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A DIFFERENTIAL QUADRATURE ANALYSIS OF VIBRATION FOR RECTANGULAR STIFFENED PLATES

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Structures consisting of thin plates stiffened by a system of ribs or diaphragms form a class of structural elements of practical importance in various engineering applications. A differential quadrature analysis of free vibration of plates with eccentric stiffeners is presented. The plate and the stiffeners are treated separately. Simultaneous governing differential equations are derived from the plate dynamic equilibrium, the stiffener dynamic equilibrium, and equilibrium and compatibility conditions along the interface of a plate segment and a stiffener. The plate and the stiffeners have displacements in three dimensions. Shear forces and in-plane forces in the plate are considered to satisfy the compatibility at the interface of a plate segment and a stiffener. Meanwhile, in-plane inertia effects in the plate and in the stiffener are ignored. The application of the differential quadrature method is demonstrated by three examples: a simply supported plate with central eccentric stiffener, a clamped square plate with central eccentric stiffener, and a double-ribbed plate with all edges clamped. The natural frequencies are compared with the experimental results, and with the results obtained by finite element analysis. Very good agreement was found.

1. INTRODUCTION

Structures consisting of thin plates stiffened by a system of ribs or diaphragms form a class of structural elements of practical importance in various engineering applications, such as aircraft, ship superstructures, bridge decks and ribbed floors. Many stiffened plates are designed to resist vibration due to dynamic loads. A brief literature survey reveals that a variety of methods has been proposed to study the vibration of stiffened plates. The most common method used in early literature was to approximate the stiffened plates as equivalent orthotropic plates, using the smeared stiffener approach. In more recent literature, with the help of high-speed computers, the plate and the stiffeners were treated separately. Such numerical methods as the finite element method and the finite difference method are widely used. Mukherjee and Mukhopadhyay [1] provided a comprehensive study of the literature.

Originated by Bellman and Casti [2], the differential quadrature (DQ) method is a numerical technique for initial- or/and boundary-value problems. It is based on the approximation of a function and hence its partial derivatives with respect to the space variables, within a domain, by a linear sum of function values at all discrete grid points. Recently, the DQ method has been applied to such structural problems as deflection, buckling, and free vibration of beams and plates. The method is accurate and computationally efficient. It has been projected as a potential alternative to the conventional numerical methods such as finite element and finite difference methods. An extensive review of the application of the DQ method in computational mechanics was

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provided by Bert and Malik [3] in 1996. In this paper, the DQ method is further extended to study the free vibration of eccentrically stiffened plates. Comparisons are made with the available numerical and experimental results. Very good agreement is achieved.

2. FORMULATION

2.1. FORMULATION OF THE PLATE AND THE BEAM

The plate and the stiffeners are formulated separately. Each plate segment is considered to be uniform and isotropic, and to satisfy the Kirchhoff hypothesis. The stiffener is considered to be a beam subjected to stress resultants from adjoining plate segments. Co-ordinates x, y, and z are for plate segments, while $\bar{x}$, $\bar{y}$, $\bar{z}$, and θ, where θ is the torsional displacement, are for stiffeners (Figure 1).

Correspondingly, the plate segments have displacements of u, v, and w, and the stiffeners have $\bar{u}$, $\bar{v}$, and $\bar{w}$. Vertical dynamic equilibrium of a plate segment leads to

$$w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy} + \frac{\rho h}{D}\ddot{w} = 0, \tag{1}$$

where a subscript represents the partial derivative with respect to the variables, ρ is the density of the plate, h is the thickness of the plate, and D is the flexural rigidity of the plate, $D = Eh^3/[12(1-\nu^2)]$, in which E is Young's modulus and ν is the Poisson ratio of the plate material.

The in-plane motions are uncoupled from the bending motion. With the in-plane inertia ignored, the differential equations governing in-plane motions are

$$u_{,xx} + \tfrac{1}{2}(1+\nu)v_{,xy} + \tfrac{1}{2}(1-\nu)u_{,yy} = 0, \tag{2}$$

$$\tfrac{1}{2}(1-\nu)v_{,xx} + \tfrac{1}{2}(1+\nu)u_{,xy} + v_{,yy} = 0. \tag{3}$$

An X-type (perpendicular to x-axis) stiffener element between the ith and the jth plate segments is shown in Figure 2. The equilibrium of forces in $\bar{x}$, $\bar{y}$, and $\bar{z}$ direction leads to

$$-EI_z\bar{u}_{,\bar{y}\bar{y}\bar{y}\bar{y}} + (N_x)_j - (N_x)_i + d[(N_{xy,y})_i + (N_{xy,y})_j] = 0, \tag{4}$$

$$2dhE\bar{v}_{,\bar{y}\bar{y}} + (N_{xy})_j - (N_{xy})_i = 0, \tag{5}$$

$$EI_x\bar{w}_{,\bar{y}\bar{y}\bar{y}\bar{y}} + 2dt\rho\ddot{\bar{w}} + (Q_x)_i - (Q_x)_j + e[(N_{xy,y})_i - (N_{xy,y})_j] = 0 \tag{6}$$

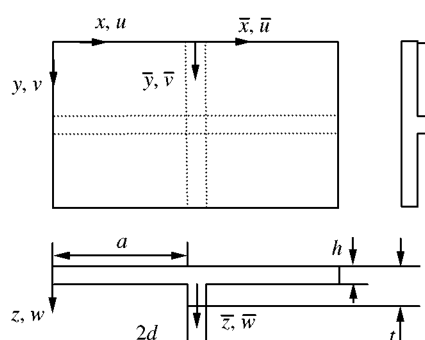

Figure 1. A typical stiffened plate.

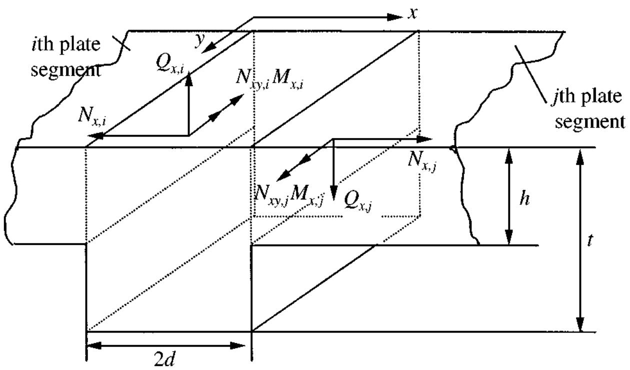

Figure 2. An X-type stiffener element.

and the equilibrium of moments in the x–z plane requires

$$J\theta_{,\bar{y}\bar{y}} + 2dt\rho\ddot{\bar{w}} + d[(Q_x)_i + (Q_x)_j] + e[(N_x)_i - (N_x)_j] + (M_x)_i - (M_x)_j = 0, \tag{7}$$

where $e = (t - h)/2$, J is the stiffener torsional stiffness, EI_z and EI_x are the stiffener bending stiffness in the $\bar{x}$–$\bar{y}$ and the $\bar{y}$–$\bar{z}$ planes respectively, the N's and Q's are in plane and transverse shear force resultants, and the M's are moment resultants. Similar equilibrium equations can be written for Y-type stiffeners (perpendicular to the y-axis).

The force and moment resultants can be expressed in terms of displacements [4].

2.2. BOUNDARY CONDITIONS

The boundary conditions for an edge of a thin plate are as follows.

(a) *Simply supported edges* (S). For $x = 0$ and a:

$$w = 0,\ w_{,xx} = 0,\ v = 0, \quad N_y = 0. \tag{8}$$

For $y = 0$ and a:

$$w = 0, \quad w_{,yy} = 0, \quad v = 0, \quad N_x = 0. \tag{9}$$

(b) *Clamped edges* (C). For $x = 0$ and a:

$$w = 0, \quad w_{,x} = 0, \quad v = 0, \quad u = 0. \tag{10}$$

For $y = 0$ and a:

$$w = 0, \quad w_{,y} = 0, \quad v = 0, \quad u = 0. \tag{11}$$

(c) *Stiffened edges* (T). The deformation at the interface where a stiffener and a plate segment join must be continuous. Therefore, at the X-type T-edges, at $x = 0$:

$$w = \bar{w} + d\theta, \quad w_{,x} = \theta, \quad u = \bar{u} - e\theta, \quad v = \bar{v} - e\bar{w}_{,\bar{y}} - d\bar{u}_{,\bar{y}}, \tag{12}$$

at $x = a$:

$$w = \bar{w} - d\theta, \quad w_{,x} = \theta, \quad u = \bar{u} - e\theta, \quad v = \bar{v} - e\bar{w}_{,\bar{y}} + d\bar{u}_{,\bar{y}}, \tag{13}$$

2.3. DQ REPRESENTATION OF GOVERNING DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

The DQ representation is demonstrated by a representative example of an X-type single-ribbed plate. The plate is divided into two plate segments 1 and 2, and a stiffener. Each plate segment is of $N_y \times N_x$ grid, and the X-type stiffener is of N_y grid. In vectorized form the displacements are

$$P = [p_{11} \quad p_{21} \cdots p_{N_y 1} \quad p_{12} \cdots p_{N_y 2} \cdots p_{N_y N_x}]^{\mathrm{T}}, \tag{14}$$

$$Q = [q_1 \cdots q_{N_y}]^{\mathrm{T}}, \tag{15}$$

where $p = w_1, u_1, v_1, w_2, u_2, v_2$, and $q = \bar{w}, \bar{u}, \bar{v}, \theta$. Also, p_{ij} is the corresponding displacement of a plate segment at (x_j, y_i), while q_i is the corresponding displacement of the stiffener at the y_i grid point. Then, a global vector, which contains all the displacements involved, can be defined as

$$V_G = [W_1 \quad W_2 \quad \bar{W} \quad U_1 \quad V_1 \quad U_2 \quad V_2 \quad \bar{U} \quad \bar{V} \quad \Theta]^{\mathrm{T}}. \tag{16}$$

Equations (1)–(7) for the plate and the stiffener can be assembled as the product of a global matrix M_G and the global vector V_G:

$$M_G V_G = \left[\frac{\rho h}{D} \omega^2 W_1 \quad \frac{\rho h}{D} \omega^2 W_1 \quad \rho \omega^2 \bar{W} \quad 0 \cdots 0 \right]^{\mathrm{T}} \tag{17}$$

or

$$M'_G V_G = [\omega^2 W_1 \quad \omega^2 W_1 \quad \omega^2 \bar{W} \quad 0 \cdots 0 \quad]^{\mathrm{T}}. \tag{18}$$

The governing differential equations (18) are replaced, at the boundary points, with the boundary conditions and rearranged; the resulting equations are

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{Bmatrix} V_{G1} \\ V_{G2} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \omega^2 \{V_{G2}\} \end{Bmatrix}. \tag{19}$$

The eigenvalues, which provide the natural frequencies, can be solved by inverse iteration with shifting.

3. NUMERICAL EXAMPLES

Three previously reported experimental and theoretical examples are analyzed by the DQ method for comparison. A 11×11 grid is taken for each plate segment, and 11 grid points for a stiffener.

The first example is a single-ribbed plate simply supported on all edges. A rectangular plate with one central X-type stiffener is considered. The geometry and the material properties are as follows: $a = 0{\cdot}6$ m, $b = 0{\cdot}41$ m, $h = 0{\cdot}00633$ m, $2d = 0{\cdot}0127$ m, $t = 0{\cdot}2855$ m, $E = 211$ GPa, $\nu = 0{\cdot}3$, $\rho = 7830$ kg/m^3. Table 1 compares the results by the DQ method and those presented by Harik and Guo (FEM) [5], Aksu (FDM) [6], Bhimaraddi *et al.* (FEM) [7], and Mukherjee and Mukhopadhyay (FEM) [8]. The agreement is excellent.

The second example is a single-ribbed square plate clamped on all edges. This example was analyzed experimentally and theoretically by Olson and Hazell [9]. Harik [5], Bhimaraddi *et al.* [7], and Mukherjee and Mukhopadhyay [8] used FEM with different elements and mesh to study its free vibration. The geometry and the material properties are as follows: $a = b = 0{\cdot}2032$ m; $h = 0{\cdot}0013716$ m, $2d = 0{\cdot}00635$ m, $t = 0{\cdot}0140716$ m, $E = 68{\cdot}7$ GPa, $\nu = 0{\cdot}29$, $\rho = 2823$ kg/m^3. The comparison of the results of the present study and the published results is given in Table 2.

The double-ribbed stiffened plate with all edges clamped, analyzed by Olson and Hazell [9] theoretically and experimentally and by Mukherjee and Mukhopadhyay [8] and Holopainen [10] theoretically is selected as the third example. The aluminum alloy plate has the geometry as shown in Figure 3, and material properties as follows: $E = 68{\cdot}9$ GPa,

TABLE 1

Eigenfrequencies (Hz) for single-ribbed plate with all edges simply supported

Mode	Reference [5]	Reference [6]	Reference [7]	Reference [8]	DQ
1	253·59	254·94	250·27	257·05	252·16
2	282·02	269·46	274·49	272·10	275·44
3	513·50	511·64	517·77	524·70	522·99

Note: All the results except for DQ are obtained from reference [5].

TABLE 2

Eigenfrequencies (Hz) for single-ribbed plate with all edges clamped

		Reference [9]				
Mode	Reference [5]	Experimental	Theoretical	Reference [7]	Reference [8]	DQ
1	697·0	689	718·1	700·4	711·8	702·1
2	730·3	725	751·4	737·0	768·2	735·1
3	927·8	1376	997·4	966·6	1016·5	1004·7
4	1302·9	2069	1419·8	1380·1	1465·2	1403·6

Note: All the results except for DQ are obtained from reference [5].

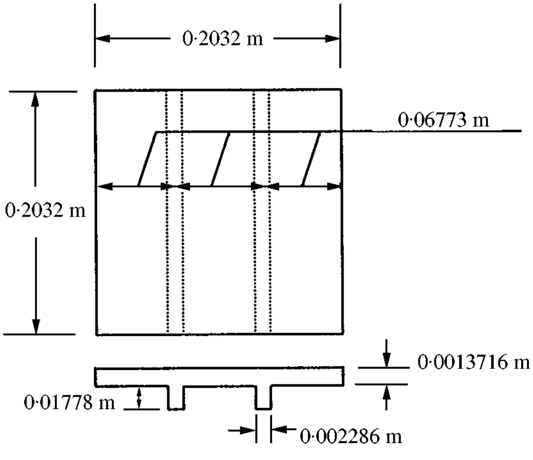

Figure 3. Double-ribbed square plate with all edges clamped.

TABLE 3

Eigenfrequencies (Hz) for double-ribbed plate with all edges clamped

		Reference [9]			
Mode	Reference [10]	Experimental	Theoretical	Reference [8]	DQ
1	943·8	909	965·3	966·4	915·9
2	1237·9	1204	1272·3	1247·7	1242·2
3	1331·0	1319	1364·3	1396·4	1344·4
4	1361·2	1506	1418·1	1481·0	1414·1

Note: All the results except DQ are obtained from reference [10].

$v = 0{\cdot}3$, $\rho = 2670\ \mathrm{kg/m^3}$. The results (see Table 3), are obtained with very good agreement with other authors' results.

4. CONCLUSIONS

A differential quadrature analysis for the free vibration of eccentrically stiffened plates is presented. The plate and stiffeners are separated at the interface with equilibrium and continuity conditions satisfied. For each plate segment or stiffener, differential quadrature equivalents are obtained to represent the governing differential equations and boundary conditions. The differential quadrature method has several advantages including numerical accuracy and computational efficiency, which are shown by several examples with different boundaries. Compared to other numerical methods such as finite difference and finite element methods, to obtain accuracy of similar level, the DQ method decreases computational efforts significantly. The results are compared with published experimental or theoretical results. Very good agreement is found.

REFERENCES

1. A. MUKHERJEE and M. MUKHOPADHYAY 1986 *The Shock and Vibration Digest* **18**, 3–8. A review of dynamic behavior of stiffened plates.
2. R. BELLMAN and J. CASTI 1971 *Journal of Mathematical Analysis and Applications* **34**, 235–238. Differential quadrature and long-term integration.
3. C. W. BERT and M. MALIK 1996 *Applied Mechanics Reviews* **49**, 1–28. Differential quadrature method in computational mechanics: a review.
4. R. SZILARD 1974 *Theory and Analysis of Plates: Classical and Numerical Methods.* Englewood Cliffs, NJ: Prentice-Hall.
5. I. E. HARIK and M. GUO 1992 *Computers and Structures* **49**, 1007–1014. Finite element analysis of eccentrically stiffened plates in free vibration.
6. G. AKSU 1982 *Journal of Applied Mechanics* **49**, 206–212. Free vibration analysis of stiffened plates by including the effect of in-plane inertia.
7. A. BHIMARADDI, A. J. CARR and P. J. MOSS 1982 *Computers and Structures* **33**, 295–305. Finite element analysis of laminated shells of revolution laminated stiffeners.
8. A. MUKHERJEE and M. MUKHOPADHYAY 1988 *Computers and Structures* **30**, 1303–1317. Finite element free vibrations of eccentrically stiffened plates.
9. M. D. OLSON and C. R. HAZELL 1977 *Journal of Sound and Vibration* **50**, 43–61. Vibration studies of some integral rib-stiffened plates.
10. T. P. HOLOPAINEN 1995 *Computers and Structures* **56**, 993–1007. Finite element free vibration analysis of eccentrically stiffened plates.