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# A UNIVERSAL PERTURBATION TECHNIQUE FOR REANALYSIS OF GYROSCOPIC SYSTEMS WITH INTERNAL AND EXTERNAL DAMPING

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## 1. INTRODUCTION

A frequent task in dynamic analysis is to determine the changes in the eigensolutions of a system after certain modifications are introduced. Since a full reanalysis is computationally expensive even for relatively simple systems such as undamped non-gyroscopic, undamped gyroscopic and damped non-gyroscopic systems, the reanalysis for gyroscopic systems with internal and external damping can be much more expensive and time-consuming [1]. Quite often, the modifications are small, and so the perturbation theory can be applied whereby the unperturbed eigensolutions are used as a basis to extract the perturbed eigensolutions of the modified system without having to repeat an entire analysis. For the dynamic analysis of gyroscopic systems with small internal damping, Meirovitch and Ryland [2] developed a perturbation theory in which the damping matrix acts as perturbations. Later, they [3] extended the approach to the case of external damping. The two studies are limited to systems with distinct complex eigenvalues. In a recent paper, Liu [4] proposed a perturbation technique for dynamic reanalysis of systems with repeated complex eigenvalues. More recently, Liu [5] put forward a universal perturbation technique for damped gyroscopic systems with distinct, repeated and closely spaced complex eigenvalues. In the latter two techniques, complete modal expansion is needed. For large complicated systems, however, complete modal expansion inevitably encounters certain difficulties and even leads to great errors under certain circumstances.

This letter presents a matrix perturbation technique for dynamic reanalysis of gyroscopic systems with internal and external damping. The subspace condensation procedure is implemented first. The lower order perturbations of eigensolutions, i.e., complex eigenvalues and the corresponding left and right eigenvectors, are determined by solving two greatly reduced generalized eigenvalue problems. The higher order perturbations of eigensolutions are then obtained by performing a singular-value decomposition procedure for a complex matrix. The proposed perturbation method is universally applicable to general damped gyroscopic systems including all the three possible cases having distinct, repeated and closely spaced complex eigenvalues. Illustrative examples covering the three different cases are presented. The perturbed eigensolutions are computed and compared with the exact solutions.

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## 2. EIGENVALUE PROBLEMS OF UNPERTURBED AND PERTURBED GENERAL DAMPED GYROSCOPIC SYSTEMS

The equation of motion for free vibration of an $n$-degree-of-freedom (d.o.f.) gyroscopic system with internal and external damping can be written in matrix form as [1]

$$\bar{M}\ddot{u} + (\bar{C} + \bar{G})\dot{u} + (\bar{K} + \bar{H})u = 0 \tag{1}$$

where $\bar{M}$, $\bar{C}$, and $\bar{K}$ are the real symmetric $n \times n$ mass, damping and stiffness matrices, respectively, $\bar{G}$ and $\bar{H}$ are the real skew symmetric $n \times n$ gyroscopic and circulatory matrices, respectively, and $u$ is the $n$-dimensional column vector of generalized co-ordinates.

Equation (1) can be expressed in state-space form

$$K_0 x = M_0 \dot{x}, \tag{2}$$

where

$$x = [\dot{u}^{\mathrm{T}}, u^{\mathrm{T}}]^{\mathrm{T}} \tag{3a}$$

is the $2n$-dimensional state vector, and

$$K_0 = \begin{bmatrix} -(\bar{C} + \bar{G}) & -(\bar{K} + \bar{H}) \\ (\bar{K} + \bar{H}) & 0 \end{bmatrix}, \tag{3b}$$

$$M_0 = \begin{bmatrix} \bar{M} & 0 \\ 0 & \bar{K} + \bar{H} \end{bmatrix} \tag{3c}$$

are real $2n \times 2n$ general matrices.

We denote the associated eigenvalues, right eigenvectors and left eigenvectors of the original (unperturbed) system represented by equation (2) by $\lambda_{i0}$, $x_{i0}$ and $y_{i0}$ ($i = 1, 2, \ldots, 2n$). They satisfy the two eigenvalue problems

$$K_0 x_{i0} = \lambda_{i0} M_0 x_{i0}, \quad i = 1, 2, \ldots, 2n, \tag{4a}$$

$$K_0^{\mathrm{T}} y_{i0} = \lambda_{i0} M_0^{\mathrm{T}} y_{i0}, \quad i = 1, 2, \ldots, 2n. \tag{4b}$$

Without loss of generality, we assume that the solution of equation (4a) or (4b) produces repeated and/or closely spaced eigenvalues. These eigenvalues can be written as

$$\lambda_{j0} \cong \lambda_{j+1,0} \cong \cdots \cong \lambda_{k0}, \tag{5}$$

which means that the original system possesses $(k - j + 1)$ repeated and/or closely spaced complex eigenvalues.

The left and right eigenvectors are bi-orthogonal and can be normalized so as to satisfy

$$y_{i0}^{\mathrm{T}} M_0 x_{j0} = \delta_{ij}, \quad i, j = 1, 2, \ldots, 2n, \tag{6}$$

where $\delta_{ij}$ is the Kronecker delta.

To obtain unique eigenvectors, let us consider the normalizing condition

$$x_{i0}^{\mathrm{T}} x_{i0} = 1, \quad i = 1, 2, \ldots, 2n. \tag{7}$$

The superiority of equation (7) is that it can produce eigenvectors in the form of complex conjugates. In very special cases, equation (7) may cease to be effective, and then another similar normalization condition can be used instead [4, 5].

The design changes in a structural system may be reflected by the changes of $K_0$ and $M_0$ in equation (2). Since the changes are usually small, the two updated matrices can be expressed as

$$K = K_0 + K_1, \tag{8a}$$

$$M = M_0 + M_1, \tag{8b}$$

with the norms of matrices $K_1$ and $M_1$ being significantly smaller than those of $K_0$ and $M_0$ respectively. This difference in magnitude can be used to define an ordering scheme in which $K_0$ and $M_0$ are $O(0)$ quantities, while $K_1$ and $M_1$ are $O(1)$ quantities [3].

The eigenvalue problems of the perturbed system and its adjoint have the form

$$Kx_i = \lambda_i M x_i, \quad i = 1, 2, \ldots, 2n, \tag{9a}$$

$$K^{\mathrm{T}} y_i = \lambda_i M^{\mathrm{T}} y_i, \quad i = 1, 2, \ldots, 2n \tag{9b}$$

respectively. Recalling equations (8a) and (8b) yields

$$(K_0 + K_1) x_i = \lambda_i (M_0 + M_1) x_i, \quad i = 1, 2, \ldots, 2n, \tag{10a}$$

$$(K_0 + K_1)^{\mathrm{T}} y_i = \lambda_i (M_0 + M_1)^{\mathrm{T}} y_i, \quad i = 1, 2, \ldots, 2n, \tag{10b}$$

where $\lambda_i$, $x_i$ and $y_i$ are the perturbed eigenvalues, right and left eigenvectors respectively. The normalization condition and bi-orthogonality property corresponding to equations (7) and (6) are

$$x_i^{\mathrm{T}} x_i = 1, \quad i = 1, 2, \ldots, 2n, \tag{11a}$$

$$y_i^{\mathrm{T}} M x_j = y_i^{\mathrm{T}} (M_0 + M_1) x_j = \delta_{ij}, \quad i, j = 1, 2, \ldots, 2n \tag{11b}$$

respectively.

## 3. LOWER ORDER PERTURBATIONS OF EIGENSOLUTIONS

For practical use, the eigenvectors corresponding to the repeated and/or closely spaced eigenvalues are usually chosen to span two complex eigensubspaces [4, 5]

$$S = [x_{j0}, x_{j+1,0}, \ldots, x_{k0}], \tag{12a}$$

$$R = [y_{j0}, y_{j+1,0}, \ldots, y_{k0}]. \tag{12b}$$

In view of equation (6), we have

$$R^{\mathrm{T}} M_0 S = I, \tag{13}$$

where $I$ is an identity matrix of order $(k - j + 1)$.

Generally speaking, the difference between the original (unperturbed) and perturbed eigenvectors may be significant, but the angle between the original and perturbed eigensubspaces can be regarded as small [5]. As a result, the orthogonal decomposition of the perturbed eigenvectors $x_i$ and $y_i$ with respect to the original eigensubspaces can be

written as

$$x_i = Sp_i + \delta x_i, \tag{14a}$$

$$y_i = Rq_i + \delta y_i, \tag{14b}$$

$$\delta x_i \perp Sp_i, \tag{14c}$$

$$\delta y_i \perp Rq_i, \tag{14d}$$

where $p_i$ and $q_i$ are $(k-j+1)$-dimensional $O(0)$ column vectors to be determined, $\delta x_i$ and $\delta y_i$ are $2n$-dimensional $O(1)$ column vectors to be determined, and $i = j, j+1, \ldots, k$. Note that, for the sake of conciseness, the range of the index $i$ will be omitted hereafter.

Substituting equations (14a) and (14b) into equations (10a) and (10b), using the variational principle, and neglecting $O(2)$ quantities, we obtain two eigenvalue problems of $(k-j+1)$th order

$$\tilde{K}p_i = \mu_i \tilde{M} p_i, \tag{15a}$$

$$\tilde{K}^{\mathrm{T}} q_i = \mu_i \tilde{M}^{\mathrm{T}} q_i, \tag{15b}$$

where $\mu_i$ is an approximation of $\lambda_i$, $\tilde{K} = R^{\mathrm{T}}(K_0 + K_1)S$ and $\tilde{M} = R^{\mathrm{T}}(M_0 + M_1)S$.

Inserting equations (14a) and (14b) into equations (11a) and (11b), respectively, and neglecting $O(2)$ quantities, we have

$$p_i^{\mathrm{T}} S^{\mathrm{T}} S p_i = 1, \tag{16a}$$

$$q_i^{\mathrm{T}} \tilde{M} p_i = 1. \tag{16b}$$

Solving equations (15a) and (15b) and recalling equations (16a) and (16b), we can obtain $(k-j+1)$ groups of unique eigensolutions: $\mu_i$, $p_i$, $q_i$. With these, the lower order perturbations of eigensolutions $\lambda_i$, $x_i$ and $y_i$ as shown in equations (14a) and (14b) can be determined. Since there is usually only a small angle between the original and the corresponding perturbed eigensubspaces, $Sp_i$ and $Rq_i$ have errors of first order and then $\mu_i$ has only an error of second order by the generalized Rayleigh's quotient theorem [5].

## 4. HIGHER ORDER PERTURBATIONS OF EIGENSOLUTIONS

To find the higher order perturbations, the perturbed eigensolutions can be expressed as

$$\lambda_i = \mu_i + \lambda_{i2} + \cdots, \tag{17a}$$

$$x_i = Sp_i + x_{i1} + x_{i2} + \cdots, \tag{17b}$$

$$y_i = Rq_i + y_{i1} + y_{i2} + \cdots. \tag{17c}$$

Comparing equations (17b) and (17c) with equations (14a) and (14b), and considering equations (14c) and (14d), we have

$$\delta x_i = x_{i1} + x_{i2} + \cdots, \tag{18a}$$

$$\delta y_i = y_{i1} + y_{i2} + \cdots, \tag{18b}$$

$$x_{ir} \perp Sp_i, \quad r = 1, 2, \ldots, \tag{18c}$$

$$y_{ir} \perp Rq_i, \quad r = 1, 2, \ldots, \tag{18d}$$

Substituting equations (17a) and (17b) into equation (10a), collecting terms of the same order, and neglecting $O(3)$ quantities, we have

$$(K_0 - \mu_i M_0) x_{i1} = -[(K_0 + K_1) - \mu_i (M_0 + M_1)] Sp_i, \tag{19a}$$

$$(K_0 - \mu_i M_0) x_{i2} = -(K_1 - \mu_i M_1) x_{i1} + \lambda_{i2} M_0 Sp_i. \tag{19b}$$

Premultiplying equation (19b) by $(Rq_i)^T$, and considering equations (18c), (6) and (13), we obtain

$$\lambda_{i2} = \frac{q_i^T R^T (K_1 - \mu_i M_1)}{q_i^T p_i} x_{i1}. \tag{20}$$

Substituting equation (17b) into equation (11a), we have

$$(Sp_i)^T x_{i1} = 0, \tag{21a}$$

$$(Sp_i)^T x_{i2} = -\tfrac{1}{2} x_{i1}^T x_{i1}. \tag{21b}$$

Combining equations (19a) and (21a) results in

$$A x_{i1} = b_1, \tag{22}$$

where

$$A = \begin{bmatrix} K_0 - \mu_i M_0 \\ p_i^T S^T \end{bmatrix}, \tag{23a}$$

$$b_1 = \begin{bmatrix} -[(K_0 + K_1) - \mu_i (M_0 + M_1)] Sp_i \\ 0 \end{bmatrix}. \tag{23b}$$

Using the singular-value decomposition for a complex matrix, we can obtain the least-squares solution of equation (22) as

$$x_{i1} = A^+ b_1, \tag{24}$$

where the superscript " $+$ " denotes the generalized inverse [6, 7].

At this stage, we have obtained $x_{i1}$ and then $\lambda_{i2}$ by recalling equation (20). Similarly, equations (19b) and (21b) can be used to find $x_{i2}$. Working with equations (10b), (11b), (17a), (17c) and (18d), and using the same approach, we can obtain $y_{i1}$ and $y_{i2}$. The perturbations of eigensolutions of third or higher orders, even though they are of little practical value, can also be found by the same procedure as above, provided that the third order terms in the associated expansions, e.g., equations (17a)–(17c), are retained.

For the case of distinct eigenvalues, i.e., when $i \neq j, j + 1, \ldots, k$, the proposed technique is also completely applicable. In fact, we only need to choose a single right and a single left eigenvector to span the two eigensubspaces $S$ and $R$ respectively.

## 5. NUMERICAL EXAMPLES

The system shown in Figure 1 consists of a mass $m$ connected to a rigid ring through three dampers and three springs. Axes $\tilde{X}$ and $\tilde{Y}$ are inertial, and axes $X$ and $Y$ are parallel to a set of axes embedded in the ring. The ring rotates with respect to $\tilde{X}$ and $\tilde{Y}$ at a constant angular velocity $\Omega$. $c_1$, $c_2$ and $c_3$ are the coefficients of viscous damping, while $k_1$, $k_2$ and $k_3$ are the stiffnesses. The angles between each set of the dampers and springs are 120°. In addition, the mass $m$ is subjected to external damping forces proportional to the absolute velocities $\dot{\tilde{X}}$ and $\dot{\tilde{Y}}$ with the proportionality constant $h$.

By means of Lagrange's equations [1], we can obtain the free vibration equation of motion in the same form as equation (1)

$$\bar{M}\ddot{u} + (\bar{C} + \bar{G})\dot{u} + (\bar{K} + \bar{H})u = 0, \tag{25}$$

where

$$u = [X, Y]^{\mathrm{T}}, \tag{26a}$$

$$\bar{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \tag{26b}$$

$$\bar{C} = \begin{bmatrix} c_1 + \frac{1}{4}(c_2 + c_3) + h & -\frac{\sqrt{3}}{4}(c_2 - c_3) \\ -\frac{\sqrt{3}}{4}(c_2 - c_3) & \frac{3}{4}(c_2 + c_3) + h \end{bmatrix}, \tag{26c}$$

$$\bar{G} = \begin{bmatrix} 0 & -2m\Omega \\ 2m\Omega & 0 \end{bmatrix}, \tag{26d}$$

$$\bar{K} = \begin{bmatrix} k_1 + \frac{1}{4}(k_2 + k_3) - m\Omega^2 & -\frac{\sqrt{3}}{4}(k_2 - k_3) \\ -\frac{\sqrt{3}}{4}(k_2 - k_3) & \frac{3}{4}(k_2 + k_3) - m\Omega^2 \end{bmatrix}. \tag{26e}$$

$$\bar{H} = \begin{bmatrix} 0 & -h\Omega \\ h\Omega & 0 \end{bmatrix}. \tag{26f}$$

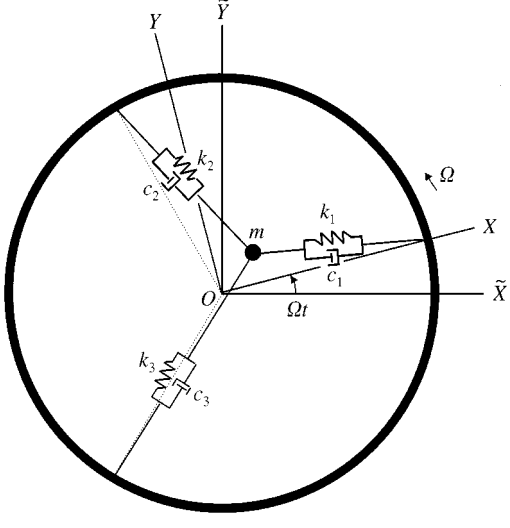


Figure 1. A general damped gyroscopic system: a mass connected to a rotating rigid ring through dampers and springs at 120° to one another.

### 5.1. CASE OF DISTINCT EIGENVALUES

Suppose that the original system has the following parameters:

$$m = 1 \text{ kg}, \quad c_1 = c_2 = c_3 = 0{\cdot}1 \text{ N s/m}, \quad k_1 = 5 \text{ N/m},$$

$$k_2 = 7 \text{ N/m}, \quad k_3 = 9 \text{ N/m}, \quad \Omega = h = 0{\cdot} \tag{27}$$

The first two eigenvalues are $\lambda_{10} = -0{\cdot}075 + 2{\cdot}96012233i$ and $\lambda_{20} = -0{\cdot}075 + 3{\cdot}49663064i$, while the eigenvalues $\lambda_{30}$ and $\lambda_{40}$ are complex conjugates of $\lambda_{10}$ and $\lambda_{20}$ respectively. Therefore, the original system has distinct eigenvalues. The eigenvalues $\lambda_{10}$ and $\lambda_{20}$ are listed in Table 1 as $O(0)$ quantities for comparison.

Assuming $c_1 = 0{\cdot}15$ N s/m, $\Omega = 0{\cdot}4$ rad/s, $h = 0{\cdot}1$ N s/m and that the other parameters in equation (27) remain unchanged, we obtain the perturbed system. The results of the perturbed eigenvalues calculated by the proposed technique and their relative errors are listed in Table 1, along with exact eigenvalues for comparison.

### 5.2. CASE OF REPEATED EIGENVALUES

If the original system has the following parameters:

$$m = 1 \text{ kg}, \quad c_1 = c_2 = c_3 = 0{\cdot}1 \text{ N s/m}, \quad k_1 = k_2 = k_3 = 5 \text{ N/m},$$

$$\Omega = h = 0, \tag{28}$$

then there are two groups of repeated eigenvalues. Now supposing that $c_1 = 0{\cdot}15$ N s/m, $\Omega = 0{\cdot}4$ rad/s, $h = 0{\cdot}1$ N s/m, and that the other parameters in equation (28) remain unchanged, we have the perturbed system. Table 2 shows the results obtained from the present technique.

### 5.3. CASE OF CLOSELY SPACED EIGENVALUES

The characteristics of the original system in this case are taken as

$$m = 1 \text{ kg}, \quad c_1 = c_2 = c_3 = 0{\cdot}1 \text{ N s/m}, \quad k_1 = 5 \text{ N/m}, \quad k_2 = k_3 = 5{\cdot}2 \text{ N/m},$$

$$\Omega = h = 0. \tag{29}$$

TABLE 1

*Eigenvalue summary for the case of distinct eigenvalues*

| | $O(0)$ | $O(0) + O(1)$ | $O(0) + O(1) + O(2)$ | Exact |
|---|---|---|---|---|
| $\lambda_1$ | $-0{\cdot}075$<br>$+ 2{\cdot}96012233i$<br>(8·190%)* | $-0{\cdot}14242515$<br>$+2{\cdot}71638284i$<br>(1·018%) | $-0{\cdot}13611622$<br>$+2{\cdot}74373990i$<br>(0·06043%) | $-0{\cdot}13454487$<br>$+2{\cdot}74320557i$ |
| $\lambda_2$ | $-0{\cdot}075$<br>$+3{\cdot}49663064i$<br>(5·986%) | $-0{\cdot}13516969$<br>$+3{\cdot}68712480i$<br>(0·6053%) | $-0{\cdot}13936644$<br>$+3{\cdot}70842626i$<br>(0·03270%) | $-0{\cdot}14045513$<br>$+3{\cdot}70896254i$ |

*Relative errors [5].

TABLE 2

*Eigenvalue summary for the case of repeated eigenvalues*

| | $O(0)$ | $O(0)+O(1)$ | $O(0)+O(1)+O(2)$ | Exact |
|---|---|---|---|---|
| $\lambda_1$ | $-0{\cdot}075$<br>$+2{\cdot}73758562i$<br>(17·83%)* | $-0{\cdot}13578823$<br>$+2{\cdot}30241572i$<br>(1·486%) | $-0{\cdot}12377856$<br>$+2{\cdot}33708297i$<br>(0·08407%) | $-0{\cdot}12471707$<br>$+2{\cdot}33535541i$ |
| $\lambda_2$ | $-0{\cdot}075$<br>$+2{\cdot}73758562i$<br>(12·89%) | $-0{\cdot}14275709$<br>$+3{\cdot}11029706i$<br>(0·8217%) | $-0{\cdot}14930673$<br>$+3{\cdot}13286889i$<br>(0·07367%) | $-0{\cdot}15028293$<br>$+3{\cdot}13496482i$ |

*Relative errors.

TABLE 3

*Eigenvalue summary for the case of closely spaced eigenvalues*

| | $O(0)$ | $O(0)+O(1)$ | $O(0)+O(1)+O(2)$ | Exact |
|---|---|---|---|---|
| $\lambda_1$ | $-0{\cdot}075$<br>$+2{\cdot}75578936i$<br>(16·34%) | $-0{\cdot}13635794$<br>$+2{\cdot}33874304i$<br>(1·441%) | $-0{\cdot}12470662$<br>$+2{\cdot}37289546i$<br>(0·07875%) | $-0{\cdot}12546119$<br>$+2{\cdot}37118448i$ |
| $\lambda_2$ | $-0{\cdot}075$<br>$+2{\cdot}79184079i$<br>(12·19%) | $-0{\cdot}14209714$<br>$+3{\cdot}14725247i$<br>(0·8032%) | $-0{\cdot}14864746$<br>$+3{\cdot}16959811i$<br>(0·07033%) | $-0{\cdot}14953881$<br>$+3{\cdot}17164571i$ |

*Relative errors.

The perturbed system is obtained by setting $c_1 = 0{\cdot}15$ N s/m, $\Omega = 0{\cdot}4$ rad/s, $h = 0{\cdot}1$ N s/m, while other parameters in equation (29) remain unchanged. The results obtained are listed in Table 3.

## 6. CONCLUSIONS

A matrix perturbation technique is presented to deal with the dynamic reanalysis of general damped gyroscopic systems. The technique can greatly reduce the computational expense because it avoids a complete computation of large-scale eigenvalue problems. Even if only part eigensolutions of the original system are available, the perturbation analysis can also be conducted because complete modal expansion has not been used in the present technique.

From the examples shown, it can be observed that, in all the three cases of eigenvalues, the first order perturbed eigensolutions obtained by the present method have satisfactory accuracy compared with the exact solutions, and the second order perturbed eigensolutions are very close to the exact solutions. Thus, it can be seen that the present technique is an effective universal perturbation technique.

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