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ON THE EIGENVALUES OF A VISCOUSLY DAMPED SIMPLE BEAM CARRYING POINT MASSES AND SPRINGS

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1. INTRODUCTION

Several authors have studied the vibration of beams with attachments. Goel [1] investigated the free vibration of a beam-mass system hinged at either end by rotational springs and arbitrarily located heavy mass. Gürgöze [2] studied the natural frequencies of restrained beam and rods with point masses, also he [3] derived the characteristic equation of a Bernoulli–Euler beam carrying springs, heavy masses and viscous dampers. Chang [4] analyzed the vibration of a mass-loaded beam with a heavy tip mass by using Laplace transform. Recently, Chang and his associate [5] adopted the same method to perform the vibration analysis of a beam with a two-degrees-of-freedom (2 d.o.fs) spring–mass system. The purpose of this study is to apply Laplace Transform to determine the eigenvalues of a uniform simple beam carrying arbitrarily located point masses, translational springs and viscous dampers.

2. METHOD OF ANALYSIS

The partial differential equation of the free bending vibration for a uniform beam with point masses, springs and viscous dampers, according to Bernoulli–Euler theory, is the well-known expression (see Figure 1)

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + m\frac{\partial^2 w(x,t)}{\partial t^2} + \sum_{i=1}^{p} m_i\delta(x-x_i)\frac{\partial^4 w(x,t)}{\partial t^2} + \sum_{j=1}^{q} k_j\delta(x-x_j)w(x,t)$$
$$+ \sum_{l=1}^{s} c_l\delta(x-x_l)\frac{\partial w(x,t)}{\partial t} = 0, \tag{1}$$

where $w(x,t)$ is the transverse displacement, m_i is the ith point mass, k_j is the stiffness of the jth translational spring, c_l is the damping constant of the lth viscous damper, m is the mass per unit length of the uniform beam, E is the Young's modulus of elasticity of the beam, I is the area moment of inertia of the beam and $\delta(\cdot)$ is the Dirac delta function.

Assume a solution of equation (1) as the form

$$w(x,t) = W(x)e^{\lambda t}, \tag{2}$$

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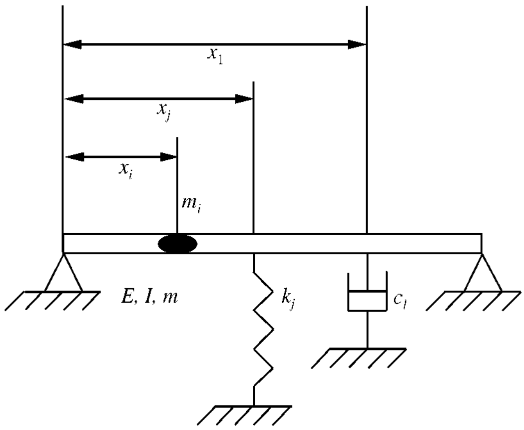

Figure 1. Viscously damped Bernoulli–Euler beam carrying point masses restrained by linear springs.

then substituting equation (2) into equation (1) gives

$$EIW'''' + m\lambda^2 W + \sum_{i=1}^{p} m_i\lambda^2\delta(x-x_i)W + \sum_{j=1}^{q} k_j\delta(x-x_j)W + \sum_{l=1}^{s} c_l\lambda\delta(x-x_l)W = 0. \tag{3}$$

The boundary conditions at $x = 0$ and L are assumed as follows:

$$W(0) = 0, \quad W''(0) = 0, \tag{4a,b}$$

$$W(L) = 0, \quad W''(L) = 0. \tag{4c,d}$$

Taking the Laplace transform on equation (3) in conjunction with the boundary condition equations (4a) and (4b), then applying the inverse Laplace transform yields

$$\begin{aligned} W(x) = {} & \frac{A}{(\sqrt{2}\beta)}\left[\cosh\left(\frac{\sqrt{2}\beta}{2}x\right)\sin\left(\frac{\sqrt{2}\beta}{2}x\right) + \sinh\left(\frac{\sqrt{2}\beta}{2}x\right)\cos\left(\frac{\sqrt{2}\beta}{2}x\right)\right] \\ & + \frac{B}{\sqrt{2}\beta^3}\left[\cosh\left(\frac{\sqrt{2}\beta}{2}x\right)\sin\left(\frac{\sqrt{2}\beta}{2}x\right) - \sinh\left(\frac{\sqrt{2}\beta}{2}x\right)\cos\left(\frac{\sqrt{2}\beta}{2}x\right)\right] \\ & + \frac{(-1)^{1/4}\beta}{2m}\sum_{i=1}^{p} m_i[\sinh(-1)^{1/4}\beta(x-x_i)) - \sin((-1)^{1/4}\beta(x-x_i))]W(x_i)H(x-x_i) \\ & + \frac{(-1)^{1/4}}{2\beta^3\alpha}\sum_{j=1}^{q} k_j[\sinh((-1)^{1/4}\beta(x-x_j)) - \sin((-1)^{1/4}\beta(x-x_j))]W(x_j)H(x-x_j) \\ & + \frac{(-1)^{1/4}}{2(m\alpha)^{0\cdot 5}\beta}\sum_{l=1}^{s} c_l[\sinh((-1)^{1/4}\beta(x-x_l)) - \sin((-1)^{1/4}\beta(x-x_l))]W(x_l)H(x-x_l), \end{aligned} \tag{5}$$

where $A = W'(0)$, $B = W'''(0)$, $\beta^4 = m\lambda^2/\alpha$, $\alpha = EI$ and $H(\cdot)$ is the Heaviside unit step function. Now substituting equation (5) into the boundary condition equations (4c) and (4d) yields the following two equations:

$$\begin{aligned}
&\frac{A}{(\sqrt{2}\beta)}\left[\cosh\left(\frac{\sqrt{2}\beta}{2}L\right)\sin\left(\frac{\sqrt{2}\beta}{2}L\right) + \sinh\left(\frac{\sqrt{2}\beta}{2}L\right)\cos\left(\frac{\sqrt{2}\beta}{2}L\right)\right] \\
&\quad + \frac{B}{\sqrt{2}\beta^3}\left[\cosh\left(\frac{\sqrt{2}\beta}{2}L\right)\sin\left(\frac{\sqrt{2}\beta}{2}L\right) - \sinh\left(\frac{\sqrt{2}\beta}{2}L\right)\cos\left(\frac{\sqrt{2}\beta}{2}L\right)\right] \\
&\quad + \frac{(-1)^{1/4}\beta}{2m}\sum_{i=1}^{p} m_i[\sinh((-1)^{1/4}\beta(L-x_i)) - \sin((-1)^{1/4}\beta(L-x_i))]W(x_i)H(L-x_i) \\
&\quad + \frac{(-1)^{1/4}}{2\beta^3\alpha}\sum_{j=1}^{q} k_j[\sinh((-1)^{1/4}\beta(L-x_j)) - \sin((-1)^{1/4}\beta(L-x_j))]W(x_j)H(L-x_j) \\
&\quad + \frac{(-1)^{1/4}}{2(m\alpha)^{0\cdot 5}\beta}\sum_{l=1}^{s} c_l[\sinh((-1)^{1/4}\beta(L-x_l)) - \sin((-1)^{1/4}\beta(L-x_l))]W(x_l)H(L-x_l) = 0,
\end{aligned} \tag{6}$$

$$\begin{aligned}
&A\left(\frac{\sqrt{2}\beta}{2}\right)\left[\sinh\left(\frac{\sqrt{2}\beta}{2}L\right)\cos\left(\frac{\sqrt{2}\beta}{2}L\right) - \cosh\left(\frac{\sqrt{2}\beta}{2}L\right)\sin\left(\frac{\sqrt{2}\beta}{2}L\right)\right] \\
&\quad + \frac{B}{\sqrt{2}\beta}\left[\cosh\left(\frac{\sqrt{2}\beta}{2}L\right)\sin\left(\frac{\sqrt{2}\beta}{2}L\right) + \sinh\left(\frac{\sqrt{2}\beta}{2}L\right)\cos\left(\frac{\sqrt{2}\beta}{2}L\right)\right] \\
&\quad + \frac{(-1)^{3/4}\beta^3}{2m}\sum_{i=1}^{p} m_i[\sinh((-1)^{1/4}\beta(L-x_i)) + \sin((-1)^{1/4}\beta(L-x_i))]W(x_i) \\
&\quad + \frac{(-1)^{3/4}}{2\beta\alpha}\sum_{j=1}^{q} k_j[\sinh((-1)^{1/4}\beta(L-x_j)) + \sin((-1)^{1/4}\beta(L-x_j))]W(x_j) \\
&\quad + \frac{(-1)^{3/4}\beta}{2(m\alpha)^{0\cdot 5}}\sum_{l=1}^{s} c_l[\sinh((-1)^{1/4}\beta(L-x_l)) + \sin((-1)^{1/4}\beta(L-x_l))]W(x_l) = 0.
\end{aligned} \tag{7}$$

Let $x = x_i$, $x = x_j$, $x = x_l$ in equation (5), thus we can obtain $W(x_i)$, $W(x_j)$ and $W(x_l)$. After inserting $W(x_i)$, $W(x_j)$ and $W(x_l)$ into equations (6) and (7) in order to eliminate $W(x_i)$, $W(x_j)$ and $W(x_l)$, therefore a system of homogeneous equations with two unknowns A and B will be achieved. The determinant of the coefficients of these equations, which is the characteristic equation, must be equal to zero for each eigenvalues λ_n ($n = 1, 2, 3, \ldots, \infty$). Finally, the characteristic equation can be expressed as

$$\begin{aligned}
&\left\{\left(\frac{1}{\sqrt{2}\beta}\right)f_1 + \left(\frac{\sqrt{2}(-1)^{1/4}}{4m}\right)\sum_{i=1}^{p} m_i h_{1i} n_{1i} \right. \\
&\quad + \left(\frac{(-1)^{1/4}}{2\alpha\beta^3}\right)\sum_{j=1}^{q} k_j h_{1i}\left[\frac{f_{2j}}{\sqrt{2}\beta} + \frac{(-1)^{1/4}\sqrt{2}}{4m}\sum_{i=1}^{p} m_i h_{2i} n_{1i}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{(-1)^{1/4}}{2(m\alpha)^{0\cdot 5}\beta}\right)\sum_{l=1}^{s} c_l n_{2l}\left[\left(\frac{1}{\sqrt{2}\beta}\right)f_{3l}+\left(\frac{(-1)^{1/4}\sqrt{2}}{4m}\right)\sum_{i=1}^{p} m_i h_{3i} n_{1i}\right.\\
&\left.+\frac{(-1)^{1/4}}{2\alpha\beta^3}\sum_{j=1}^{q} k_j f_{6j}\left(\frac{1}{\sqrt{2}\beta}f_{2j}+\frac{(-1)^{1/4}\sqrt{2}}{4m}\sum_{i=1}^{p} m_i h_{2i} n_{1i}\right)\right]\\
&\times\left\{\left(\frac{1}{\sqrt{2}\beta}\right)f_1+\frac{(-1)^{3/4}\sqrt{2}}{4m}\sum_{i=1}^{p} m_i h_{2i} n_{3i}+\left(\frac{(-1)^{3/4}}{2\beta\alpha}\right)\sum_{j=1}^{q} k_j f_{4j}\left[\left(\frac{1}{\sqrt{2}\beta^3}\right)h_{4j}\right.\right.\\
&\left.+\frac{(-1)^{1/4}\sqrt{2}}{8m}\sum_{i=1}^{p} m_i h_{2i} n_{3i}\right]+\frac{(-1)^{3/4}\beta}{2\sqrt{m\alpha}}\sum_{l=1}^{s} c_l n_{4l}\left[\frac{1}{\sqrt{2}\beta^3}f_{5l}+\frac{(-1)^{1/4}}{2\sqrt{2}\beta^2 m}\sum_{i=1}^{p} m_i h_{3i} n_{3i}\right.\\
&\left.\left.+\frac{(-1)^{1/4}}{2\beta^3\alpha}\sum_{j=1}^{q} k_j f_{6j}\left(\frac{1}{\sqrt{2}\beta^3}h_{4j}+\frac{(-1)^{1/4}}{2\sqrt{2}\beta^2}\sum_{i=1}^{p} m_i h_{2i} n_{3i}\right)\right]\right\}\\
&-\left\{\frac{1}{\sqrt{2}\beta^3}n_5+\frac{(-1)^{1/4}}{2\sqrt{2}\beta^2 m}\sum_{i=1}^{p} m_i h_{1i} n_{3i}+\frac{(-1)^{1/4}}{2\beta^3\alpha}\sum_{j=1}^{q} k_j h_{5j}\right.\\
&\times\left[\frac{1}{\sqrt{2}\beta^3}h_{4j}+\frac{(-1)^{1/4}}{2\sqrt{2}\beta^2 m}\sum_{i=1}^{p} m_i h_{2i} n_{3i}\right]+\frac{(-1)^{1/4}}{2\beta\sqrt{m\alpha}}\sum_{l=1}^{s} c_l n_{2l}\\
&\times\left[\frac{1}{\sqrt{2}\beta^3}f_{5l}+\frac{(-1)^{1/4}}{2\sqrt{2}\beta^2}\sum_{i=1}^{p} m_i h_{3i} n_{3i}+\frac{(-1)^{1/4}}{2\beta^3\alpha}\sum_{j=1}^{q} k_j f_{6j}\right.\\
&\left.\left.\times\left(\frac{1}{\sqrt{2}\beta^3}h_{4j}+\frac{(-1)^{1/4}}{2\sqrt{2}\beta^2 m}\sum_{i=1}^{p} m_i h_{2i}\right)\right]\right\}\left\{\frac{\beta}{4}h_6+\frac{(-1)^{3/4}\beta^2}{2\sqrt{2m}}\sum_{i=1}^{p} m_i h_{1i} n_{1i}\right.\\
&+\frac{(-1)^{3/4}}{2\beta\alpha}\sum_{j=1}^{q} k_j n_{6j}\left[\frac{1}{\sqrt{2}\beta}f_{2j}+\frac{\sqrt{2}(-1)^{1/4}}{4m}\sum_{i=1}^{p} m_i h_{2i} n_{1i}\right]\\
&+\frac{(-1)^{3/4}\beta}{2\sqrt{m\alpha}}\sum_{l=1}^{s} c_l n_{4l}\left[\frac{1}{\sqrt{2}\beta}f_{3l}+\frac{\sqrt{2}(-1)^{1/4}}{4m}\sum_{i=1}^{p} m_i h_{3i} n_{1i}\right.\\
&\left.\left.+\frac{(-1)^{1/4}}{2\beta^3\alpha}\sum_{j=1}^{q} k_j f_{6j}\left(\frac{1}{\sqrt{2}\beta}f_{2j}+\frac{\sqrt{2}(-1)^{1/4}}{4m}\sum_{i=1}^{p} m_i h_{2i} n_{1i}\right)\right]\right\}=0, \qquad (8)
\end{aligned}
$$

where $f_1, f_{2j}, \ldots, f_{6j}, h_{1i}, \ldots, h_{5j}, h_6, n_{1j}, \ldots, n_{6j}$ are functions of β, α, x_i, x_j, x_l and λ_n, which are presented in Appendix A. The eigenvalues λ_n can be calculated readily after equation (8) has been solved.

3. NUMERICAL RESULTS AND DISCUSSIONS

Some numerical examples are presented to demonstrate the validity of the proposed method. The system parameters used here are as follows: beam mass density $m = 341{\cdot}615$ slug/ft ($1{\cdot}6363 \times 10^4$ kg/m); beam Young's modulus $E = 576{,}000$ kips/ft^2

TABLE 1

Eigenvalues λ_n for $x_{i1} = 0$, $m_1 = 0$

Mode (n)	$\lambda_n = \mathrm{i}\omega_n$ (present)	$\omega_n = \frac{(\beta_n L)^2}{L^2}\sqrt{\frac{EI}{m}}$ (Ref. [1])
1	$\pm$ 1·356286e + 002i	1·356286e + 002
2	$\pm$ 5·425147e + 002i	5·425147e + 002
3	$\pm$ 1·220658e + 003i	1·220658e + 003

TABLE 2

Eigenvalues λ_n for $x_{i1} = L/2$, $m_1 = 0{\cdot}01mL$

Mode (n)	$\lambda_n = \mathrm{i}\omega_n$ (present)	$\omega_n = \frac{(\beta_n L)^2}{L^2}\sqrt{\frac{EI}{m}}$ (Ref. [1])
1	$\pm$ 1·342920e + 002i	1·342920e + 002
2	$\pm$ 5·425147e + 002i	5·425147e + 002
3	$\pm$ 1·208820e + 003i	1·208820e + 003

TABLE 3

Eigenvalues λ_n for $x_{i1} = L/2$, $m_1 = 0{\cdot}1mL$

Mode (n)	$\lambda_n = \mathrm{i}\omega_n$ (present)	$\omega_n = \frac{(\beta_n L)^2}{L^2}\sqrt{\frac{EI}{m}}$ (Ref. [1])
1	$\pm$ 1·237859e + 002i	1·237859e + 002
2	$\pm$ 5·425147e + 002i	5·425147e + 002
3	$\pm$ 1·127885e + 003i	1·127885e + 003

($2{\cdot}756 \times 10^{10}$ N/m^2); beam length $L = 50$ ft (15·24 m). For simplicity, p, q and s are all considered as one in the numerical computations. There are infinite number of roots in equation (8), and the first three roots are evaluated by using Secant method in conjunction with MATLAB. There are six parameters in the characteristic equation: m_1, k_1, c_1, x_{i1}, x_{j1} and x_{l1} denoting individually the point mass, spring constant, damping constant, locations of the point mass, spring and damper.

First of all, the eigenvalues of the beam carrying only the point mass is calculated. Tables 1–4 show the comparisons between the results calculated according to the present method and those of the others. As it can be seen, the eigenvalues λ_n shows an excellent agreement between the results from the proposed method and those of the other methods for different magnitude of the heavy mass. Moreover, it is very interesting to notice that the second eigenvalues are almost identical for various heavy masses. This phenomenon can be illustrated since the heavy mass is located at the middle, which happens to be the node of the second mode; therefore, it does not have any impact on the second natural frequency. Secondly, the eigenvalues of the beam with only the translation spring attached are evaluated and presented in Tables 5–7. As it can be detected from Table 7, the first eigenvalue of the system with spring ($k_1 = 23mL\omega_1^2$) is exactly four times the second

TABLE 4

Eigenvalues λ_n for $x_{i1} = L/2$, $m_1 = mL$

Mode (n)	$\lambda_n = i\omega_n$ (present)	$\omega_n = \frac{(\beta_n L)^2}{L^2}\sqrt{\frac{EI}{m}}$
1	$\pm$ 0·780493e + 002i	—
2	$\pm$ 5·425147e + 002i	—
3	$\pm$ 9·329262e + 002i	—

TABLE 5

Eigenvalues λ_n for $x_{j1} = L/2$, $k_1 = 0{\cdot}1mL\omega_1^2$

Mode (n)	$\lambda_n = i\omega_n$ (present)	$\omega_n = \frac{(\beta_n L)^2}{L^2}\sqrt{\frac{EI}{m}}$
1	$\pm$ 1·485370e + 002i	—
2	$\pm$ 5·425147e + 002i	—
3	$\pm$ 1·222167e + 003i	—

TABLE 6

Eigenvalues λ_n for $x_{j1} = L/2$, $k_1 = mL\omega_1^2$

Mode (n)	$\lambda_n = i\omega_n$ (present)	$\omega_n = \frac{(\beta_n L)^2}{L^2}\sqrt{\frac{EI}{m}}$
1	$\pm$ 2·326012e + 002i	—
2	$\pm$ 5·425147e + 002i	—
3	$\pm$ 1·235927e + 003i	—

TABLE 7

Eigenvalues λ_n for $x_{j1} = L/2$, $k_1 = 23mL\omega_1^2$

Mode (n)	$\lambda_n = i\omega_n$ (present)	$\omega_n = \frac{(\beta_n L)^2}{L^2}\sqrt{\frac{EI}{m}}$
1	$\pm$ 5·425147e + 002i	5·425147e + 002
2	$\pm$ 5·425147e + 002i	—
3	$\pm$ 1·597886e + 003i	—

eigenvalue of the system in Table 7. Physically speaking, the system in Table 7 can be considered as a beam attached by a spring with infinite spring constant, so that the span length of this system is reduced to be half of the original length of the system without any spring attached. Once again, the second eigenvalues of the system remains almost the same despite the spring constants being different; the phenomenon again is quite understood since the spring is located at the middle of the beam, which is the node of the second mode.

TABLE 8

Eigenvalues λ_n for $x_{j1} = L/2$, $c_1 = 0{\cdot}1mL\omega_1$

Mode (n)	λ_n
1	$-1{\cdot}357088e+001 \pm 1{\cdot}349884e+002i$
2	$-3{\cdot}617202e-014 \pm 5{\cdot}425147e+002i$
3	$-1{\cdot}356018e+001 \pm 1{\cdot}220340e+003i$

TABLE 9

Eigenvalues λ_n for $x_{j1} = L/2$, $c_1 = 0{\cdot}2mL\omega_1$

Mode (n)	λ_n
1	$-2{\cdot}719003e+001 \pm 1{\cdot}330399e+002i$
2	$-5{\cdot}545447e-012 \pm 5{\cdot}425147e+002i$
3	$-2{\cdot}710418e+001 \pm 1{\cdot}219386e+003i$

TABLE 10

Eigenvalues λ_n for $x_{j1} = L/2$, $c_1 = 0{\cdot}3mL\omega_1$

Mode (n)	λ_n
1	$-4{\cdot}090648e+001 \pm 1{\cdot}296947e+002i$
2	$-1{\cdot}331735e-012 \pm 5{\cdot}425147e+002i$
3	$-4{\cdot}061522e+001 \pm 1{\cdot}217792e+003i$

TABLE 11

Eigenvalues λ_n for $x_{i1} = L/2$, $x_{j1} = L/2$, $x_{l1} = L/2$, $m_1 = 0{\cdot}1mL$, $k_1 = 0{\cdot}1mL\omega_1^2$, $c_1 = 0{\cdot}1mL\omega_1$

Mode (n)	λ_n
1	$-1{\cdot}130626e+001 \pm 1{\cdot}351799e+002i$
2	$-4{\cdot}177324e-012 \pm 5{\cdot}425147e+002i$
3	$-8{\cdot}482803e+000 \pm 1{\cdot}128716e+003i$

In Tables 8–10, the eigenvalues of the system carrying the damper are obtained for different damping constants, it should be noted that the computed eigenvalues are complex numbers which are quite reasonable. Also it can be concluded that the second eigenvalues of the structure are unchanged for different damping constants. Finally, as presented in Table 11, the first three eigenvalues are calculated for the case with $m_1 = 0{\cdot}1mL$, $k_1 = 0{\cdot}1mL\omega_1^2$, $c_1 = 0{\cdot}1mL\omega_1$, $x_{i1} = L/2$, $x_{j1} = L/2$ and $x_{l1} = L/2$. In Figure 2, the undamped first natural frequency of the system, carrying a heavy mass $m_1 = mL$, is plotted against the position of the point mass. As it can be seen from the plot that the first natural frequency reaches the minimum value as the position of the mass is located at the middle, which is definitely

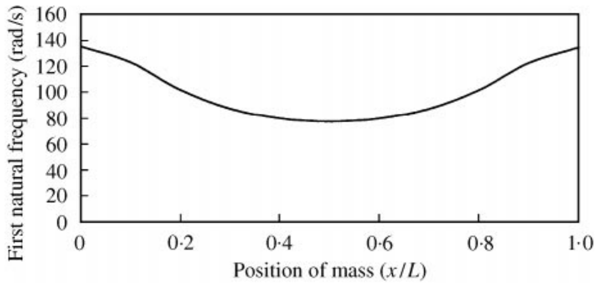

Figure 2. First natural frequency versus the position of mass.

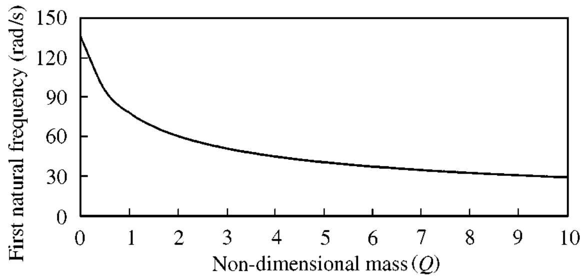

Figure 3. First natural frequency versus the non-dimensional mass.

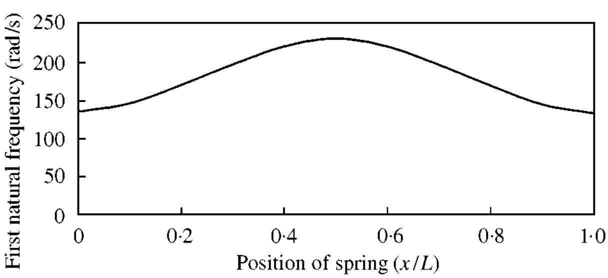

Figure 4. First natural frequency versus the position of spring.

rational since it produces largest static deflection in the middle. In Figure 3 the first natural frequency of the system with a point mass located at the middle of the beam is depicted for various non-dimensional mass $Q = m_1/mL$, it is quite acceptable that the fundamental natural frequency gets smaller as non-dimensional mass Q gets larger. In Figure 4, the undamped first natural frequency of the system, carrying a spring with spring constant $k_1 = mL\omega_1^2$, is depicted with respect to the position of the spring. It is noted that the first natural frequency becomes a maximum as the spring is located at the middle of the beam. It can be explained that the smaller static deflection of the beam is obtained when the spring is attached at the middle rather than at any other places of the beam. As it can be found from Figure 5, the first natural frequency of the system is presented for various non-dimensional spring $G = k_1/mL\omega_1^2$, it is noted that the first natural frequency becomes larger as non-dimensional spring G gets larger.

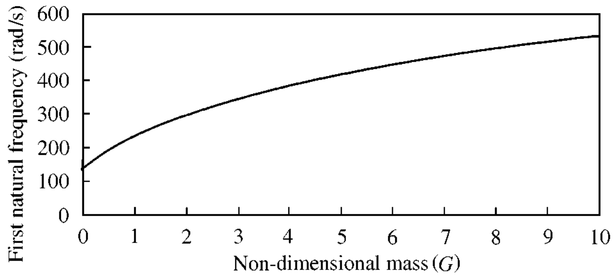

Figure 5. First natural frequency versus the non-dimensional spring.

4. CONCLUSIONS

In this paper, the natural frequencies of a Bernoulli–Euler beam carrying arbitrarily located point masses, translational springs and viscous dampers are determined by using Laplace transform with respect to the spatial variable. The various cases of different location and magnitude of spring constants and point masses are investigated to model different structural systems, it is noted that the proposed approach plays an important role in performing the analysis and design of the structures.

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APPENDIX A

$$f_1 = \cosh\left(\frac{\sqrt{2}\beta}{2}L\right)\sin\left(\frac{\sqrt{2}\beta}{2}L\right) + \sinh\left(\frac{\sqrt{2}\beta}{2}L\right)\cos\left(\frac{\sqrt{2}\beta}{2}L\right),$$

$$f_{2j} = \cosh\left(\frac{\sqrt{2}\beta}{2}x_j\right)\sin\left(\frac{\sqrt{2}\beta}{2}x_j\right) + \sinh\left(\frac{\sqrt{2}\beta}{2}x_j\right)\cos\left(\frac{\sqrt{2}\beta}{2}x_j\right),$$

$$f_{3l} = \cosh\left(\frac{\sqrt{2}\beta}{2}x_l\right)\sin\left(\frac{\sqrt{2}\beta}{2}x_l\right) + \sinh\left(\frac{\sqrt{2}\beta}{2}x_l\right)\cos\left(\frac{\sqrt{2}\beta}{2}x_l\right),$$

$$f_{4j} = \sinh((-1)^{1/4}\beta(L - x_j)) + \sin((-1)^{1/4}\beta(L - x_j)),$$

$$f_{5l} = \cosh\left(\frac{\sqrt{2}\beta}{2}x_l\right)\sin\left(\frac{\sqrt{2}\beta}{2}x_l\right) - \sinh\left(\frac{\sqrt{2}\beta}{2}x_l\right)\cos\left(\frac{\sqrt{2}\beta}{2}x_l\right),$$

$$f_{6j} = \sinh((-1)^{1/4}\beta(x_l - x_j)) - \sin((-1)^{1/4}\beta(x_l - x_j)),$$

$$h_{1i} = \sinh((-1)^{1/4}\beta(L - x_i)) - \sin((-1)^{1/4}\beta(L - x_i)),$$

$$h_{2i} = \sinh((-1)^{1/4}\beta(x_j - x_i)) - \sin((-1)^{1/4}\beta(x_j - x_i)),$$

$$h_{3i} = \sinh((-1)^{1/4}\beta(x_l - x_i)) - \sin((-1)^{1/4}\beta(x_l - x_i)),$$

$$h_{4j} = \cosh\left(\frac{\sqrt{2}\beta}{2}x_j\right)\sin\left(\frac{\sqrt{2}\beta}{2}x_j\right) - \sinh\left(\frac{\sqrt{2}\beta}{2}x_j\right)\cos\left(\frac{\sqrt{2}\beta}{2}x_j\right),$$

$$h_{5j} = \sinh((-1)^{1/4}\beta(L - x_j)) - \sin((-1)^{1/4}\beta(L - x_j))$$

$$h_6 = \sinh\left(\frac{\sqrt{2}\beta}{2}L\right)\cos\left(\frac{\sqrt{2}\beta}{2}L\right) - \cosh\left(\frac{\sqrt{2}\beta}{2}L\right)\sin\left(\frac{\sqrt{2}\beta}{2}L\right),$$

$$n_{1i} = \cosh\left(\frac{\sqrt{2}\beta}{2}x_i\right)\sin\left(\frac{\sqrt{2}\beta}{2}x_i\right) + \sinh\left(\frac{\sqrt{2}\beta}{2}x_i\right)\cos\left(\frac{\sqrt{2}\beta}{2}x_i\right),$$

$$n_{2l} = \sinh((-1)^{1/4}\beta(L - x_l)) - \sin((-1)^{1/4}\beta(L - x_l)),$$

$$n_{3i} = \cosh\left(\frac{\sqrt{2}\beta}{2}x_i\right)\sin\left(\frac{\sqrt{2}\beta}{2}x_i\right) - \sinh\left(\frac{\sqrt{2}\beta}{2}x_i\right)\cos\left(\frac{\sqrt{2}\beta}{2}x_i\right),$$

$$n_{4l} = \sinh((-1)^{1/4}\beta(L - x_l)) + \sin((-1)^{1/4}\beta(L - x_l)),$$

$$n_5 = \cosh\left(\frac{\sqrt{2}\beta}{2}L\right)\sin\left(\frac{\sqrt{2}\beta}{2}L\right) - \sinh\left(\frac{\sqrt{2}\beta}{2}L\right)\cos\left(\frac{\sqrt{2}\beta}{2}L\right),$$

$$n_{6j} = \sinh((-1)^{1/4}\beta(L - x_j)) + \sin((-1)^{1/4}\beta(L - x_j)).$$