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SCATTERING OF AN ACOUSTIC PLANE WAVE BY A CORRUGATED CYLINDER

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An incident time harmonic scalar wave ψ_i impinging on a cylindrical surface S gives rise to a time harmonic scattered wave ψ_s and the total field $\psi = \psi_i + \psi_s$ is solution of the Helmholtz equation in the domain D surrounding S,

$$\Delta\psi + k^2\psi = 0, \tag{1}$$

where k is the wave number of the incident field. In addition, ψ satisfies some boundary condition on S. To investigate the properties of the scattered wave, a versatile technique, with many applications to different configurations of cylindrical surfaces [1, 2], consists in expanding ψ_i and ψ_s in series of Bessel and of Hankel functions and in matching the coefficients of these expansions to satisfy the boundary condition on S.

A different point of view is adopted, upon considering that one has in fact to solve a boundary value problem of Helmholtz's equation for which integral equations with Green functions as kernels have been developed [3]. A circular cylinder is considered with axis along oz and radius a (see Figure 1) and $\psi_i = \exp(ikx)$ with time-dependence $\exp(i\omega t)$. So, one has to deal with a two-dimensional (2D) problem and can use the cylindrical co-ordinates $\mathbf{r} = (r, \phi)$. S is assumed perfectly reflecting and smooth so that ψ and G satisfy on S the Neumann boundary conditions

$$[\partial_r \psi(\mathbf{r})]_{r=a} = 0, \quad [\partial_r G(\mathbf{r}, \mathbf{r}')]_{r=a} = 0, \tag{2}$$

but one is mainly interested in a weakly corrugated perfectly conducting cylinder to be defined later. For the boundary conditions (2), the conventional integral equation of the 2D-Helmholtz equation [4, 5] due to Weber [6] takes the simple form

$$\psi(\mathbf{r}) = -\int_0^{2\pi} - \mathrm{d}\phi' \, [\psi(\mathbf{r}')\partial_{r'} G(\mathbf{r}, \mathbf{r}')]_{r'=a} \quad r \geqslant a. \tag{3}$$

To get the Green function satisfying equation (2), one starts with the Green function $G^o(\mathbf{r}, \mathbf{r}')$ for the un-bounded 2D domain which is [5, 6] the Hankel function $\mathrm{i}\mathrm{H}_0^{(1)}(k|\mathbf{r} - \mathbf{r}'|)/4$ that one writes, by using a well-known expansion of $\mathrm{H}_0^{(1)}$ [7] as

$$4G^o(\mathbf{r}, \mathbf{r}') = \mathrm{i} \sum_{n=-\infty}^{\infty} \mathrm{H}_n(kr')\mathrm{J}_n(kr) \exp[\mathrm{i}n(\phi - \phi')], \tag{4}$$

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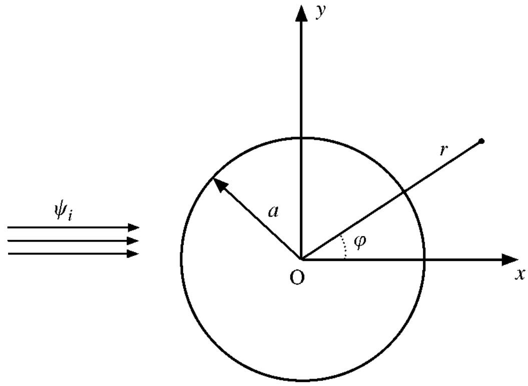

Figure 1. Geometric configuration.

in which J_n and $\mathrm{H}_n (= \mathrm{H}_n^{(1)})$ are the Bessel and Hankel functions. Then, $G(\mathbf{r}, \mathbf{r}') = G^o(\mathbf{r}, \mathbf{r}') + g(\mathbf{r}, \mathbf{r}')$ in which g is a solution of the 2D-Helmholtz equation such that G satisfies equation (2) and

$$4g(\mathbf{r}, \mathbf{r}') = -\mathrm{i} \sum_{n=-\infty}^{\infty} \mathrm{H}_n(kr')\mathrm{H}_n(kr)\mathrm{J}_n'(ka)/\mathrm{H}_n'(ka) \exp[\mathrm{i}n(\phi - \phi')]. \tag{5}$$

So finally

$$4G(\mathbf{r}, \mathbf{r}') = \mathrm{i} \sum_{n=-\infty}^{\infty} \mathrm{H}_n(kr')[\mathrm{J}_n(kr) - \mathrm{H}_n(kr)\mathrm{J}_n'(ka)/\mathrm{H}_n'(ka)] \exp[\mathrm{i}n(\phi - \phi')]. \tag{6}$$

And since $\exp(\mathrm{i}kx) = \sum_m \mathrm{iJ}_m(kr) \exp(\mathrm{i}m\phi)$ [5], one proves easily that the solution of the integral equation (3) is

$$\psi(r) = \sum_{m=-\infty}^{\infty} \mathrm{i}^m [\mathrm{J}_m(kr) - H_m(kr)\mathrm{J}_m'(ka)/\mathrm{H}_m'(ka)] \exp(\mathrm{i}m\phi), \tag{7}$$

which represents the total field for a plane wave $\exp(\mathrm{i}kx)$ incident perpendicularly to the z-axis of a perfectly reflecting circular smooth cylinder [5].

The surface of the cylinder is supposed to be described by a function $b = a + \varepsilon(\phi)$ in which the roughness function $\varepsilon(\phi)$ is small enough to make negligible the ε^2-terms. So, one has just to change a into b in relations (2) and (3) so that the integral equation becomes

$$\psi(\mathbf{r}) = -\int_0^{2\pi} \mathrm{d}\phi' [\psi(\mathbf{r}')\partial_{r'} G(\mathbf{r}, \mathbf{r}')]_{r'=b}, \quad r \geqslant b. \tag{8}$$

To get an approximate solution of the integral equation (8), a first order expansion of the integrand neglecting the ε^2-terms is used. So

$$[\psi(\mathbf{r}')]_{r'=b} = [\psi(\mathbf{r}')]_{r'=a} + \varepsilon(\phi')[\partial_{r'}\psi(r')]_{r'=a}, \quad = [\psi_0(\mathbf{r}')]_{r'=a}, \tag{9a}$$

since according to equation (2) the second term is zero, also denoting by $\psi_0(\mathbf{r})$ the solution (7) when $\varepsilon = 0$ and

$$[\partial_{r'} G(\mathbf{r}, \mathbf{r}')]_{r'=b} = [\partial_{r'} G(\mathbf{r}, \mathbf{r}')]_{r'=a} + \varepsilon(\phi')[\partial_{r'}^2 G(\mathbf{r}, \mathbf{r}')]_{r'=a}. \tag{9b}$$

Substituting equations (9a) and (9b) into equation (8) gives

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) - \int_0^{2\pi} - \mathrm{d}\phi' [\psi_0(\mathbf{r}')\partial^2_{r'} G(\mathbf{r}, \mathbf{r}')]_{r'=a}, \tag{10}$$

since $[\psi_0(\mathbf{r}')]_{r'=a}$ is the solution of the integral equation (3) while according to equation (6)

$$4[\partial^2_{r'} G(\mathbf{r}, \mathbf{r}')]_{r'=a} = \mathrm{i}ak^2 \sum_n [\mathrm{J}_n(kr)\mathrm{H}''_n(ka) - \mathrm{H}_n(kr)\mathrm{J}'_n(ka)\mathrm{H}''_n(ka)/\mathrm{H}'_n(ka)] \exp[\mathrm{i}n(\phi - \phi')]. \tag{11}$$

Now one obtains from equation (7) $[\psi(\mathbf{r}')]_{r'=a} = \sum_m \mathrm{i}^m w_m(ka) \exp(\mathrm{i}m\phi')/\mathrm{H}'_m(ka)$, in which the Wronskian $w_m(ka) = J_m(ka)\mathrm{H}'_m(ka) - \mathrm{H}_m(ka)J'_m(ka) = 2\mathrm{i}/\pi ka$ [7], so

$$[\psi(\mathbf{r}')]_{r'=a} = (2\mathrm{i}/\pi ka) \sum_m \mathrm{i}^m \exp(\mathrm{i}m\phi')/\mathrm{H}'_m(ka). \tag{12}$$

Substituting equations (11) and (12) into equation (9) gives

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + k/2\pi \int_0^{2\pi} \mathrm{d}\phi' \varepsilon(\phi') \sum_{m,n} \mathrm{i}^m F_{mn}(a, r) \exp[\mathrm{i}n\phi + \mathrm{i}(m-n)\phi'], \tag{13}$$

$$F_{m,n}(a, r) = [\mathrm{J}_n(kr)\mathrm{H}''_n(ka) - H_n(kr)\mathrm{J}'_n(ka)\mathrm{H}''_n(ka)/\mathrm{H}'_n(ka)]/\mathrm{H}'_m(ka). \tag{13a}$$

Exchanging integration and summation in equation (13) gives finally

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + (k/2\pi) \sum_{m,n} \mathrm{i}^m F_{m,n}(a, r) \exp(\mathrm{i}n\phi) \int_0^{2\pi} \mathrm{d}\phi' \varepsilon(\phi') \exp[\mathrm{i}(m-n)\phi']. \tag{14}$$

For a perfectly reflecting corrugated cylinder, one may write

$$\varepsilon(\phi) = \rho[2 - \exp(\mathrm{i}p\phi) - \exp(-\mathrm{i}p\phi)], \tag{15}$$

in which ρ is a length, small with respect to the radius of the cylinder and p an integer. With equation (15) one obtains from equation (14) the approximation

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + k\rho \sum_m [2F_{m,m}(a, r) - F_{m,m+p}(a, r)\exp(\mathrm{i}p\phi) - F_{m,n-p}(a, r)$$
$$\times \exp(-\mathrm{i}p\phi)]\,\mathrm{i}^m \exp(\mathrm{i}m\phi) \tag{16}$$

for the total field outside a weakly corrugated perfectly conducting circular cylinder on which the harmonic plane wave $\exp(\mathrm{i}kx)$ impinges.

One could also consider a perfectly conducting rough cylinder with a roughness function depending on a random number p, for instance $\varepsilon(\phi) = \rho\sin(p\phi)$. These results may be generalized to problems with boundary conditions more general than conditions (1), in particular for cylinders with a surface impedance Z so that one has $[\partial_r\psi + \mathrm{i}kZ\psi)_{r=a} = 0$ and $[\partial_r G + \mathrm{i}kZG]_{r=a} = 0$. The integral equation (3) becomes

$$\psi(\mathbf{r}) = -\int_0^{2\pi} \mathrm{d}\phi' [\psi(\mathbf{r}')\partial_{r'} G_M(\mathbf{r}, \mathbf{r}')]_{r'=a}, \tag{17}$$

where G_M is obtained from equation (6) by changing $\mathrm{J}'_n(ka)/\mathrm{H}'_n(ka)$ into $\Omega\mathrm{J}_n(ka)/\Omega\mathrm{H}_n(ka)$ in which Ω is the operator $\partial_r + \mathrm{i}kZ$.

For instance, if Z depends only on frequency [8] and if the real and imaginary parts R and X of Z can be expanded in even and odd powers, respectively, of ω, as

$$Z(\omega) = R + \mathrm{i}X = R_0 + R_2\omega^2 + \cdots + \mathrm{i}(X_1\omega + X_3\omega^3 + \cdots), \tag{18}$$

one would use similar expansions for ψ and G in order to obtain for every power of ω an integral equation and one would solve successively this system of equations.

To obtain a tractable approximation of the scattered wave by a corrugated perfectly reflecting cylinder, one may use the Debye approximations of the Bessel and Hankel functions [7], and provided that ka is large enough, one may truncate the infinite series in equation (15) after M, the integer part of ka [9]. Methods of summing the coefficients have been discussed by Jobst [10].

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