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PROPAGATION AND SCATTERING OF ACOUSTIC WAVES IN A TURBULENT MEDIUM

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Formulae for differential and total scattering cross-sections per unit volume of a medium have been derived using the von Kármán correlation function of medium inhomogeneities. The scattering cross-sections have been studied in terms of their dependence on the mean scale of inhomogeneities in comparison to the wavelength, and also on the parameter ν of the von Kármán correlation function. It has been shown that the effective medium acoustic refractive index does not depend on the form of its fluctuations correlation function if the mean scale of inhomogeneities is large in comparison to the wavelength of acoustic waves, and if the Bourret approximation for the effective wave number operator is used.

1. INTRODUCTION

Problems connected with elastic wave propagation in randomly inhomogeneous media have been intensively studied in the last years [1]. This is due to a large extent to their wide-ranging applications in various branches of science: acoustics, biophysics, optics, geophysics, hydroacoustics, physics of the atmosphere, etc. [2, 3]. The scattering of acoustics waves by medium inhomogeneities and interferences of the primary wave with the scattered waves, as well as interferences between the scattered waves, cause stochastic changes of the amplitude and phase of the wave, and changes in the effective velocity of wave propagation [4–6] as well as in the character of wave energy transport.

The character of wave energy transport in random media depends on the value of the mean free path l of the wave in comparison with the distance travelled by the wave [1, 7, 8]. The mean free path of the wave has been defined by Sornette [1] as the reciprocal of the total scattering cross-section per unit volume of the medium. Three different regimes of wave energy transport in randomly inhomogeneous media can be distinguished: wave transport of energy with effective wave velocity, $c(\omega)$, different from velocity in an homogeneous medium [1], diffusive transport of wave energy, and the regime of wave energy localization. The wave energy transport is propagative if the distance travelled by the wave is smaller than the elastic mean free path l of the wave. In the case of distances travelled by the wave being larger than l, the character of energy propagation is diffusive with a coefficient of diffusion $D_0 = c(\omega)\, l(\omega)/3$ [1, 9], where ω denotes the cyclic frequency of the wave. In strongly inhomogeneous media the so-called Anderson localization of wave energy can take place [7–11] provided the Ioffe–Regel condition $\lambda \approx l$ is fulfilled. The

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Anderson wave energy localization is caused by the coherent interferences of waves scattered backwards at an angle of 180°. These interferences cause the diffusion coefficient of the wave energy to decrease to zero.

This work is devoted to studies of the acoustic waves scattering by turbulent media in which the correlation of fluctuations in the acoustic refractive index is described by the von Kármán function. The influence of other forms of correlation functions on the effective acoustic refractive index of a medium has also been studied using the Green function method and the Bourret approximation for calculation of the effective wave number operator.

2. STATISTICAL CHARACTERISTICS OF RANDOMLY INHOMOGENEOUS MEDIA

The acoustical properties of a random medium can be characterized by the mean-square fluctuations $\langle \varepsilon^2 \rangle$ of the medium acoustic refractive index and by the autocorrelation function [2] of these fluctuations $\langle \varepsilon(\mathbf{x}, t)\varepsilon(\mathbf{x} + \mathbf{r}, t) \rangle$, where the bracket $\langle \ \rangle$ denotes averaging over statistical ensembles. The refractive index is defined by $n = c_0/c = 1 + \varepsilon$, where c_0 is a reference wave speed, c is the acoustic wave speed, and ε denotes the fluctuating part of the refractive index. In the following analysis, it is assumed that the field is a temporally stationary and spatially homogeneous one. In this case, the autocorrelation function is a function of the magnitude of r only and can be written as $\int_0 \langle \varepsilon(o)\varepsilon(r) \rangle$. If the inhomogeneities in a medium are caused by turbulence, correlation between the inhomogeneties is described by the von Kármán [5, 12, 18] function $\Psi(r)$,

$$\Psi(r) = \frac{\langle \varepsilon^2 \rangle}{2^{\nu-1}\Gamma(\nu)} \left(\frac{r}{a}\right)^{\nu} \mathrm{K}_{\nu}\left(\frac{r}{a}\right), \tag{1}$$

where $\Gamma(\nu)$ denotes the Euler gamma function, ν is a number (the parameter of the von Kármán function) and a denotes the radius of correlation of inhomogeneities, i.e., the mean distance over which fluctuations in the acoustic refractive index are correlated. $\mathrm{K}_{\nu}(r/a)$ is a Bessel function of second kind of imaginary argument.

In considerations of elastic waves propagation in random media, the Fourier transform $\Phi(\kappa)$ of the correlation function $\Psi(r)$ is also needed:

$$\Phi(\kappa) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi(r)\, \mathrm{e}^{\mathrm{i}\boldsymbol{\kappa}\mathbf{r}}\, \mathrm{d}^3\,\mathbf{r}. \tag{2}$$

In isotropic turbulent media formula (2), after introducing a spherical co-ordinate system and integration over the angles, takes the form

$$\Phi(\kappa) = \frac{1}{2\pi^2\kappa} \int_0^{\infty} \sin(\kappa r)\, \Psi(r)\, r\, \mathrm{d}r, \tag{3}$$

and one obtains the following formula for the Fourier transform of the von Kármán function [12, 13]:

$$\Phi(\kappa) = \frac{\langle \varepsilon^2 \rangle \Gamma(\nu + 3/2)\, a^3}{\pi^{3/2}\Gamma(\nu)(1 + \kappa^2 a^2)^{\nu + 3/2}}\,. \tag{4}$$

3. THE DIFFERENTIAL SCATTERING CROSS-SECTION PER UNIT VOLUME OF A RANDOM MEDIUM

The differential scattering cross-section is defined as the ratio of wave energy flux scattered by unit volume of a medium per unit solid angle in the direction Θ, to the incident energy flux per unit surface area [2, 13, 14]. In the case of acoustic waves, the differential scattering cross-section of unit volume of a randomly inhomogeneous medium is given by the formula [2, 15]

$$\sigma(\Theta) = 2\pi k^4 \Phi(2k \sin(\Theta/2)) \cos^2 \Theta, \tag{5}$$

where $k = \omega/c_0$ is the wave number, Θ is the scattering angle and Φ denotes the Fourier transform of the correlation function of the inhomogeneities. From formulae (4) and (5) the following expression for the differential scattering cross-section is obtained:

$$\sigma(\Theta) = \frac{2\langle\varepsilon^2\rangle \Gamma(\nu + 3/2)\, k^4 a^3 \cos^2 \Theta}{\pi^{1/2}\Gamma(\nu)(1 + 4k^2a^2 \sin^2(\Theta/2))^{\nu+3/2}}. \tag{6}$$

In Figure 1, plots of the above formula have been presented as a function of ka, for various values of angle Θ and $\nu = \frac{1}{3}$, normalized to the average value of the square of the refractive index fluctuations $\langle\varepsilon^2\rangle$. The normalization has been introduced because there is little data on values of $\langle\varepsilon^2\rangle$ in the literature. (The only values the authors have identified are to be found in references [14, p. 235, 16, p. 7]). The value $\nu = \frac{1}{3}$ has been chosen, because this value is often quoted in the literature, for example [5, p. 18, 13, p. 40]. One can see from Figure 1, that $\sigma(\Theta)a/\langle\varepsilon^2\rangle$ depends weakly on the scattering angle Θ for $ka < 1$, while for $ka > 1$ the dependence is strong. The value of $\sigma(\Theta)a/\langle\varepsilon^2\rangle$ decreases with increasing scattering angle Θ in the range $0° < \Theta < 90°$, and it increases as the scattering angle further

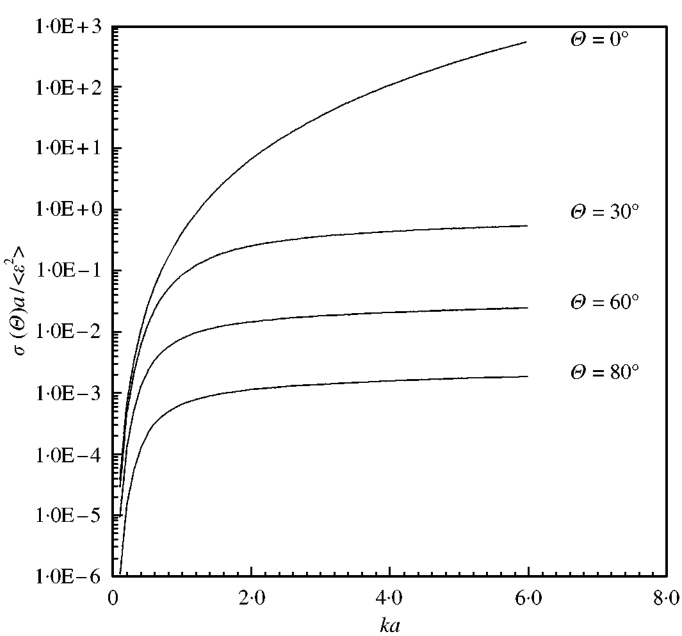

Figure 1. Differential scattering cross-section per unit volume of a turbulent medium, normalized to the average value of the square of the refractive index fluctuations, for a von Kármán correlation function with $\nu = 1/3$ and various values of scattering angle.

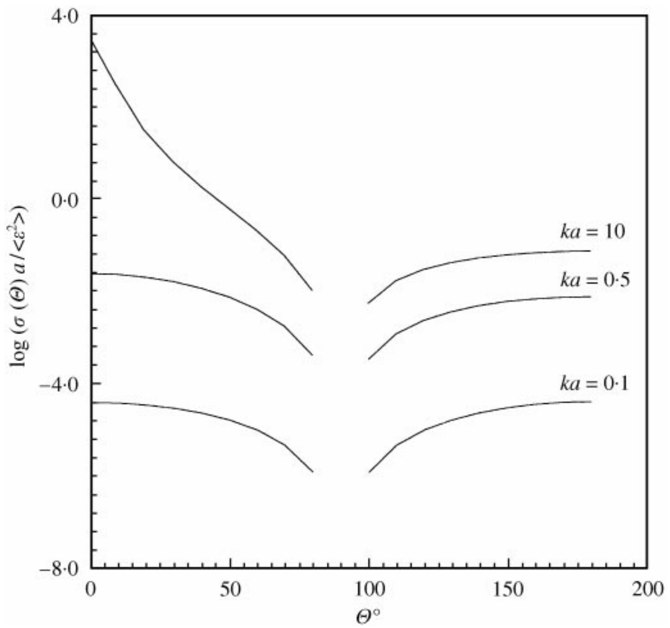

Figure 2. Differential scattering cross-section per unit volume of a turbulent medium normalized to the average value of the square of the refractive index flluctuations, as a function of scattering angle for a von Kármán correlation function with $v = \frac{1}{3}$, and for various values of the acoustic radius, ka.

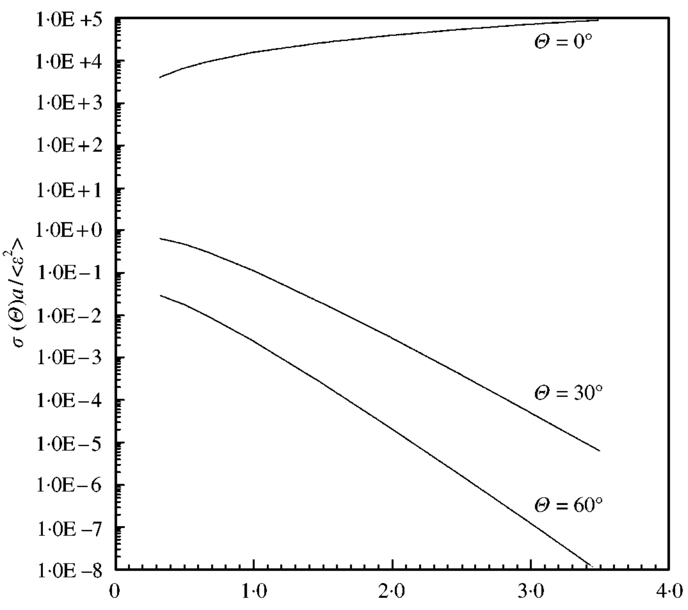

Figure 3. Differential scattering cross-section per unit volume of a turbulent medium, normalized to the average value of the square of the refractive index fluctuations, as a function of the parameter v of the von Kármán correlation function, for various scattering angles and $ka = 10$.

increases in the range $90^\circ < \Theta < 180^\circ$, as can be seen in Figure 2. Values of $\sigma(\theta)a/\langle\varepsilon^2\rangle$ increase as the parameter, v, of the von Kármán correlation function increases, for small scattering angles Θ, and decreases for larger scattering angles, as can be seen from Figure 3.

4. THE TOTAL ACOUSTIC SCATTERING CROSS-SECTION PER UNIT VOLUME OF A TURBULENT MEDIUM

The total scattering cross-section per unit volume of a turbulent medium (which was defined in section 2 as the reciprocal of the mean-free path of the wave) is obtained by integrating the differential scattering cross-section over the solid angle 4π. Introducing the wave number of the scattered wave $k_s = 2k\sin(\Theta/2)$, an element of solid angle $\mathrm{d}\Omega = \sin\Theta\,\mathrm{d}\Theta\,\mathrm{d}\phi$ can be written as a function of k_s: $\mathrm{d}\Omega = k_s\,\mathrm{d}k_s\,\mathrm{d}\phi/k^2$ and equation (5) takes the form

$$\sigma(k_s) = 2\pi k^4\left(1 - \frac{k_s^2}{k^2} + \frac{k_s^4}{4k^4}\right)\Phi(k_s), \tag{7}$$

while the total scattering cross-section is

$$\sigma_s = \frac{2\pi}{k^2}\int_0^{2k}\sigma(k_s)\,\Phi(k_s)\,k_s\,\mathrm{d}k_s, \tag{8}$$

where the Fourier transform of the medium inhomogeneities correlation function is given by formula (4).

By making the appropriate calculations the following formula has been obtained for the total scattering cross-section:

$$\frac{\sigma_s a}{\langle\varepsilon^2\rangle} = 2\pi A k^2 a^2\left(\frac{1-M^{2\nu+1}}{2\nu+1}(1-k^{-2}a^{-2}) - \frac{1-M^{2\nu-1}}{k^2a^2(2\nu-1)} + \frac{1-M^{2\nu-3}}{k^4a^4(2\nu+1)(2\nu-3)}\right)$$
$$- 2\pi A k^2 a^2\left(\frac{4M^{2\nu+1}}{2\nu+1} + \frac{1-M^{2\nu-1}}{k^4a^4(4\nu^2-1)}\right), \tag{9}$$

where $A = 2\Gamma(\nu+3/2)/\pi^{1/2}\Gamma(\nu)$ and $M = (1+4k^2a^2)^{-1/2}$. This appears to be a novel result in the literature. However, its complexity suggest that a computational approach is appropriate for further understanding.

In Figure 4 plots of $\sigma_s a/\langle\varepsilon^2\rangle$, are presented as functions of ka, for various values of the parameter, ν, of the von Kármán correlation function. They can be useful in the estimation of the structure of turbulence, giving the possibility of estimating the value of the parameter ν from an experimental plot of $\sigma_s a/\langle\varepsilon^2\rangle$ as a function of ka. The authors are unaware of any such plots in the literature, or of any published data that could be used to construct them.

5. VELOCITY OF PROPAGATION OF ACOUSTIC WAVES IN A TURBULENT MEDIUM FOR $ka \gg 1$

Using the Green function method in studies of acoustic wave propagation in randomly inhomogeneous media one obtains the following formula for the complex effective wave number [6, 7]:

$$k_e = k\left(1 + \frac{\pi k}{4}\int_0^\infty x\ln\left(\frac{2k+x}{2k-x}\right)^2\Phi(x)\,\mathrm{d}x + \frac{\mathrm{i}\pi^2k^2}{2}\int_0^{2k}\Phi(x)\,x\,\mathrm{d}x\right). \tag{10}$$

Here k denotes the wave number in an homogeneous medium, and $\Phi(x)$ is the Fourier transform of the autocorrelation function of fluctuations in the medium refractive index for

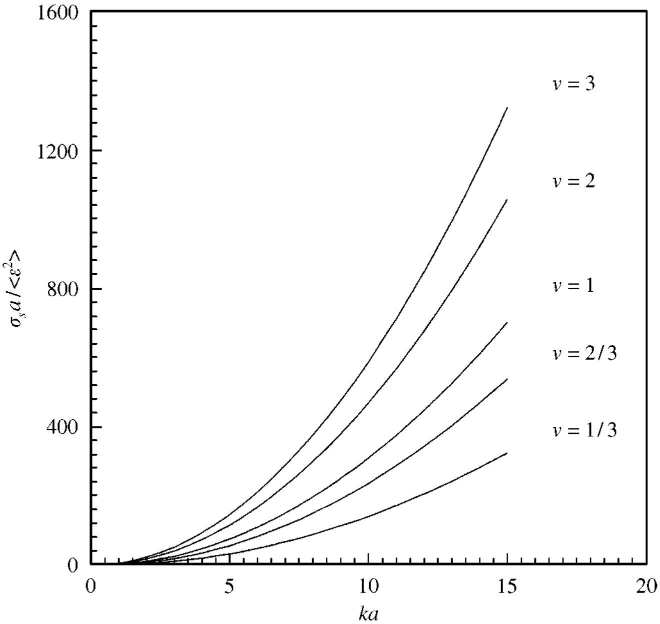

Figure 4. Total scattering cross-section per unit volume of a turbulent medium as a function of *ka*, for various values of parameter *v* of the von Kármán correlation function, normalized to the average value of the square of refractive index fluctuations.

acoustic waves. In deriving the above formula the Bourret approximation [8, 17] has been used, i.e., the effective wave number operator $Q(r)$ has been approximated by the first term of its series expansion:

$$Q(r) = k^4 G_0(\mathbf{r}, \mathbf{r}_0)\,\Psi(r) \tag{11}$$

where

$$G_0(\mathbf{r}, \mathbf{r}_0) = (-\exp(ik\,|\mathbf{r}-\mathbf{r}_0|))/(4\pi\,|\mathbf{r}-\mathbf{r}_0|). \tag{12}$$

From equation (10) one can obtain the complex refractive index of a medium, $n = k_e/k = n_1 + \mathrm{i}\, n_2$ where n_1 describes changes in the velocity of the acoustic wave, while n_2 is connected with the attenuation of the mean acoustic field in a randomly inhomogeneous medium. If the mean scale of inhomogeneities is larger than the wavelength of acoustic radiation, using the approximation [17]

$$\frac{1}{4}\ln\left(\frac{2k+x}{2k-x}\right)^2 \approx \frac{x}{2k}$$

one obtains [17] from formula (10) the following equation:

$$n_1 = 1 + \frac{\pi}{2}\int_0^\infty \Phi(\kappa)\kappa^2\,\mathrm{d}\kappa. \tag{13}$$

By using the above equation, the real part of the acoustic refractive index has been calculated for randomly inhomogeneous media with continuous changes in the refractive index, for which the correlation function has a Gaussian form [12]:

$$\Psi(r) = \langle \varepsilon^2 \rangle \,\mathrm{e}^{-r^2/a^2}, \qquad \Phi(\kappa) = \frac{\langle \varepsilon^2 \rangle}{8\pi^{3/2}} \,\mathrm{e}^{-\kappa^2 a^2/4}, \tag{14}$$

and also for media with discontinuous changes in the acoustic refractive index whose fluctuations have a correlation function of the form [12]

$$\Psi(r) = \langle \varepsilon^2 \rangle \,\mathrm{e}^{-r/a}, \qquad \Phi(\kappa) = \frac{\langle \varepsilon^2 \rangle a^3}{\pi^2 (1 + \kappa^2 a^2)}. \tag{15}$$

In both cases one obtains the result

$$n_1 = 1 + \langle \varepsilon^2 \rangle / 8. \tag{16}$$

For a turbulent medium with a von Kármán correlation function of fluctuations in the acoustic refractive index, equations (4) and (13) yield

$$n_1 = 1 + \frac{\Gamma(\nu + 3/2)\langle \varepsilon^2 \rangle}{\pi^{1/2}\Gamma(\nu)} \int_0^\infty \frac{x^2}{(1 + x^2)^{\nu + 3/2}} \,\mathrm{d}x. \tag{17}$$

The integral in the above formula has been calculated by using the Euler integral of the second kind (the so-called beta function) [18], defined as

$$B(x, y) = \int_0^\infty \frac{t^{x-1}}{(1 + t)^{x+y}} \,\mathrm{d}t. \tag{18}$$

It is easy to show that

$$\int_0^\infty \frac{x^2}{(1 + x^2)^{\nu + 3/2}} \,\mathrm{d}x = \frac{1}{2} B\left(\frac{3}{2}, \nu\right). \tag{19}$$

By using the relation between Euler beta and gamma functions [18, 19]

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}, \tag{20}$$

the following equation has been obtained:

$$\int_0^\infty \frac{x^2}{(1 + x^2)^{\nu + 3/2}} \,\mathrm{d}x = \frac{1}{2} \frac{\Gamma(3/2)\Gamma(\nu)}{\Gamma(3/2 + \nu)}, \tag{21}$$

and subsequently, from formulae (16) and (20)

$$n_1 = 1 + \frac{\langle \varepsilon^2 \rangle}{4\pi^{1/2}} \Gamma(3/2). \tag{22}$$

However, $\Gamma(\frac{3}{2}) = \sqrt{\pi/2}$ [19], so equation (21) leads to the formula

$$n_1 = 1 + \langle \varepsilon^2 \rangle / 8, \tag{23}$$

the same result as equation (16).

6. CONCLUSIONS

The above analysis shows that the real part of the acoustic refractive index of a randomly inhomogeneous media does not appear to depend on the form of the correlation function of the medium inhomogeneities if the wavelength of the applied acoustic radiation is small in comparison with the mean scale of inhomogeneities (in isotropic random media), and if the Bourret approximation in the Green function method is used. Measurement of effective speed of acoustic waves in random media makes it possible to estimate the value of the mean-square fluctuation $\langle \varepsilon^2 \rangle$ in the refractive index, if $ka > 1$. It results from equation (23) that the effective speed of acoustic waves in a randomly inhomogeneous medium is smaller than that in an homogeneous one.

The differential scattering cross-section per unit volume of a turbulent medium (with a von Kármán correlation function) increases with increasing ka values. If $ka < 1$, the increase in $\sigma(\Theta)a/\langle \varepsilon^2 \rangle$ with increasing ka values is especially rapid as can be seen in Figures 1 and 2. The total scattering cross-section per unit volume for an acoustic wave of unit volume of a turbulent medium also increases with increasing ka values. The rate of this increase is dependent on the value of the parameter, v, of the von Kármán correlation function. This, in principle, makes it possible to estimate the parameter v experimentally.

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