)^2 - (y)^2]^2 + [\eta_r(x_r)^2]^2\}^{-1}, \tag{7}$$

$$y = \omega/\omega_o, \qquad \bar{m}_r = m_r/M, \qquad x_r = \omega_r/\omega_o, \quad 1 \leqslant r \leqslant R. \tag{8}$$

The quantity x_r defines the resonance frequency distribution of the satellite oscillators. It is emphasized that in the present paper this distribution, and not the spatial distribution of the satellite oscillators, is in reference. The satellite oscillators are attached on a single-velocity platform; that platform constitutes the mass of the master oscillator. The velocity in the plane of the platform is uniform so that each attached sprung mass (satellite oscillator) perceives the same velocity at the point of attachment. Whether the spatial distribution of the sprung masses on the platform is made to coincide with the resonance frequency

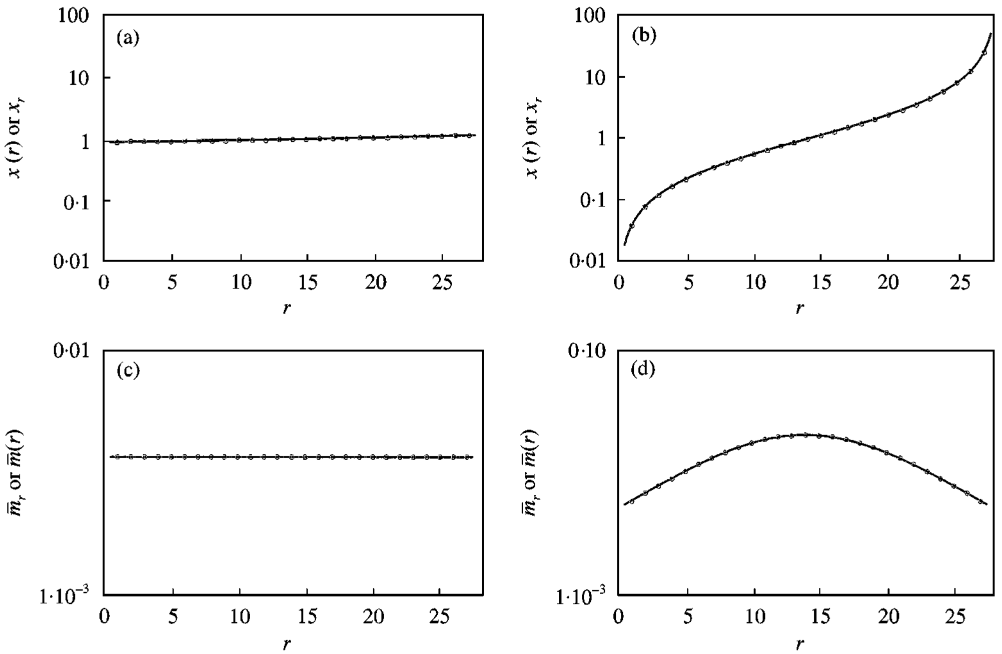

Figure 2. Resonance frequency distribution, x_r, and normalized mass distribution, $\bar{m}_r$, as functions of the discrete r (distinct dots), and $x(r)$ and $\bar{m}(r)$, as functions of the continuous r (solid line curves). Number R of satellite oscillators is twenty seven, ($R = 27$) and mass ratio (M_s/M) is a tenth (10^{-1}). (a) x_r as specified in equation (9a): ○, (x_r); ——, $\{x(r)\}$. (b) x_r as specified in equation (9b): ○, (x_r); ——, $\{x(r)\}$. (c) $\bar{m}_r$ as specified in equation (9c): ○, $(\bar{m}_r)$; ——, $\{\bar{m}(r)\}$. (d) $\bar{m}_r$ as specified in equation (9d): ○, $(\bar{m}_r)$; ——, $\{\bar{m}(r)\}$.

distribution is moot in this paper (cf. Figure 1). In some publications, variations on this theme are implied but are not always strictly defined. Thus, the problem of *a priori* assuming a spatially discrete distribution of satellite oscillators (sprung masses) versus a spatially continuous distribution was briefly discussed in reference [8]. In this reference, the satellite oscillators all possess the same resonance frequency. In reference [6], on the other hand, a single-coupled structure with multiple resonance frequencies is used.

It may be useful to exemplify much of the analytical processes. For this purpose, two distinct resonance frequency distributions for the satellite oscillators; i.e.., x_r as a function of r, are depicted in Figure 2. In Figure 2(a) and 2(b) the resonance frequency distributions are [5, 7]

$$\omega_r/\omega_o = x_r = [1 + \{1 - 2\bar{r}\}\gamma(\bar{R})]^{-1/2}, \quad 1 \leqslant r \leqslant R \tag{9a}$$

and

$$\omega_r/\omega_o = x_r = \bar{r}(1 - \bar{r})^{-1}, \quad 1 \leqslant r \leqslant R, \tag{9b}$$

respectively, where

$$\bar{r} = r(R + 1)^{-1}, \qquad \bar{R} = R(R + 1)^{-1}, \qquad \gamma(\bar{R}) = (\gamma/2\bar{R}) < (1/2). \tag{10a}$$

In keeping with references [7, 5], a normalized mass distributions for the satellite oscillators; i.e., $\bar{m}_r$ as a function of r, are also introduced and displayed in Figure 2. In

Figure 2(c) and 2(d) the normalized mass distributions are

$$\bar{m}_r = (M_s/M)(R)^{-1}, \qquad (M_s/M) = \sum_1^R \bar{m}_r, \tag{9c}$$

$$\bar{m}_r = M_s/M\,[2/\pi(R+1)]\,[(1-\bar{r})^2 + (\bar{r})^2]^{-1}, \qquad M_s/M = \sum_1^R \bar{m}_r, \tag{9d}$$

respectively, where M_s/M is the mass ratio of the total mass of the satellite oscillators to that of the mass of the master oscillator. In this paper, this mass ratio is set at one-tenth; $M_s/M = (10)^{-1}$. It is noted in this connection, that in the analysis presented in this paper, *randomly picked* distributions of resonance frequencies x_r and of masses m_r are admissible, as long as they are exactly and orderly specified [9].

The impedance $Z_o(\omega)$, stated in equation (5), comprises the self-impedance $(\mathrm{i}\omega M)$ $[1-(y)^{-2}(1+\mathrm{i}\eta_o)]$ of the master oscillator and the sum over the impedances contributed by the satellite oscillators to which the master oscillator is coupled [7]. Typically, the impedance $(\mathrm{i}\omega M)\ A(y, \bar{m}_r, x_r, \eta_r)$ is that contributed by the rth oscillator; it is a function of the normalized frequency $y = \omega/\omega_o$ and is a functional of the normalized mass distribution $\bar{m}_r = m_r/M$, the normalized resonance frequency $x_r = \omega_r/\omega_o$ and the loss factor η_r of the rth satellite oscillator. As equation's (1a) and (8) state, the normalizing frequency ω_o is the resonance frequency of the master oscillator in isolation. Under the same cover, the response $V_r(\omega)$, of the mass m_r of the rth satellite oscillator, is related to the response $V_o(\omega)$, of the mass M of the master oscillator, by a factor that is a function of the normalized frequency $y = (\omega/\omega_o)$ and is a functional of two of the three parameters; namely, of x_r and η_r. The relationship between $V_r(\omega)$ and $V_o(\omega)$ is stated in equation (4).

Clearly r is a discrete variable. Can an *artificially* assigned r set to be a continuous variable? This proposed continuity of r is exemplified by the solid-line curves in Figure 2. It is convenient, in this connection, to normalize (scale) the continuous variable r by the number of satellite oscillators R *plus* unity 1, as suggested by equations (9) and (10). The normalized r is designated $\bar{r}$ and the dependent parameters are then expressed in the forms

$$\bar{m}_r \Rightarrow \bar{m}(\bar{r}), \qquad x_r \Rightarrow x(\bar{r}), \qquad \eta_r \Rightarrow \eta(\bar{r}). \tag{11}$$

The scaled forms of x_r, namely, $x(\bar{r})$, in equations (9a) and (9b), are depicted as functions of $\bar{r}$ in Figure 3(a) and 3(b) respectively. In this vein, equations (4)–(7) may be recast, without further ado, in the forms

$$Z_o(\omega)\,V_o(\omega) = P_e(\omega), \qquad V(\bar{r}, \omega) = B\{y, x(\bar{r}), \eta(\bar{r})\}\,V_o(\omega), \tag{12}$$

where

$$Z_o(\omega) = (\mathrm{i}\omega M)\{1-(y)^{-2}[1-s(y)] - \mathrm{i}(y)^{-2}[\eta_o + \eta_s(y)]\},$$

$$[s(y) - \mathrm{i}\eta_s(y)] = (y)^2 \int_{(\varepsilon)}^{(R+\varepsilon)} \mathrm{d}r\, A\{y, \bar{m}(\bar{r}), x(\bar{r}), \bar{\eta}(\bar{r})\}, \tag{13}$$

$$A\{y, \bar{m}(\bar{r}), x(\bar{r}), \eta(\bar{r})\} = \bar{m}(\bar{r})\,B\{y, x(\bar{r}), \eta(\bar{r})\},$$

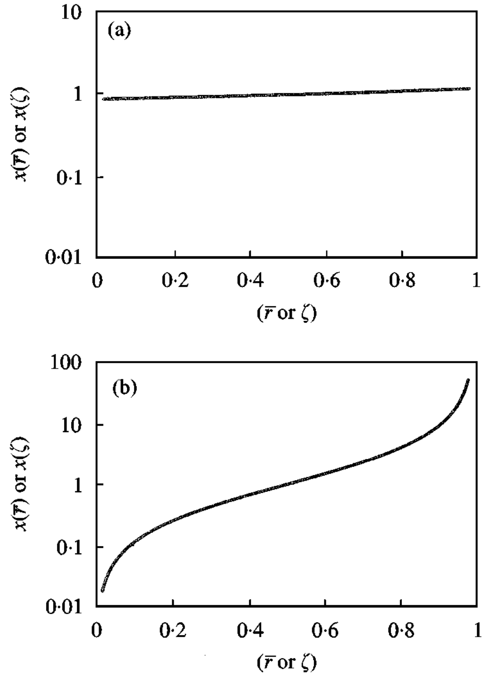

Figure 3. Resonance frequency distribution; $x(\bar{r})$ or $x(\zeta)$ as a function of the normalized continuous variable $\bar{r}$ or ζ, respectively (cf. Figure 2, ($R = 27$). (a) $x(\zeta)$ as specified in equation (16a): ——, $\{x(\bar{r})\}$ or $\{x(\zeta)\}$. (b) $x(\zeta)$ as specified in equation (16b): ——, $\{x(\bar{r})\}$ or $\{x(\zeta)\}$.

$$[s(y) - \mathrm{i}\eta_s(y)] = (y)^2 \int_{(\bar{\varepsilon})}^{(\bar{R}+\bar{\varepsilon})} \mathrm{d}\bar{r}\, A\{y, \mu(\bar{r}), x(\bar{r}), \eta(\bar{r})\}, \tag{14a}$$

$$A\{y, \mu(\bar{r}), x(\bar{r}), \eta(\bar{r})\} = \mu(\bar{r})\, B\{y, x(\bar{r}), \eta(\bar{r})\}, \tag{14b}$$

$$B\{y, x(\bar{r}), \eta(\bar{r})\} = \{x(\bar{r})\}^2 [(1 + \{\eta(\bar{r})\}^2)\{x(\bar{r})\}^2 - (y)^2 - \mathrm{i}\eta(\bar{r})(y)^2] \times \{[\{x(\bar{r})\}^2 - (y)^2]^2 + [\eta(\bar{r})\{x(\bar{r})\}^2]^2\}^{-1}, \tag{15}$$

$$\bar{m}_r \Rightarrow \mu(\bar{r})(R+1)^{-1}, \quad \bar{\varepsilon} = \varepsilon(R+1)^{-1} \quad \bar{R} = R(R+1)^{-1}. \tag{10b}$$

The parameter $\bar{\varepsilon}$ designates the end-effects that are introduced in the transition from the discrete to the continuous domains; e.g., $\bar{\varepsilon} = \varepsilon(R+1)^{-1}$, with $\varepsilon \approx \frac{1}{2}$, say (cf. Figure 2).

Again, consider the mechanical system comprising of a master oscillator, with mass M and stiffness K, that is coupled with a set of satellite oscillators, as defined in equation (1) and sketched in Figure 1. However, now the resonance frequencies $\{\omega(\zeta)\}$ of the satellite oscillators are assumed to be *a priori* continuously, but not necessarily uniformly distributed in the interval $0 < \zeta < 1$ [5]. Example of two typical resonance frequency distributions; i.e., $x(\zeta)$ as functions of ζ, are depicted in Figure 3. In Figure 3(a) and (3b) the normalized resonance frequency distributions are

$$\omega(\zeta)/\omega_o = x(\zeta) = [1 + (1 - 2\zeta)\gamma(\bar{R})]^{-(1/2)}, \qquad (\bar{R}) < (1/2), \quad 0 < \zeta < 1, \tag{16a}$$

$$\omega(\zeta)/\omega_o = x(\zeta) = (\zeta)(1-\zeta)^{-1}, \qquad 0 < \zeta < 1, \tag{16b}$$

respectively, and $\gamma(\bar{R})$ is defined in equation (10) [5, 7] (cf. equations (9a) and (9b), respectively, and equation (10)). The corresponding normalized mass distributions $\mu(\zeta)$ of the satellite oscillators are [5, 7]

$$\mu(\zeta) = (M_s/M)(\bar{R})^{-1}, \quad 0 < \zeta < 1, \tag{16c}$$

$$\mu(\zeta) = (M_s/M)(2/\pi)[(1-\zeta)^2 + (\zeta)^2]^{-1}, \qquad 0 < \zeta < 1, \tag{16d}$$

(cf. equations (9c) and (9d), respectively, and equation (10).) The linear equations of motion of the master oscillator *in situ* and of a typical satellite oscillator *in situ* are

$$Z_o(\omega)\, V_o(\omega) = P_e(\omega), \qquad V(\zeta, \omega) = B\{y, x(\zeta), \eta(\zeta)\}\, V_o(\omega), \tag{17}$$

where

$$Z_o(\omega) = (\mathrm{i}\omega M)\{1 - (y)^{-2}[1 - s^c(y)] - \mathrm{i}(y)^{-2}[\eta_o + \eta_s^c(y)]\}, \tag{18}$$

$$[s^c(y) - \mathrm{i}\eta_s^c(y)] = (y)^2 \int_0^1 \mathrm{d}\zeta\, A\{y, \mu(\zeta), x(\zeta), \eta(\zeta)\}, \tag{19}$$

$$A\{y, \mu(\zeta), x(\zeta), \eta(\zeta)\} = \mu(\zeta)\, B\{y, x(\zeta), \eta(\zeta)\},$$

$$B\{y, x(\zeta), \eta(\zeta)\} = \{x(\zeta)\}^2[\{1 + \eta^2(\zeta)\}\{x(\zeta)\}^2 - (y)^2 - \mathrm{i}\eta(\zeta)(y)^2]$$

$$\times \{[\{x(\zeta)\}^2 - (y)^2]^2 + [\eta(\zeta)\{x(\zeta)\}^2]^2\}^{-1}, \tag{20}$$

$$y = \omega/\omega_o, \qquad x(\zeta) = \omega(\zeta)/\omega_o, \quad 0 < \zeta < 1. \tag{21}$$

A question arises: under what conditions can the linear equations (4)–(7) be asymptotically circumvented into the linear equations (12)–(15) and equations (12)–(15) into the linear equations (17)–(20), respectively, and are these conditions significant? This question has been partially answered in previous publications; in this paper, a more definitive answer is sought [2, 7]. However, prior to this pursuit, one may inquire: suppose these asymptotic circumventions were found permissible as these are stated, can one perform the integration in the continuous domain that, in fact, replaces the summation in the discrete domain?

3. EVALUATION OF THE INDUCED LOSS FACTOR $\eta_s(y)$ IN EQUATIONS (13)–(15)

The following quantities are defined first:

$$z(\bar{r}) = [x(\bar{r})/y], \qquad [\partial\{x(\bar{r})\}/\partial\bar{r}]^{-1} = f(\bar{r}). \tag{22}$$

Then, from equations (13)–(15), one obtains

$$\eta_s(y) = (y)^3 \int_{z(\bar{\varepsilon})}^{z(\bar{R}+\bar{\varepsilon})} \mathrm{d}z(\bar{r})[f(\bar{r})\mu(\bar{r})][\{z(\bar{r})\}^2\eta(\bar{r})]\{[\{z(\bar{r})\}^2 - 1]^2 + [\{z(\bar{r})\}^2\eta(\bar{r})]^2\}^{-1}. \tag{23}$$

Were the factor $[f(\bar{r})\mu(\bar{r})]$ to be a well-behaved function of $\bar{r}$ in the range $\bar{\varepsilon} < \bar{r} < \bar{R} + \bar{\varepsilon}$ and were $\eta(\bar{r})$ to be small compared with unity, as a function of $\bar{r}$ in that same range, the integral in equation (23) immediately yields the result

$$\eta_s(y) = (\pi/2)(y)^3[f(\bar{r}_o)\mu((\bar{r}_o)], \qquad z(\bar{r}_o) = 1, \quad (\bar{\varepsilon}) < \bar{r}_o < (\bar{R} + \bar{\varepsilon}), \tag{24}$$

which, clearly, is independent of $\eta(\bar{r})$ provided these loss factors are set small compared with unity in the range $(\bar{\varepsilon}) < \bar{r} < \bar{R} + \bar{\varepsilon}$ [2]. Employing the normalized resonance frequency distributions stated in equations (9a) and (9b), one finds

$$f(\bar{r}_o) = [\gamma(\bar{R})(y)^3]^{-1}, \qquad [1 + (\gamma/2)]^{-(1/2)} < y < [1 - (\gamma/2)]^{-(1/2)}, \tag{25a}$$

$$f(\bar{r}_o) = (1 + y)^{-2}, \qquad (2R + 1)^{-1} < y < (2R + 1), \tag{25b}$$

respectively, where γ is defined in equation (10) and is chosen, herein, to be equal to 0·6. Similarly, employing the normalized mass distributions stated in equations (9c) and (9d), one finds

$$\mu(\bar{r}_o) \Rightarrow (M_s/M)(\bar{R})^{-1}, \qquad (\bar{\varepsilon}) < (\bar{r}_o < (\bar{R} + \bar{\varepsilon}), \tag{25c}$$

$$\mu(\bar{r}_o) \Rightarrow (M_s/M)(2/\pi)(1 + y)^2[1 + (y)^2]^{-1}, \qquad (\bar{\varepsilon}) < \bar{r}_o < (\bar{R} + \bar{\varepsilon}). \tag{25d}$$

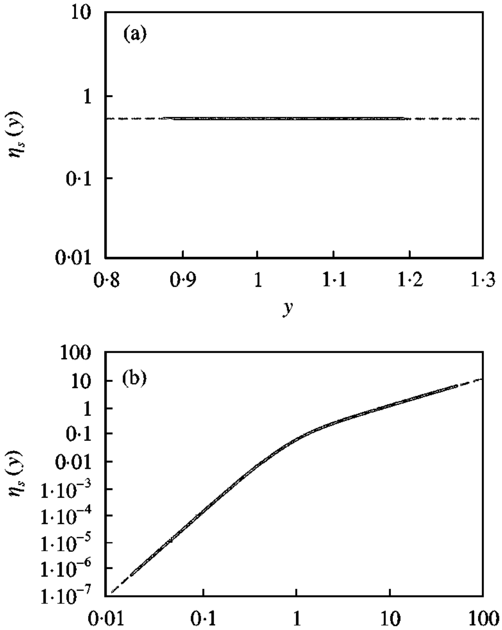

Figure 4. Induced loss factor $\eta_s(y)$, as a function of the normalized frequency y, resulting from *ad hoc* replacement of a summation by an integration ($R = 27$, $(M_s/M) = 10^{-1}$). (a) For a normalized resonance frequency x_r and a normalized mass distribution $\bar{m}_r$ as specified in equations (9a) and (9c) respectively: ——, within permissible limits; - - - - - -, beyond permissible limits. (b) For a normalized resonance frequency x_r and a normalized mass distribution $\bar{m}_r$ as specified in equations (9b) and (9d) respectively: ——, within permissible limits; - - - - - -, beyond permissible limits.

From equations (24) and (25) one readily derives, for the two examples herein considered, the results

$$\eta_s(y) = (M_s/M)[\pi/\{2\gamma(\bar{R})\}], \qquad [1+(\gamma/2)]^{-(1/2)} < y < [1-(\gamma/2)]^{-(1/2)}, \tag{26a}$$

$$\eta_s(y) = (M_s/M)(y)^3[1+(y)^2]^{-1}, \qquad [(2R+1)]^{-1} < y < [(2R+1)]. \tag{26b}$$

It is recognized that $\eta_s(y)$ is largely meaningful only at and in the vicinity of y equal to unity, $y \approx 1$, so that the restricted range of y in equation (26) is not physically significant. For the two examples, stated in equations (26a) and (26b), $\eta_s(y)$ is displayed in Figure 4(a) and (4b) respectively. The dashed-line curves in these figures are merely free extrapolations beyond the permissible ranges.

Since it emerges that $\eta_s(y)$ is independent of the loss factors $\eta(\bar{r})$, provided $\eta(\bar{r}) \ll 1$ in the range $(\bar{\varepsilon}) < \bar{r} < (\bar{R} + \bar{\varepsilon})$, these loss factors of the satellite oscillators may, indeed, be set equal to zero and nobody would be the wiser. Is that true?

Finally, it is remarked that the difference between the induced loss factor $\eta_s(y)$, stated in equation (14), and $\eta_s^c(y)$ stated in equation (19), is the number R of satellite oscillators. Clearly, equation (19) can be derived from equation (14) by allowing R to increase enough to render $(\bar{R} + \bar{\varepsilon}) \to 1$ and $(\bar{\varepsilon}) \to 0$.

4. MODAL OVERLAP CONDITION FOR RELATING THE DISCRETE AND THE CONTINUOUS DOMAINS

To establish the relationship between the discrete and the continuous domains one may define two distinct frequency bands. The first, designated $(\delta\omega_r)$, defines the fair frequency territory occupied by each satellite oscillator with respect to its adjacent neighbors; this frequency bandwidth is centered on the resonance frequency of that satellite oscillator. The expression for this bandwidth is

$$\delta\omega_r = \{(1/2)(\omega_{r+1} - \omega_{r-1})[U(R-1+\varepsilon-r) - U(2-\varepsilon-r)] + (\omega_2-\omega_1)U(1+\varepsilon-r)$$
$$+ (\omega_R - \omega_{R-1})U(r+\varepsilon-R)\}U(r-\varepsilon)U(R+\varepsilon-r), \tag{27a}$$

where $U(\alpha)$ is the usual unit step function and ε is less than unity, say equal to $\frac{1}{2}$, again. The inverse of the frequency bandwidth $\delta\omega_r$ is commonly referred to as the *modal density* n_r of the satellite oscillators; i.e.,

$$n_r = (\delta\omega_r)^{-1}, \quad \varepsilon \leqslant r \leqslant R+\varepsilon, \tag{28}$$

where the inversion of $\delta\omega_r$ is carefully executed with respect to the unit step functions [10, 11]. The second frequency band, designated $\Delta\omega_r$, defines the inherent frequency bandwidth of the rth satellite oscillator; this bandwidth accounts for the inherent damping of this satellite oscillator. The measure of the damping, in turn, defines the loss factor η_r. The bandwidth $\Delta\omega_r$ is then stated in terms of the loss factor η_r and the resonance frequency ω_r in the form

$$\Delta\omega_r = (\omega_r\eta_r)U(r-\varepsilon)U(R+\varepsilon-r). \tag{27b}$$

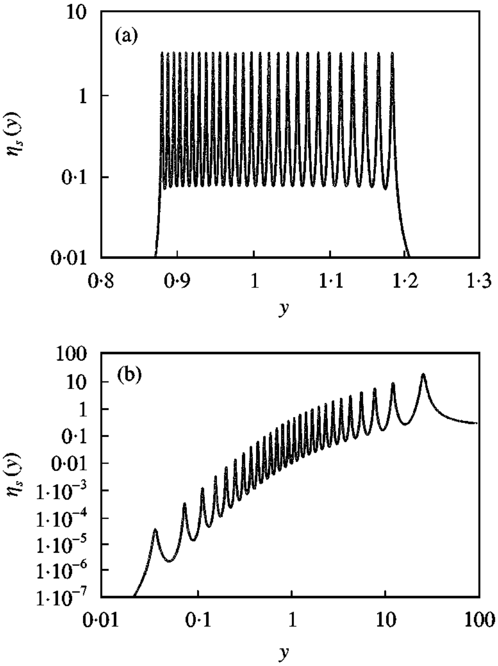

Figure 5. Induced loss factor $\eta_s(y)$, as a function of the normalized frequency y, for a modal overlap parameter b_r that is less than unity, $b_r \approx (10)^{-1}$ ($R = 27$, $(M_s/M) = 10^{-1}$). (a) For a normalized resonance frequency x_r and a normalized mass distribution $\bar{m}_r$ as specified in equations (9a) and (9c) respectively. (b) For a normalized resonance frequency x_r and a normalized mass distribution $\bar{m}_r$ as specified in equations (9b) and (9d) respectively.

The ratio b_r of these two frequency bandwidths defines a *modal overlap parameter*

$$\{b_r = (\Delta\omega_r/\delta\omega_r) = (n_r\omega_r\eta_r)\}\, U(r-\varepsilon)U(R+\varepsilon-r), \tag{29}$$

where use is made of equations (27) and (28).

For a modal overlap parameter b_r that is less than unity, $b_r < 1$, adjacent satellite oscillators reside outside each other's bandwidths. Consequently, the influence of the satellite oscillators on the response of the master oscillator, as a function of y, $y = (\omega/\omega_o)$, can be identified individually; each contribution associated with a satellite oscillator stands out prominently from others. An example of such an influence in terms of evaluating the induced loss factor $\eta_s(y)$, as a function of y with b_r less than unity, is depicted in Figure 5 [5, 7]. On the other hand, for a modal overlap parameter b_r that exceeds the value of unity, $b_r > 1$, adjacent satellite oscillators reside within each other's bandwidths. Therefore, their influence on the response of the master oscillator is largely continuous as a function of (y), $y = (\omega/\omega_o)$. The more the modal overlap parameter b_r exceeds the value of unity, the more the continuity [7]. An example of such an influence, in terms of evaluating the induced loss factor $\eta_s(y)$, as a function of y with b_r exceeding unity, is depicted in Figure 6 [5, 7]. Comparing Figure 4 with Figures 5 and 6, respectively, serves to expose the issue that is central to this paper. Figure 4(a) and 4(b), respectively, overlap prime portions of Figure 6(a) and 6(b) only. These prime portions are defined within the range set on y in equations (26a) and (26b) respectively. No such overlap exists between Figure 4(a) and 4(b) and Figure 5(a) and 5(b) respectively. Nonetheless, if the undulations in Figure 5 are appropriately

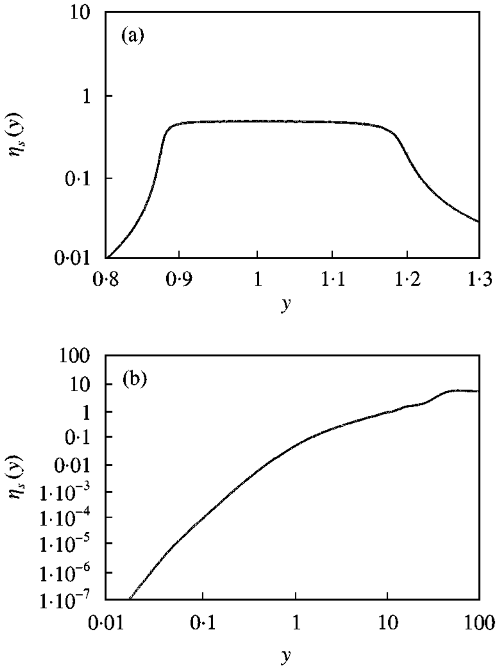

Figure 6. Induced loss factor $\eta_s(y)$, as a function of the normalized frequency y, for a modal overlap parameter b_r that exceeds unity, $b_r \approx 2$ ($R = 27$, $(M_s/M) = 10^{-1}$). For a normalized resonance frequency x_r and a normalized mass distribution $\bar{m}_r$ as specified (a) in equations (9a) and (9c) respectively. (b) For a normalized resonance frequency x_r and a normalized mass distribution $\bar{m}_r$ as specified in equations (9b) and (9d) respectively.

averaged, the overlap between Figures 4 and 6 is extended to include the mean values of Figure 5 [7, 12] (cf. Appendix A). When b_r becomes small compared with unity the mean values alone do not reflect the presence of undulations and, therefore, Figure 4 cannot serve to substitute for Figure 5. The information that lies in Figure 5 can hardly be derived from Figure 4; Figure 4 is akin to Figure 6 but not to Figure 5. Since Figure 4 is derived by replacing the summation by integration, it is concluded that this replacement is commensurate with the imposition of modal overlap parameters that exceed unity, $b_r > 1$ (cf. Appendix A). An imposition of this kind ensures that the evaluations of mean-value data do not conceal information that may render the data, at best, ambiguous and, at worst, misleading [13].

When the modal overlap parameter b_r exceeds unity, the *modal overlap condition* is said to be satisfied. When the modal overlap condition is satisfied, the linear dynamic description $[S(y) - i\eta_s(y)]$, comprising the sum of the individual linear dynamic descriptions of all the satellite oscillators; e.g., typically the term $[(y)^2 A(y, \bar{m}_r, x_r, \eta_r)]$ contributed to the sum by the rth satellite oscillator, may be, well nigh, evaluated by integration:

$$[s(y) - i\eta_s(y)] = (y)^2 \sum_{1}^{R} A(y, \bar{m}_r, x_r, \eta_r) \Rightarrow (y)^2 \int_{(\varepsilon)}^{(R+\varepsilon)} \mathrm{d}r\, A\{y, \bar{m}(r), x(r), \eta(r)\}, \quad (30a)$$

where r in the second of equation (30a), is a continuous and dimensionless variable [2] (cf. equations (6) and (14a)). It is emphasized that the transition, described in equation (30a), is validated if, and only if, it is implicitly understood that b_r, as stated in equation (29), exceeds unity, namely

$$\{b_r = (n_r\omega_r\eta_r) > 1\}\, U(r-\varepsilon)U(R+\varepsilon-r). \tag{31a}$$

The assignment of continuity to r, in equation (30a), renders the normalized resonance frequency $x_r \Rightarrow x(r)$ a continuous function of (r), as exemplified in Figure 2(a) and 2(b) [14]. Similarly, $\bar{m}_r \Rightarrow \bar{m}(r)$ and $\eta_r \Rightarrow \eta(r)$ (cf. Figure 2(c) and 2(d)). In this rendering, equations (27a) and (27b) may be approximated, by smoothing, in the forms

$$n(r) = [\delta\omega(r)]^{-1} \approx [\partial\omega(r)/\partial r]^{-1}, \qquad \Delta\omega(r) = \omega(r)\eta(r), \quad (\varepsilon) < r < (R+\varepsilon), \tag{32a, b}$$

respectively, where one recognizes that $[\partial\omega(r)/\partial r]$ is, by definition, a positive quantity and equation (28) is referenced. From equation (22), one finds that

$$[\partial\omega(r)/\partial r] \equiv [(R+1)f(\bar{r})/\omega_0]^{-1}. \tag{32c}$$

Equation (31a) may then be expressed in the form

$$\{(R+1)\eta(\bar{r}) = [b(\bar{r})][\partial \ln\{x(\bar{r})\}/\partial\bar{r}]\}\, U(\bar{r}-\bar{\varepsilon})U(\bar{R}+\bar{\varepsilon}-\bar{r}), \tag{31b}$$

where, again, $\bar{\varepsilon} = \varepsilon(R+1)^{-1}$, $\bar{R} = R(R+1)^{-1}$, and $\bar{r} = r(R+1)^{-1}$. The so normalized continuous r may be further designated ζ and hence, in that vein, equations (31) and (32) are to be recast with $\bar{r} \rightarrow (\zeta)$; e.g., in this normalization equation (31b) is written in the form

$$\{([R+1)\eta(\zeta)] = [b(\zeta)][\partial \ln\{\omega(\zeta)\}/\partial\zeta]\}\, U(\zeta-\bar{\varepsilon})U(\bar{R}+\bar{\varepsilon}-\zeta), \tag{31c}$$

where

$$\zeta = r(R+1)^{-1}, \quad \bar{\varepsilon} = \varepsilon(R+1)^{-1}, \qquad \bar{R} = R(R+1)^{-1}. \tag{33a}$$

The scaling is illustrated in Figure 3 for the examples depicted in Figure 2(a) and 2(b) (cf. equations (9) and (16)). Employing this scaling in equation (30a), one obtains

$$[s(y) - \mathrm{i}\eta_s(y)] = (y)^2 \int_{(\varepsilon)}^{(R+\varepsilon)} \mathrm{d}r\, A\{y, \bar{m}(r), x(r), \eta(r)\}$$

$$\Rightarrow (y)^2 \int_{(\bar{\varepsilon})}^{(\bar{R}+\bar{\varepsilon})} \mathrm{d}\zeta\, A\{y, \mu(\zeta), x(\zeta), \eta(\zeta)\}, \tag{30b}$$

where ζ and $\bar{\varepsilon}$ are defined in equation (33a) (cf. equations (10) and (14)). If one further assumes that the number R of the satellite oscillators is large compared with unity, equation (30b) becomes

$$[s(y) - \mathrm{i}\eta_s(y)] \Rightarrow (y)^2 \int_0^1 \mathrm{d}\zeta\, A\{y, \mu(\zeta), x(\zeta), \eta(\zeta)\} = [s^c(y) - \mathrm{i}\eta_s^c(y)], \tag{30c}$$

where $[s^c(y) - \mathrm{i}\eta_s^c(y)]$ is defined in equation (19) and it follows from equation (33a) that

$$\bar{\varepsilon} \rightarrow 0, \quad (\bar{R}+\bar{\varepsilon}) \rightarrow 1, \quad \bar{R} \rightarrow 1, \quad \text{as } R \gg 1. \tag{33b}$$

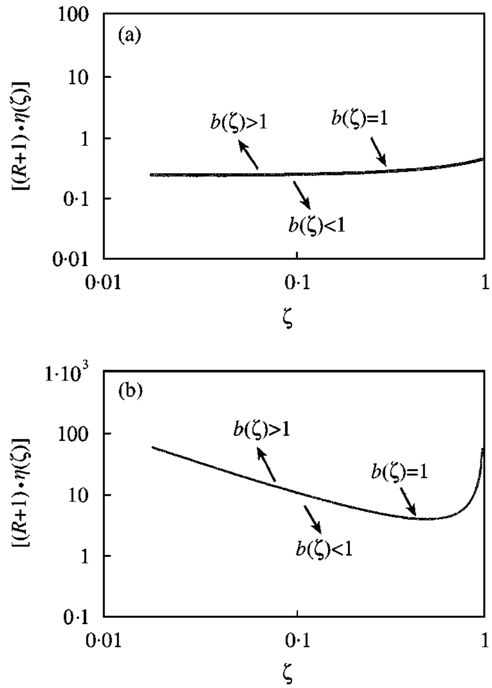

Figure 7. Lower limit imposed on the value of the quantity $[(R+1)\eta(\zeta)]$, as a function of ζ, in order to just satisfy the modal overlap condition; i.e., $b_r \approx 1$, thereby just permitting, without qualification, for an integration to replace a summation (cf. Figures 5 and 6), ($R = 27$). For a resonance frequency distribution as stated in (a) equation (16a), (b) equation (16b).

The quantity $[\partial \ln\{\omega(\zeta)\}/\partial\zeta] \equiv [\partial \ln(\{x(\zeta)\}/\partial\zeta]$, in equation (31c), is the local slope in figures of which Figure 3(a) and 3(b) serve as two specific examples [14]. It is noted that equation (31c) may be cast in the alternate form as

$$\{\eta(r) = b(r)[\partial \ln\{\omega(r)\}/\partial r] \equiv b(r)[\partial \ln\{x(r)\}/\partial r]\}\ U(r-\varepsilon)U(R+\varepsilon-r). \tag{31d}$$

In equation (31d), the local slope is that of curves like those exemplified in Figure 2(a) and 2(b) for the continuous r [14]. Again, it is emphasized that the relationship between Figure 2(a) and 2(b) and Figure 3(a) and 3(b), respectively, is merely the scale factor $(R+1)$. In that sense equation (31c) is merely the normalized version of equation (31d) and, as such, the former is to be preferred when comparing situations of varying numbers of satellite oscillators [5]. Using equation (9) for example and equations (31) and (33) for guidance, the modal overlap equations yield for the two examples

$$\left\{[R+1)\eta(\zeta)] = [b(\zeta)]\begin{Bmatrix}[\gamma(\bar{R})\{x(\zeta)\}^2]\\ [\zeta(1-\zeta)]^{-1}\end{Bmatrix}\right\} U(\zeta-\bar{\varepsilon})U(\bar{R}+\bar{\varepsilon}-\zeta), \tag{34a,b}$$

where in the case of equation (34a) it is recognized that

$$[x(\zeta)]^{-2} = [1+(1-2\zeta)\gamma(\bar{R})], \qquad \gamma(\bar{R}) = (\gamma/2\bar{R}) < (1/2), \tag{35}$$

(cf. equation (16a)). Equations (34a) and (34b) are depicted graphically in Figure 7(a) and 7(b) respectively. These equations and figures exemplify that $\eta(\zeta)$ cannot be independently set equal to zero without qualification. The lower limit that can be assigned to $\eta(\zeta)$ is dependent on the number R of satellite oscillators plus unity. The value of $(R+1)\eta(\zeta)$ may be large or small compared with unity, but it must remain finite if equations (30b) and (30c) are to be validated. The only way to allow for infinitesimal $\eta(\zeta)$'s; i.e., nearly undamped satellite oscillators, is to explicitly or implicitly render $(R+1)^{-1}$ proportionately infinitesimal. In particular, the number of satellite oscillators needs to approach infinity at least at the same rate that $\eta(\zeta)$ approaches infinitesimal values. Thus, once the modal overlap condition is satisfied, the satellite oscillators collectively, and without further qualifications, possess enough damping to account for the physical presence of the induced loss factor. On the other hand, when the modal overlap condition is not satisfied, averaging the undulations in the induced damping yields values that match those obtained for a modal overlap condition that is satisfied. Again, indicating that the satellite oscillators posseses enough damping to account for the physical presence of the induced loss factor [4]. This convergence of the undulations does not, however, license one to cavalierly introduce vanishing loss factors; the convergence may be conditional or may be misinterpreted [5].

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REFERENCES

1. A. PIERCE, V. W. SPARROW and D. A. RUSSELL 1995 *Journal of Acoustics and Vibration* **117**, 339–348. Fundamental structural-acoustic idealizations for structures with fuzzy internals.
2. M. STRASBERG and D. FEIT 1996 *Journal of the Acoustical Society of America* **99**, 335–344. Vibration damping of large structures by attached small resonant structures.
3. R. L. WEAVER 1996 *Journal of the Acoustical Society of America* **99**, 2528–2529. Mean and mean-square responses of a prototypical master/fuzzy structure.
4. YU. A. KOBELEV 1987 *Soviet Physics Acoustics* **33**, 295–296. Absorption of sound waves in a thin layer.
5. R. J. NAGEM, I. VELIJKOVIC and G. SANDRI 1997 *Journal of Sound and Vibration* **207**, 429–434. Vibration damping by a continuous distribution of undamped oscillators.
6. M. STRASBERG 1996 *Journal of the Acoustical Society of America* **100**, 3456–3459. Continuous structure as "fuzzy" substructures.
7. G. MAIDANIK and K. J. BECKER 1998/1999 *Journal of the Acoustical Society of America* **104**, 2628–2637. Noise control of a master harmonic oscillator coupled to a set of satellite harmonic oscillators, *Journal of the Acoustical Society of America* **106**, 3109–3118. Characterization of multiple-sprung mass for wideband noise control, *Journal of the Acoustical Society of America* **106**, 3119–3127. Criteria for designing multiple-sprung masses for wideband noise control.
8. G. MAIDANIK and J. DICKEY 1988 *Journal of Sound and Vibration* **123**, 309–314. Singly and regularly ribbed panels.
9. M. STRAUSBERG 2000 *Journal of the Acoustical Society of America* **107**, 2885. When is a "fuzzy" not a fuzzy (continue)?
10. R. H. LYON 1975 *Statistical Energy Analysis of Dynamic Systems*. Cambridge: MIT; R. H. Lyon and R. G. Dejung 1995 *Theory and Application of Statistical Energy Analysis*. Boston: Butterworth-Heinemann.
11. L. CREMER, M. HECKL and E. UNGAR 1988 *Structure-Borne Sound, Structural Vibrations and Sound Radiation at Audio Frequencies*. Berlin: Springer-Verlag; Second edition.
12. E. SKUDRZYK 1980 *Journal of the Acoustical Society of America* **67**, 1105–1135. The mean-value method of predicting the dynamic response of complex vibrators.
13. G. MAIDANIK 1974 *Journal of the Acoustical Society of America* **55**, 170–183. Influence of deviation and variations in transducers on the filtering actions of spectral filters.

14. N. B. HAASER, J. P. LASALLE and J. A. SULLIVAN 1965 *Introduction to Analysis*, Vol. 1, Chapter 13. Boston: University of Notre Dame, GINN and Co.
15. M. J. BRENNAN 1997 *Noise Control Engineering Journal* **45**, 201–207. Wideband vibration neutralizer.

APPENDIX A: A FEW ADDITIONAL REMARKS

In a steady state analysis of the dynamic characteristics of the response of structural complexes, many data sets are presented in terms of appropriately averaged values. Such presentations are ultimately acceptable as long as these mean values are all the information that is either available, sought or both. The loss factor $\eta_s(y)$ presented in Figure 6, which pertains to a modal overlap parameter b_r that is equal to two, $b_r = 2$, simultaneously fits the two categories. On the other hand, the loss factor $\eta_s(y)$ depicted in Figure 5, which pertains to a modal overlap parameter b_r that is equal to one-tenth, $b_r = 10^{-1}$, does not meet either categories. Figure 4, however, cast the data in Figure 5 in terms of mean values, thus meeting the availability category only; the data sought may demand some form of accounting for the undulations. The undulations are excursions in the values of $\eta_s(y)$, as a function of the normalized frequency y, between resonance peaks and anti-resonance nadirs. Since the size of the excursions are determined by the modal overlap parameter b_r, can a measure of the undulations be devised as a function of this parameter? It is noted that the smaller the modal overlap parameters are compared with unity, the larger the excursions are. One is reminded in this connection, however, that to a first order of approximation results obtained from averaging the undulations; e.g., the appropriately averaged values in Figure 5, as well as the results obtained by performing the integration in equation (23), e.g., the data depicted in Figure 4, are found to be independent of the individual loss factors of the satellite oscillators [1–7]. Therefore, to that approximation the mean values also lack dependence on the modal overlap parameter b_r; e.g., although there is no demand for equal values of this parameter, there is an overlap between Figures 4 and 6. On the other hand, although the values of the modal overlap parameters may be identical in Figures 4 and 5, there is no overlap between prime regions in these figures, except in a qualified sense. Indeed, when mean values only are to be reported, to avoid "a tail wagging a dog", it is to be assumed that the modal overlap parameters are set in excess of unity, unless they are otherwise specifically stated [13]. Again, when values are derived by appropriately averaging undulating data, the value of the modal overlap parameter b_r must accompany the specification of these data; e.g., when Figure 4 is meant to represent the mean values of Figure 5, the modal overlap parameter b_r must be stated as one-tenth, $b_r = 10^{-1}$. Otherwise, Figure 4 is meant to represent Figure 6 (for which $b_r = 2$) and not, at all, Figure 5.

In this connection, it transpires that the first order approximation just discussed is better the larger the number R of the satellite oscillators and the closer the modal overlap parameter b_r is to unity. Indeed, for low values of R and for large values of b_r, erosions may beset the exact data obtained via the summation in equation (6) [7]. Such simultaneous decreases in the number of satellite oscillators and increases in the modal overlap parameters may not only render the exact data dependent on these quantities, but may also violate the requirement that the contributions to the integral be dominated by the resonances in the integrand (cf. equations (23) and (24)). Reference [7] suggests that to ensure a first order approximation that is free of erosions one must require that $\bar{b}_r = [b_r(R+1)^{-1}] \ll 1$ [15]. On the other hand, equation (31a) suggests that to ensure that mean values are the prime data, b_r must exceed unity, $b_r > 1$.