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PRECISE TIME-STEP INTEGRATION FOR THE DYNAMIC RESPONSE OF A CONTINUOUS BEAM UNDER MOVING LOADS

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The dynamic response of a non-uniform continuous Euler–Bernoulli beam is analyzed with Hamilton's principle and the eigenpairs are obtained by the Ritz method. A high-precision integration method is used to calculate the dynamic responses of this beam. Numerical results show that the method is more accurate in the prediction of the vibration responses under the moving loads than the Newmark method.

1. INTRODUCTION

The dynamic response of a beam under moving load has been widely studied. Wu and Dai [1] used the transfer matrix method and mode superposition technique to determine the dynamic responses of multi-span non-uniform beams under moving loads. Lee [2] considered the intermediate point supports in the form of linear springs of large stiffness. Richer *et al.* [3] investigated the continuum discretization for finite element models in analyzing a moving load on an elastic beam. Henchi and Fafard [4] used an exact dynamic stiffness element in the finite element approximation to study the dynamic response of multi-span structures under a convoy of moving loads. A dynamic model coupled with a fast Fourier transformation algorithm is developed. The vibration of a multi-span non-uniform beam subjected to moving loads is analyzed by using modified beam vibration functions as the assumed modes [5].

The central difference, Newmark and Wilson-θ methods are used in the computation in all the above works. This paper applies the precise time-step integration technique [6] to study the dynamic responses of a non-uniform continuous beam under a system of moving loads. The analysis is based on Hamilton's principle with the intermediate point supports represented by very stiff linear springs. The eigenpairs are calculated with a new approach using the Ritz method. Numerical results are presented for loads moving with constant or varying speed across the beam. Comparison with the exact solution and the results by Zheng *et al.* [5] shows that this technique is accurate and practical.

2. EQUATIONS OF MOTION

Figure 1 shows a continuous Euler–Bernoulli beam with $(Q-1)$ intermediate points supports under N_f moving loads. The beam is constrained at these supports. The loads $P_s(t)$ $(s = 1, 2, \ldots, N_f)$ are moving at a prescribed velocity v along the axial direction from left to right. The initial location of the first moving load is assumed at the left end of the beam and

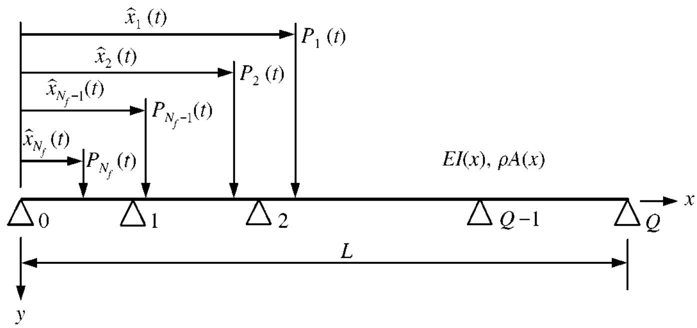

Figure 1. A continuous beam with $(Q-1)$ intermediate point supports under N_f moving forces.

l_s is the distance between $P_s(t)$ and $P_1(t)$. The load locations $\hat{x}_s(t)$ $(s = 1, 2, \dots, N_f)$ are described as follows when the moving speed is constant

$$\hat{x}_s(t) = vt - l_s \quad (l_s/v < t < (L + l_s)/v), \tag{1}$$

where L is the total length of the beam. For a more general formulation when the loads are moving at varying speed with a constant acceleration, the load locations can be written as

$$\begin{aligned} \hat{x}_s(t) &= v_0 t + \tfrac{1}{2} a t^2 - l_s, \quad (t_{s1} < t < t_{s2}), \\ t_{s1} &= \left(\sqrt{v_0^2 + 2al_s} - v_0\right)/a, \\ t_{s2} &= \left(\sqrt{v_0^2 + 2a(l_s + L)} - v_0\right)/a, \end{aligned} \tag{2}$$

where t_{s1}, t_{s2} are the time instances when $P_s(t)$ gets on and off the beam respectively, a is the constant acceleration and v_0 is the initial velocity of the group of forces.

By separation of variables, the vertical deformation of the beam $y(x, t)$ can be expressed as

$$y(x, t) = \sum_{i=1}^{n} q_i(t) Y_i(x) \quad \{Y_i(x),\ i = 1, 2, \dots, n\}, \tag{3}$$

where $\{Y_i(x), i = 1, 2, \dots, n\}$ are the vibration modes which satisfy the boundary conditions and $\{q_i(t), i = 1,2, \dots, n\}$ are the generalized co-ordinates.

From the Euler–Lagrange equation of the load on beam system, the equation of motion of a damped system can be written as follows:

$$\sum_{j=1}^{n} m_{ij} \ddot{q}_j(t) + \sum_{j=1}^{n} C_{ij} \dot{q}_j(t) + \sum_{j=1}^{n} k_{ij} q_j(t) = f_i(t) \quad (i = 1, 2, \dots, n), \tag{4}$$

where

$$\begin{aligned} m_{ij} &= \int_0^L \rho A(x) Y_i(x) Y_j(x) \, \mathrm{d}x, \\ k'_{ij} &= \int_0^L EI(x) Y''_i(x) Y''_j(x) \, \mathrm{d}x, \\ k''_{ij} &= k \sum_{l=1}^{Q-1} Y_i(x_l) Y_j(x_l), \end{aligned} \tag{5}$$

$$k_{ij} = k'_{ij} + k''_{ij},$$

$$f_i(t) = \sum_{l=1}^{N_f} P_l(t) Y_i(\hat{x}_l(t)) \quad (i = 1, 2, \dots, n, j = 1, 2, \dots, n)$$

and ρ is the density, $A(x)$ is the cross-sectional area, E is Young's modulus, $I(x)$ is the moment of inertia of the beam cross-section, x_i ($i = 0, 1, 2, \dots, Q$) are the co-ordinates of intermediate point supports or end supports, k is the vertical stiffness used to model the point constraints and C_{ij} is the damping coefficient.

Equation (4) can also be rewritten as

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = F(t), \tag{6}$$

where

$$M = \{m_{ij}, i = 1, 2, \dots, n; j = 1, 2, \dots, n\}, \quad K = \{k_{ij}, i = 1, 2, \dots, n; j = 1, 2, \dots, n\},$$
$$C = \{C_{ij}, i = 1, 2, \dots, n; j = 1, 2, \dots, n\}, \quad q(t) = \{q_1(t), q_2(t), \dots, q_n(t)\}, \tag{7}$$
$$F(t) = \{f_1(t), f_2(t), \dots, f_n(t)\}.$$

3. NATURAL FREQUENCIES AND THE ASSUMED MODE SHAPES

The maximum potential and the maximum kinetic energies can be written by applying the Ritz method as

$$U_{max} = \tfrac{1}{2} \int_0^L EI(x)(Y_i''(x))^2 \, \mathrm{d}x,$$
$$T_{max} = \frac{\omega^2}{2} \int_0^L \rho A(x)(Y(x))^2 \, \mathrm{d}x, \tag{8}$$

in which

$$\omega^2 = \frac{E}{\rho} \frac{\int_0^L I(x)(Y''(x))^2 \, \mathrm{d}x}{\int_0^L A(x)(Y(x))^2 \, \mathrm{d}x}, \tag{9}$$

where ω is the angular frequency. For a single-span simply supported beam with uniform cross-section, the vibration functions are

$$Y_{Ui}(x) = \sin\left(\frac{i\pi x}{L}\right) \quad (i = 1, 2, \dots, n), \tag{10}$$

where n is the number of vibration modes. A previous work [7] models the deflection curve in terms of a polynomial function. A new form for the deflection curve is assumed in this work which leads to a very simple solution of the eigenpairs

$$Y(x) = \sum_{i=1}^{n} \alpha_i Y_{Ui}(x), \tag{11}$$

where $\{\alpha_i = 1, 2, \dots, n\}$ are constant coefficients. Equation (11) can be proved to satisfy the boundary conditions of the non-uniform beam. The constant $\{\alpha_i, i = 1, 2, \dots, n\}$ can be found to make the integral

$$Z = \int_0^L [EI(x)(Y''(x))^2 - \omega^2 \rho A(x)(Y(x))^2]\,dx \tag{12}$$

a minimum by substituting equations (11) and (5) into equation (12), and letting

$$\frac{\partial Z}{\partial \alpha_i} = 0 \quad (i = 1, 2, \dots, n),$$

we have

$$(K^o - \omega^2 M^o)\alpha = 0, \tag{13}$$

where K^o and M^o are $n \times n$ matrices. $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is an $n \times 1$ vector, and the matrix components k_{ij}^o and m_{ij}^o are

$$\begin{aligned} k_{ij}^o &= \int_0^L EI(x)\, Y''_{Ui}(x)\, Y''_{Uj}(x)\,dx \\ m_{ij}^o &= \int_0^L \rho A(x)\, Y_{Ui}(x)\, Y_{Uj}(x)\,dx \end{aligned} \quad (i = 1, 2, \dots, n, j = 1, 2, \dots, n). \tag{14}$$

Equation (13) can therefore be written as

$$(B - \omega^2 I)\alpha' = 0, \tag{15}$$

where

$$\boldsymbol{B} = \boldsymbol{K}^o \boldsymbol{M}^{o^{-1}}, \quad \alpha' = \boldsymbol{M}^o \alpha. \tag{16}$$

Here ω^2, α' are determined as the eigenvalues and the eigenvectors of matrix $\boldsymbol{B}$ in equation (15). Then α can be calculated from equation (16), and hence $Y(x)$ from equation (11). The natural frequencies and mode shape functions are therefore determined. It is noted that equation (16) gives the exact solution on the eigenpairs for a uniform continuous beam.

4. THE HIGH-PRECISION INTEGRATION SCHEME

According to the precise time-step integration method [6], the equation of motion of the beam in equation (6) can be written as

$$\dot{u} = Hu + f, \tag{17}$$

where u is the response vector of size $2n \times 1$, H is a $2n \times 2n$ matrix, and f is the force vector of size $2n \times 1$, with

$$u = \begin{bmatrix} q(t) \\ p(t) \end{bmatrix},$$

$$H = \begin{bmatrix} -\dfrac{M^{-1}C}{2} & M^{-1} \\ -\left(K - \dfrac{CM^{-1}C}{4}\right) & -\dfrac{CM^{-1}}{2} \end{bmatrix}, \tag{18}$$

$$f = \begin{bmatrix} 0 \\ F(t) \end{bmatrix} = \begin{bmatrix} 0 \\ A(t) \end{bmatrix} P(t),$$

$$A(t) = \begin{bmatrix} Y_1(\hat{x}_1(t)) & Y_1(\hat{x}_2(t)) & \cdots & Y_1(\hat{x}_{Nf}(t)) \\ Y_2(\hat{x}_1(t)) & Y_2(\hat{x}_2(t)) & \cdots & Y_2(\hat{x}_{Nf}(t)) \\ \cdots & \cdots & \cdots & \cdots \\ Y_n(\hat{x}_1(t)) & Y_n(\hat{x}_2(t)) & \cdots & Y_n(\hat{x}_{Nf}(t)) \end{bmatrix},$$

$$p(t) = M\dot{q}(t) + \frac{Cq(t)}{2}.$$

Matrix $A(t)$ is obtained from equation (11). Equation (17) can be written into discrete equations using the exponential matrix representation. Integrating equation (17), we can have

$$u(t) = \mathrm{e}^{H(t-t_0)} u(t_0) + \int_{t0}^{t} \mathrm{e}^{H(t-\tau)} f(\tau)\,\mathrm{d}\tau. \tag{19}$$

Expressing equation (19) in discrete form

$$u((j+1)h) = \mathrm{e}^{Hh} u(jh) + \int_{jh}^{(j+1)h} \mathrm{e}^{H((j+1)h-\tau)} f(\tau)\,\mathrm{d}\tau, \tag{20}$$

where h is the time step of integration. The force $f(\tau)$ is assumed to be constant within the time interval from jh to $(j+1)h$,

$$\begin{aligned} u((j+1)h) &= \mathrm{e}^{Hh} u(jh) + \left[\int_0^h \mathrm{e}^{H\tau'}\,\mathrm{d}\tau'\right] f(jh) \\ &= \mathrm{e}^{Hh} u(jh) + H^{-1}[\mathrm{e}^{Hh} - I] f(jh), \end{aligned} \tag{21}$$

and the final discrete model for the $(j+1)$th step is rewritten as

$$u_{j+1} = \exp(Hh) u_j + H^{-1}(\exp(Hh) - I) f_j \quad (j = 0, 1, 2, \ldots). \tag{22}$$

The precision of integration depends on the accuracy of $\exp(Hh)$. The $2N$ algorithm presented by Zhong *et al.* [8] is used and $\exp(Hh)$ has the form

$$\begin{aligned} \exp(Hh) &= \left[\exp\left(H\frac{h}{N_t}\right)\right]^{N_t} \\ &= [\exp(H\Delta t)]^{N_t}, \end{aligned} \tag{23}$$

where $\Delta t = h/N_t$, $N_t = 2^N$ and N can be any positive integer. Since h is not large and Δt would be extremely small, a truncated Taylor expansion of $\exp(H\Delta t)$ may be used.

$$\begin{aligned} \exp(H\Delta t) &\approx I + H\Delta t + \frac{(H\Delta t)^2}{2!} + \frac{(H\Delta t)^3}{3!} + \frac{(H\Delta t)^4}{4!} \\ &= I + R_O, \end{aligned} \tag{24}$$

where

$$R_0 = H\Delta t + \frac{(H\Delta t)^2}{2!} + \frac{(H\Delta t)^3}{3!} + \frac{(H\Delta t)^4}{4!},$$

and R_i at the ith step of computation can be proved to take the form

$$R_i = 2R_{i-1} + R_{i-1}R_{i-1} \quad (i = 1, 2, \ldots, N).$$

Then

$$\exp(Hh) = [\exp(H\Delta t)]^{N_t} \approx I + R_N \tag{25}$$

The term $\exp(Hh)$ can be computed from equations (24) and (25), and the vibration response can be calculated from equation (22). The computed results are compared with those from the Newmark method using the same time step h in equation (23), and the number of data points in the computation should be a multiple of two. It is noted that the accuracy of $\exp(Hh)$ and the vibration response u depend on the size of time step $\Delta t = h/N_t$ adopted, and there is no convergence error involved in the final results.

5. SIMULATION AND RESULTS

5.1. RELIABILITY AND ACCURACY OF THE PROPOSED METHOD

A simply supported uniform beam subjected to the excitation of a moving load is considered. The cross-sectional area and the material density of the beam are, respectively, $1{\cdot}146 \times 10^{-3}$ m^2 and 7700 kg/m^3. The overall length is one meter and Young's modulus is $2{\cdot}07 \times 10^{-5}$ MPa. The speed of the moving load is 17·3 m/s. Computation of the responses was done using the first 12 vibration modes with the integration time step h equal to $9{\cdot}0315 \times 10^{-4}$ s. The number of data used is 64 and $N = 9$. Δt equals $h/2^9 = 1{\cdot}76396 \times 10^{-6}$ s, and the Taylor series expansion for $\exp(H\Delta t)$ contains infinitesimal approximation errors. Figure 2 shows the results obtained from using the proposed method, the Newmark method and the exact solution [9]. The deflection under the moving load has been normalized with the static deflection when the load is at midspan. A comparison of the computation error and computer time from the precise method and the Newmark method is also presented in Table 1. The computation error is defined as

$$Error = \frac{\|x - x_{exact}\|}{\|x_{exact}\|} 100\%,$$

where x and x_{exact} are the computed result and the exact solution respectively.

Both the curves from the precise integration method and the Newmark method closely match with the exact solution. The computation errors are almost the same for both methods as seen in Table 1. Table 1 also shows that the computation errors from both methods are comparable for different time steps of integration, but the computer time required in the precise method is only one-quarter of that in the Newmark method. The precise method also gives a larger error than the Newmark method when a very large time step is used, as seen in the last row of Table 1. This would indicate that a large time step should go together with a larger N value in the computation.

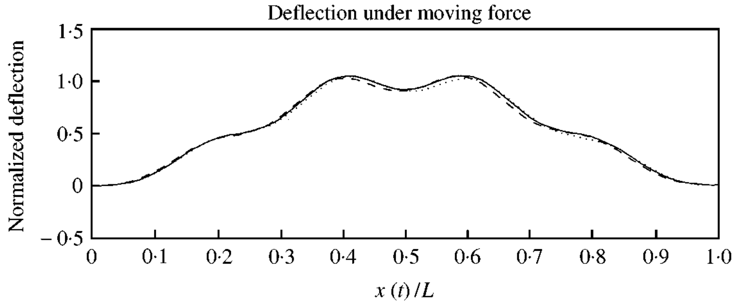

Figure 2. Deflection of beam at midspan under the moving load: ——, exact solution; – – –, precise integration; · · · ·, Newmark method.

TABLE 1

Comparison between two Methods

No. of data	N	Time step(s)	Precise method		Newmark method	
			Error (%)	Time (s)	Error (%)	Time (s)
500	9	$1·1561 \times 10^{-4}$	1·4910	2·25	1·5250	10·65
200	9	$2·8902 \times 10^{-4}$	1·5411	0·99	1·6806	4·01
128	9	$4·5159 \times 10^{-4}$	1·6078	0·60	1·9091	2·52
64	9	$9·0318 \times 10^{-4}$	2·5074	0·25	2·8021	1·31
32	9	0·0018	7·2589	0·16	4·9796	0·66

5.2. VIBRATION RESPONSE OF MULTI-SPAN NON-UNIFORM BEAM UNDER MOVING LOADS

Consider the problem of moving loads on a three-span girder of variable depth as shown in Figure 3. The material density and Young's modulus are, respectively, 2400 kg/m^3 and 30 000 MPa. The group of forces consists of four forces each of 450 kN with a spacing as shown in Figure 3, and it travels at a constant speed of 17 m/s across the bridge. The vertical stiffness representing the intermediate supports is taken as $1·0 \times 10^{16}$ kNm. The first 12 modes are used in the calculation with the integration time step equal to $1·47058 \times 10^{-2}$ s. 256 data are used with $N = 8$ which cover the duration when the forces are on the beam. Lee's method [2] is modified to solve this problem with a non-uniform beam. The deflections at the middle of the three spans obtained from the modified Lee's method, Zheng's method [5] and the present method are shown in Figure 4. They are all very close to each other, indicating that the accuracy of the proposed approach is comparable to those in existing methods.

6. CONCLUSIONS

A method is proposed to study the dynamic responses of a non-uniform continuous beam under the action of moving forces. A new approach to determine the eigenpairs using the Ritz method is proposed. A precise integration method is applied to obtain the response time histories in which a more economical computer effort could be achieved with the same time step of integration compared with the Newmark method.

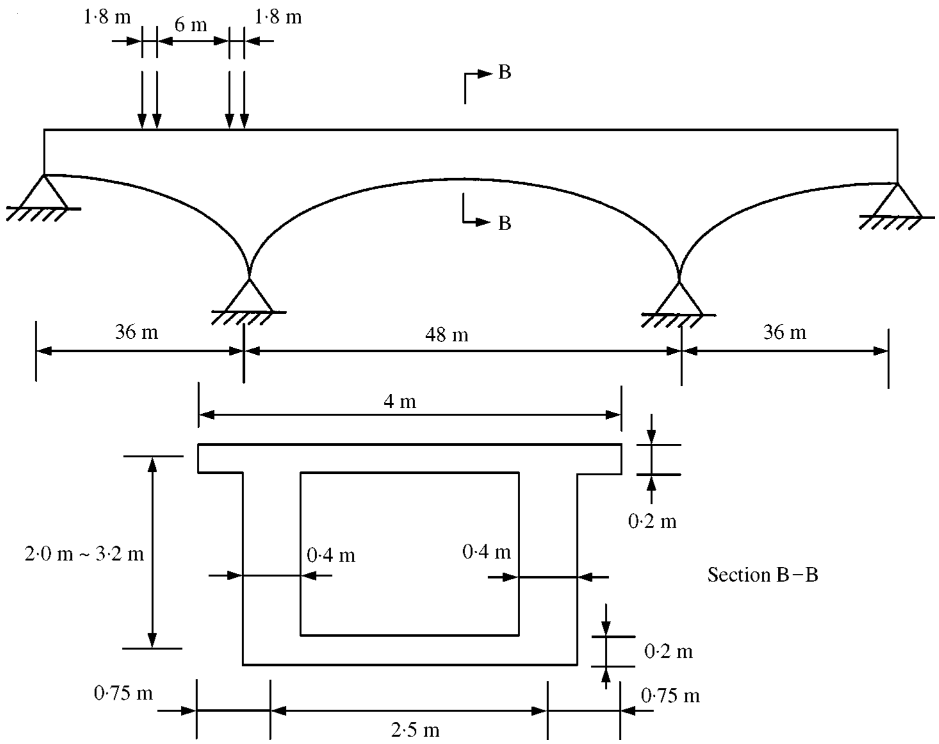

Figure 3. Continuous girder with varying depth.

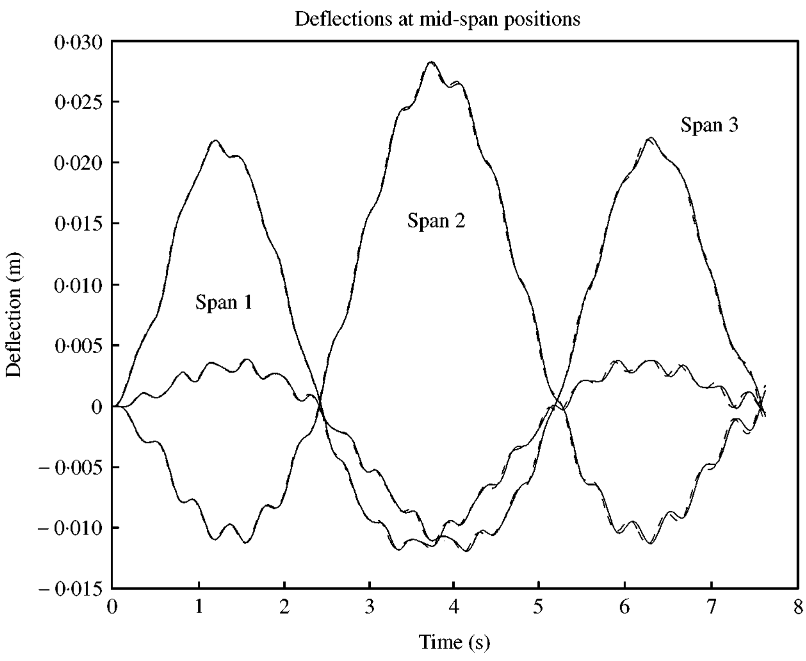

Figure 4. Deflection at the middle of the three spans: ——, modified Lee's method; – – –, precise integration; · · · ·, Zheng's method.

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