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TRANSIENT ACOUSTIC DIFFRACTION AND RADIATION BY AN AXISYMMETRICAL ELASTIC SHELL: A NEW STATEMENT OF THE BASIC EQUATIONS AND A NUMERICAL METHOD BASED ON POLYNOMIAL APPROXIMATIONS

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This work deals with an axisymmetrical shell composed of a cylindrical shell closed by two hemispherical shells made of the same material and with the same thickness. The shell is immersed into a homogeneous perfect fluid extending to infinity. The first part is devoted to the establishment of the equations governing the shell vibrations. The method used, which, in the authors' opinion, is not quite new, is based on the expansion of the elasticity equations into a Taylor series of the transverse variable: by using the same degree of expansion, the equations obtained for the cylindrical part and for the spherical parts are consistent (they correspond to the Donnell and Mushtari approximation). The first interest of this analysis is that the continuity conditions along the junction lines between the cylindrical and the spherical parts are immediately obtained. The main problem is to obtain the boundary conditions satisfied by the hemispherical shells displacement at the apexes. Indeed, due to the use of spherical co-ordinates—which is a quite natural choice—the coefficients of the equations become singular at the apexes and boundary conditions are required to express that an apex is a mechanically regular point. The method that is used here enables one to obtain such a result which, to the authors' knowledge, is new. The transient response of the system shell/external fluid is sought as a series of its resonance modes, that is its free oscillations. The main difficulty is to obtain a numerical approximation of the resonance modes: their calculation leads to solving the Fourier transform of the system of homogeneous equations. The numerical method for solving the problem is the following. The acoustic pressure is described by a hybrid layer potential, the density of which is approximated by a linear combination of orthogonal polynomials. Each component of the shell displacement is approximated by a linear combination of polynomial functions: these functions are chosen as linear combinations of orthogonal polynomials which satisfy the same continuity and boundary conditions as the shell displacement components. In the first step, the resonance frequencies are calculated. Then the coefficients of the corresponding resonance mode expansion are deduced. The validity and the efficiency of this approach will be shown in a second article through comparisons between numerical predictions and experimental results.

1. INTRODUCTION

The structure considered here is often called *Line* 2′ in the French literature. It is composed of a thin elastic cylinder closed at both ends by two thin elastic hemispherical end-caps. It is immersed in a perfect fluid which extends to infinity. Its interior is a vacuum.

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The study of the vibro-acoustics response of such a structure has various motivations. When the fluid is water, if the ratio of the total length to the diameter is about 2, the structure corresponds to a model of an underwater mine. For larger values of this ratio, it corresponds to the idealization of a torpedo or a submarine. A long *Line* 2′ structure filled up with air and immersed in the same gas can be considered as the idealization of an airplane fuselage.

Although few publications are devoted to the transient acoustic diffraction by *Line* 2′ shells, different approaches have already been used to solve closely related problems. A coupled finite element—boundary element method is applied to the analysis, in the frequency domain, of the acoustic diffraction by spherical shells in reference [1] and a *Line* 2′ shell in reference [2]. More details are given about the numerical method in reference [3]. In the field of acoustic scattering, the resonances of *Line* 2′ structures can also be estimated by the phase-matching method [4] from the propagation of "surface waves" which produce, for certain frequencies called the resonance frequencies, standing waves over the circumference of the structure.

The present work deals with the response of the structure to a transient incident wave. This is not at all a restriction: the method which is proposed applies for any transient excitation and can easily be extended to any random excitation (as, for example, the wall pressure induced by an external turbulent flow). Basically, the response of the system is sought as a series expansion of the resonance modes, that is to say of the free oscillations. The advantage of the resonance modes, compared with the *in vacuo* structure modes, is that there are intrinsically related to the physical properties of both the structure and the fluid. The main numerical difficulty is to compute these modes (at least, a sufficient number of them).

The second section is devoted to the establishment of a thin-shell approximation of the elasticity equations. The first motivation is that it is necessary to use the same degree of approximation for the cylindrical and the hemispherical parts: the method used here leads to the Donnell and Mushtari approximation. The main point concerns the spherical end-caps. Indeed, it is natural to express the shell equations in spherical co-ordinates. But at the apexes—the points $\theta = 0$ and π in spherical co-ordinates—boundary conditions are required to express that these points are *ordinary* points, that is to say points where displacement components, forces, momentum, etc. are finite. This result which, to our knowledge, is new, is deduced from the expression of the shell strain energy.

In the next section, it is shown that the response of the system vibrating structure/surrounding fluid can be expressed in terms of the resonance modes. By using a Fourier series with respect to the angular variable of the natural cylindrical co-ordinate system and a boundary integral representation of the diffracted acoustic pressure, the equations governing the resonance mode are reduced to a sequence of systems of integro-differential equations of one variable only.

In section 4, a numerical method is proposed: it is based on polynomial approximations. The layer density used for the acoustic pressure representation is expanded into a truncated series of Legendre polynomials. The components of the shell displacement are expanded into truncated series of polynomial functions. These functions are finite linear combinations of Legendre polynomials which are chosen to satisfy (1) the continuity conditions along the junction lines between the cylindrical shell and the spherical ones, and (2) the boundary conditions at the apexes. The coefficients of these truncated series are solutions of a system of Ritz–Galerkin equations. But, due to a classical property of orthogonal polynomials, the Ritz–Galerkin equations are replaced by a system of collocation equations, whose matrix is less time consuming to compute.

In the last section, numerical results are mentioned which can be found in reference [5] and, with much more detail, in reference [6]. These results will be presented in a forthcoming paper.

2. EQUATIONS GOVERNING THE SHELL VIBRATIONS

2.1. THE *LINE* 2′ SHELL: GEOMETRY AND ASSOCIATED CO-ORDINATE SYSTEMS

The structure is composed of three elementary thin shells Σ_1, Σ_2 and Σ_3. The elements Σ_1 and Σ_3 are two identical hemispherical end-caps which close the extremities of a cylindrical element Σ_2. They all have the same mean radius R and the same thickness h which is assumed to be a few percent of R; the length of the cylindrical part is $2L$.

The three elements are made of the same material, characterized by a density ϱ_s, a Young's modulus E and the Poisson ratio ν. A co-ordinate system is associated with each element (see Figure 1).

The method used here to establish the equations which govern the vibrations of the cylindrical and the spherical elements is based on the following assumptions.

1. The thickness h of the shell is small compared to its other dimensions and to the wavelengths involved.
2. As a consequence, it is assumed that the various mechanical quantities—displacement components, strain and stress tensors components—can be approximated by a low order Taylor series of the transverse variable r which varies from $-h/2$ to $h/2$.

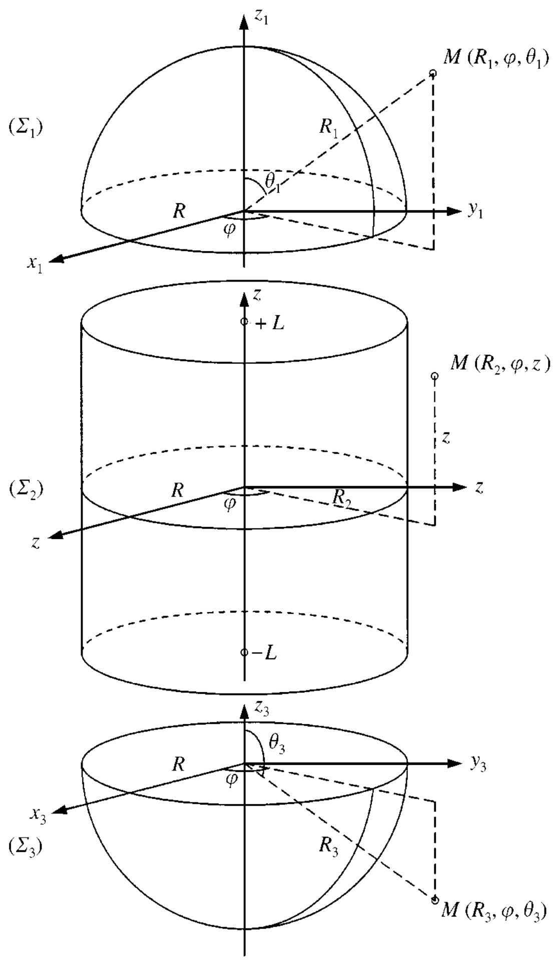

Figure 1. Geometry of *Line* 2′ and the three co-ordinate systems.

3. The external surfaces of the shell $r = \pm h/2$ are free (the forces are negligible compared to the stresses' approximations).

Then, the virtual work theorem is applied to obtain the energetic form of the equations governing the structural vibrations. The continuity conditions along the common boundaries of the cylindrical and the hemispherical parts are immediately obtained; the boundary conditions at the apexes require a more detailed analysis of the stresses. The details of these calculations can be found in references [6, 7]. The approximations developed here can be found, together with more accurate equations, in many textbooks, such as reference [8] or [9].

2.2. EQUATIONS GOVERNING THE CYLINDRICAL SHELL VIBRATIONS

The elastic solid occupies the three-dimensional domain Ω_2 defined in cylindrical co-ordinates by

$$\Omega_2 \equiv \{R_2 = R + r \text{ with } -h/2 < r < +h/2,\ 0 \leqslant \varphi < 2\pi,\ -L < z < +L\}.$$

Use is made of the notation $f_{,x} = \partial f/\partial x$ for the derivative of a function f with respect to the variable x. The displacement of a point of the solid is denoted $U\mathbf{e}_z + V\mathbf{e}_\varphi + W\mathbf{e}_r$. The strain tensor $\mathscr{D}_{ij}$ is thus

$$\begin{aligned}
&\mathscr{D}_{zz} = U_{,z}, && \mathscr{D}_{z\varphi} = \frac{1}{2}\left[\frac{U_{,\varphi}}{R+r} + V_{,z}\right],\\
&\mathscr{D}_{\varphi\varphi} = \frac{V_{,\varphi}}{R+r} + \frac{W}{R+r}, && \mathscr{D}_{zr} = \frac{1}{2}[W_{,z} + U_{,r}],\\
&\mathscr{D}_{rr} = W_{,r}, && \mathscr{D}_{\varphi r} = \frac{1}{2}\left[V_{,r} - \frac{V}{R+r} + \frac{W_{,\varphi}}{R+r}\right].
\end{aligned} \tag{1}$$

Hooke's law relates this tensor to the stress tensor $\mathscr{S}_{ij}$ by

$$\begin{aligned}
&\mathscr{S}_{zz} = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\mathscr{D}_{zz} + \nu(\mathscr{D}_{\varphi\varphi} + \mathscr{D}_{rr})], && \mathscr{S}_{z\varphi} = \frac{E}{1+\nu}\mathscr{D}_{z\varphi},\\
&\mathscr{S}_{\varphi\varphi} = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\mathscr{D}_{\varphi\varphi} + \nu(\mathscr{D}_{rr} + \mathscr{D}_{zz})], && \mathscr{S}_{\varphi r} = \frac{E}{1+\nu}\mathscr{D}_{\varphi r},\\
&\mathscr{S}_{rr} = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\mathscr{D}_{rr} + \nu(\mathscr{D}_{zz} + \mathscr{D}_{\varphi\varphi})], && \mathscr{S}_{rz} = \frac{E}{1+\nu}\mathscr{D}_{rz}.
\end{aligned} \tag{2}$$

The hypothesis that h is small suggests looking for truncated Taylor series for the shell displacement and the stress tensor:

$$\begin{aligned}
U &= U^0(z,\varphi) + rU^1(z,\varphi) + \mathcal{O}(r^2),\\
V &= V^0(z,\varphi) + rV^1(z,\varphi) + \mathcal{O}(r^2),\\
W &= W^0(z,\varphi) + rW^1(z,\varphi) + \mathcal{O}(r^2),\\
\mathscr{S}_{ij} &= \mathscr{S}^0_{ij}(z,\varphi) + r\mathscr{S}^1_{ij}(z,\varphi) + \mathcal{O}(r^2), \quad \text{with } i, j = r, \varphi, z.
\end{aligned}$$

The condition of a "free boundary" for $r = \pm h/2$, together with the approximation $1/(R+r) \simeq R^{-1}(1 - r/R)$ are used. They lead to

$$U^1 = -W^0_{,z}, \quad V^1 = (1/R)(V^0 - W^0_{,\varphi}), \quad W_1 = 0.$$

The only unknown functions are thus

$$u = U^0, \quad v = V^0, \quad w = W^0.$$

The strain tensor components are thus approximated by

$$d_{zz} = u_{,z} - rw_{,zz}, \qquad d_{z\varphi} = \frac{1}{2}\left(\frac{u_{,\varphi}}{R} + v_{,z} - \frac{2r}{R} w_{,z\varphi}\right),$$

$$d_{\varphi\varphi} = \frac{1}{R}\left(v_{,\varphi} + w - \frac{r}{R} w_{,\varphi\varphi}\right), \; d_{zr} = 0,$$

$$d_{rr} = -\frac{v}{1-v}(d_{zz} + d_{\varphi\varphi}), \quad d_{\varphi r} = 0. \tag{3}$$

The corresponding stress tensor components are

$$\sigma_{zz} = \frac{E}{1-v^2}(d_{zz} + vd_{\varphi\varphi}), \quad \sigma_{z\varphi} = \frac{E}{1+v} d_{z\varphi},$$

$$\sigma_{\varphi\varphi} = \frac{E}{1-v^2}(d_{\varphi\varphi} + vd_{zz}), \quad \sigma_{zr} = 0,$$

$$\sigma_{rr} = 0, \qquad \sigma_{\varphi r} = 0 \tag{3'}$$

and the approximation of the potential energy density is expressed as

$$\begin{aligned} \mathrm{d}\mathscr{E} = \frac{E}{2(1-v^2)}\Bigg[\Bigg(u^2_{,z} + \left(\frac{v_{,\varphi} + w}{R}\right)^2 + 2v\frac{u_{,z}(v_{,\varphi} + w)}{R} \\ + \frac{1-v}{2}\left(\frac{u_{,\varphi}}{R} + v_{,z}\right)^2\Bigg) + r^2\Bigg(w^2_{,zz} + \frac{w^2_{,\varphi\varphi}}{R^4} + \frac{2v}{R^2} w_{,zz}w_{,\varphi\varphi} \\ + (1-v)\left(\frac{w_{,z\varphi}}{R}\right)^2 + (1-v)\left(\frac{w_{,\varphi z}}{R}\right)^2\Bigg) + r(\cdots)\Bigg]. \end{aligned} \tag{3''}$$

In this expression, the tensor components $d_{z\varphi}$ (resp. $\sigma_{z\varphi}$) and $d_{\varphi z}$ (resp. $\sigma_{\varphi z}$) have been distinguished though their expressions are the same: indeed, they represent different physical quantities which, in the final expression of the virtual work theorem, give different boundary terms.

Because the (u, v, w) do not depend on r, this expression can be integrated analytically with respect to this variable; the only terms to be accounted for are those which involve an even power (0 and 2) in r; the odd powers give an integral which is zero. Thus, the approximation of the potential energy of the shell is given by

$$\mathscr{E} = \frac{Eh}{2(1-\nu^2)} \int_0^{2\pi} R\,\mathrm{d}\varphi \int_{-L}^{L} \mathrm{d}z \left\{ u_{,z}^2 + \left(\frac{v_{,\varphi}+w}{R}\right)^2 + 2\nu \frac{u_{,z}(v_{,\varphi}+w)}{R} + \frac{1-\nu}{2}\left(\frac{u_{,\varphi}}{R}+v_{,z}\right)^2 + \frac{h^2}{12}\left(w_{,zz}^2 + \left(\frac{w_{,\varphi\varphi}}{R^2}\right)^2 + \frac{2\nu}{R^2} w_{,zz}w_{,\varphi\varphi} + (1-\nu)\left(\frac{w_{,z\varphi}}{R}\right)^2 + (1-\nu)\left(\frac{w_{,\varphi z}}{R}\right)^2 \right]\right\}. \qquad (4)$$

The kinetic energy is

$$\mathscr{C} = \frac{\varrho_s h}{2} \int_0^{2\pi} R\,\mathrm{d}\varphi \int_{-L}^{L} (\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\,\mathrm{d}z,$$

where $\dot{g}$ denotes the time derivative of any function g. Let f be the density of a force exerted on the shell with components (f_z, f_φ, f_r). The virtual work theorem leads to the following variational equation for the thin shell:

$$\begin{aligned}
&\int_0^{2\pi} R\,\mathrm{d}\varphi \int_{-L}^{L} \mathrm{d}z \left\{ \frac{Eh}{1-\nu^2} \left\{ -\delta u \left(u_{,zz} + \frac{1-\nu}{2R^2} u_{,\varphi\varphi} + \frac{1+\nu}{2R} v_{,\varphi z} + \frac{\nu}{R} w_{,z} \right) \right. \right. \\
&\quad - \delta v \left(\frac{1+\nu}{2R} u_{,z\varphi} + \frac{v_{,\varphi\varphi}}{R^2} + \frac{1-\nu}{2} v_{,zz} + \frac{1}{R^2} w_{,\varphi} \right) \\
&\quad \left. + \delta w \left[\frac{\nu}{R} u_{,z} + \frac{v_{,\varphi}}{R^2} + \frac{w}{R^2} \right] + \frac{h^2}{12} \left(w_{,zzzz} + \frac{2}{R^2} w_{,zz\varphi\varphi} + \frac{1}{R^4} w_{,\varphi\varphi\varphi\varphi} \right) \right]\Bigg\} \\
&\quad \left. + \varrho_s h (\ddot{u}\,\delta u + \ddot{v}\,\delta v + \ddot{w}\,\delta w) \right\} \\
&\quad + \frac{Eh}{1-v^2} \int_0^{2\pi} R\,\mathrm{d}\varphi \left\{ \delta u \left[u_{,z} + \frac{\nu}{R}(v_{,\varphi} + w) \right] + \delta v \frac{1-\nu}{2} \left(\frac{u_{,\varphi}}{R} + v_{,z} \right) \right. \\
&\quad + \frac{h^2}{12} \left[-\delta w \left(w_{,zzz} + \frac{w_{,\varphi\varphi z}}{R^2} \right) + \frac{1-\nu}{R^2} \delta w_{,\varphi} w_{,\varphi z} \right. \\
&\quad \left. \left. + \delta w_{,z} \left(w_{,zz} + \frac{\nu}{R^2} w_{,\varphi\varphi} \right) \right] \right\}_{z=-L}^{z=+L} \\
&= \int_0^{2\pi} R\,\mathrm{d}\varphi \int_{-L}^{L} \mathrm{d}z \,(f_z\,\delta u + f_\varphi\,\delta v + f_r\,\delta w). \qquad (5)
\end{aligned}$$

This integral relationship must be satisfied for any virtual displacement vector $(\delta u, \delta v, \delta w)$: this can be achieved if the surface integrals and the boundary integrals are equal to zero independently.

The boundary integrals represent the energy which is lost (or received) by the shell along the circles $z = \pm L$. This approximation is known as the *Donnell* and *Mushtari cylindrical shell equation.*

One can now recall the physical meaning of the various terms in the boundary integrals. The coefficients of the displacement components δu, δv and δw are force line densities:

$$F_{zz} = \frac{Eh}{1-\nu^2}\left[u_{,z} + \frac{\nu}{R}(v_{,\varphi} + w)\right],$$

$$F_{z\varphi} = \frac{Eh}{1-\nu^2}\frac{1-\nu}{2}\left(\frac{u_{,\varphi}}{R} + v_{,z}\right),$$

$$F_{zr} = -\frac{Eh^3}{12(1-\nu^2)}\left(w_{,zzz} + \frac{w_{,\varphi\varphi z}}{R^2}\right). \tag{6}$$

The term proportional to $\delta w_{,\varphi}/R$, the derivative of the virtual normal displacement in the φ direction, is a twisting momentum line density:

$$M_{z\varphi} = \frac{Eh^3}{12(1-\nu^2)}\frac{1-\nu}{R}w_{,\varphi z}. \tag{6'}$$

Finally, the term proportional to $\delta w_{,z}$, the z-derivative of the virtual normal displacement, is a bending momentum line density:

$$M_{zz} = \frac{Eh^3}{12(1-\nu^2)}\left(w_{,zz} + \frac{\nu}{R^2}w_{,\varphi\varphi}\right). \tag{6''}$$

2.3. EQUATIONS GOVERNING THE SPHERICAL SHELL VIBRATIONS

In what follows, the variable θ denotes θ_1 or θ_3 and varies from ϑ_a to ϑ_b. For Σ_1, one takes $\vartheta_a > 0$ and let it tend to 0, while $\vartheta_b = \pi/2$; for Σ_2, one takes $\vartheta_b < \pi$ and let it tend to π, while $\vartheta_a = \pi/2$.

Let Σ be the spherical surface and Ω be the three-dimensional domain defined in spherical co-ordinates by

$$\Sigma \equiv \{\rho = R, 0 \leqslant \varphi < 2\pi, \vartheta_a < \theta < \vartheta_b\},$$

$$\Omega \equiv \{R - h/2 < \rho = R + r < R + h/2, 0 \leqslant \varphi < 2\pi, \vartheta_a < \theta < \vartheta_b\}.$$

The displacement of a point of the solid is denoted by $U\mathbf{e}_\theta + V\mathbf{e}_\varphi + W\mathbf{e}_r$. The strain tensor $\mathscr{D}_{ij}$ is given by

$$\mathscr{D}_{\theta\theta} = \frac{U_{,\theta}}{R+r} + \frac{W}{R+r}, \qquad \mathscr{D}_{\theta\varphi} = \frac{1}{2}\frac{1}{R+r}\left[\frac{U_{,\varphi}}{\sin\theta} + V_{,\theta} - V\cot g\theta\right],$$

$$\mathscr{D}_{\varphi\varphi} = \frac{U\operatorname{cotg}\theta}{R+r} + \frac{1}{R+r}\frac{V_{,\varphi}}{\sin\theta} + \frac{W}{R+r}, \qquad \mathscr{D}_{\theta r} = \frac{1}{2}\left[\frac{W_{,\theta}}{R+r} + U_{,r} - \frac{U}{R+r}\right],$$

$$\mathscr{D}_{rr} = W_{,r}, \qquad \mathscr{D}_{\varphi r} = \frac{1}{2}\left[\frac{1}{R+r}\frac{W_{,\varphi}}{\sin\theta} + V_{,r} - \frac{V}{R+r}\right]. \tag{7}$$

By using Hooke's law, the stress tensor is written as

$$\mathscr{S}_{\theta\theta} = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\mathscr{D}_{\theta\theta} + \nu(\mathscr{D}_{\varphi\varphi} + \mathscr{D}_{rr})], \quad \mathscr{S}_{\theta\varphi} = \frac{E}{1+\nu}\mathscr{D}_{\theta\varphi},$$

$$\mathscr{S}_{\varphi\varphi} = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\mathscr{D}_{\varphi\varphi} + \nu(\mathscr{D}_{rr} + \mathscr{D}_{\theta\theta})], \quad \mathscr{S}_{\varphi r} = \frac{E}{1+\nu}\mathscr{D}_{\varphi r},$$

$$\mathscr{S}_{rr} = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\mathscr{D}_{rr} + \nu(\mathscr{D}_{\theta\theta} + \mathscr{D}_{\varphi\varphi})], \quad \mathscr{S}_{rz} = \frac{E}{1+\nu}\mathscr{D}_{rz}. \tag{8}$$

As done for the cylindrical shell, the displacement and the stress tensor components are sought as truncated Taylor series:

$$U = U^0(\theta,\varphi) + rU^1(\theta,\varphi) + \mathscr{O}(r^2),$$

$$V = V^0(\theta,\varphi) + rV^1(\theta,\varphi) + \mathscr{O}(r^2),$$

$$W = W^0(\theta,\varphi) + rW^1(\theta,\varphi) + \mathscr{O}(r^2),$$

$$\mathscr{S}_{ij} = \mathscr{S}_{ij}^0(\theta,\varphi) + r\mathscr{S}_{ij}^1(\theta,\varphi) + \mathscr{O}(r^2), \quad \text{with } i,j = r,\theta,\varphi.$$

Use is made of the approximation

$$\frac{1}{R+r} = \frac{1}{R}\left(1 - \frac{r}{R}\right) + \mathscr{O}(r^2).$$

The free boundary conditions for $r = \pm h/2$ lead to

$$\mathscr{S}_{ir}^0(\theta,\varphi) = \mathscr{S}_{ir}^1(\theta,\varphi) = 0, \quad i = r,\theta,\varphi,$$

which implies that the components U^1 and V^1 are expressed in terms of $u = U^0$, $v = V^0$ and $w = W^0$ as

$$U^1 = \frac{U^0 - W^0_{,\theta}}{R}, \qquad V^1 = \frac{1}{R}\left(V^0 - \frac{W^0_{,\varphi}}{\sin\theta}\right).$$

The approximations d_{ij} and σ_{ij} of the strain and stress tensors components are

$$d_{\theta\theta} = \frac{1}{R}\left[u_{,\theta} + w - \frac{r}{R}w_{,\theta\theta}\right], \qquad d_{\varphi r} = 0,$$

$$d_{\varphi\varphi} = \frac{1}{R}\left[u\,\mathrm{cotg}\,\theta + \frac{v_{,\varphi}}{\sin\theta} + w - \frac{r}{R\sin\theta}\left(w_{,\theta}\cos\theta + \frac{w_{,\varphi\varphi}}{\sin\theta}\right)\right],$$

$$d_{\theta\varphi} = \frac{1}{2R}\left[\frac{u_{,\varphi}}{\sin\theta} - v\,\mathrm{cotg}\,\theta + v_{,\theta} + \frac{2r}{R\sin\theta}(w_{,\varphi}\,\mathrm{cotg}\,\theta - w_{,\theta\varphi})\right],$$

$$d_{rr} = -\frac{\nu}{1-\nu}(d_{\theta\theta} + d_{\varphi\varphi}), \qquad d_{\theta r} = 0, \tag{9}$$

$$\sigma_{\theta\theta} = \frac{E}{1-v^2}(d_{\theta\theta} + vd_{\varphi\varphi}), \quad \sigma_{\theta\varphi} = \frac{E}{1+v^2}(1-v)\,d_{\theta\varphi},$$

$$\sigma_{\varphi\varphi} = \frac{E}{1-v^2}(d_{\varphi\varphi} + vd_{\theta\theta}), \quad \sigma_{\theta r} = 0,$$

$$\sigma_{rr} = 0, \qquad \sigma_{\varphi r} = 0. \tag{9'}$$

The potential energy of the shell is thus approximated by

$$\begin{aligned}\mathscr{E} = \frac{Eh}{2(1-v^2)}\int_0^{2\pi} \mathrm{d}\varphi \int_{\vartheta_a}^{\vartheta_b} \sin\theta\,\mathrm{d}\theta \Bigg\{ \left(u \operatorname{cotg}\theta + u_{,\theta} + \frac{v_{,\varphi}}{\sin\theta} + 2w\right)^2 \\ - 2(1-v)(u_{,\theta} + w)\left(u\operatorname{cotg}\theta + \frac{v_{,\varphi}}{\sin\theta} + w\right) + \frac{1-v}{2}\left(\frac{u_{,\varphi}}{\sin\theta} - v\operatorname{cotg}\theta + v_{,\theta}\right)^2 \\ + \frac{h^2}{12R^2}\Bigg[\left(w_{,\theta}\operatorname{cotg}\theta + w_{,\theta\theta} + \frac{w_{,\varphi\varphi}}{\sin^2\theta}\right)^2 - 2(1-v)w_{,\theta\theta}\left(w_{,\theta}\operatorname{cotg}\theta + \frac{w_{,\varphi\varphi}}{\sin^2\theta}\right) \\ + \frac{(1-v)}{\sin^2\theta}(w_{,\varphi}\operatorname{cotg}\theta - w_{,\theta\varphi})^2 + \frac{(1-v)}{\sin^2\theta}(w_{,\varphi}\operatorname{cotg}\theta - w_{,\varphi\theta})^2\Bigg]\Bigg\}.\end{aligned}$$

Its kinetic energy has a form similar to that of the cylindrical shell.

Upon assuming that the shell is excited by a force with density $(f_\theta, f_\varphi, f_r)$, the virtual work theorem leads to the variational form of the spherical shell equation:

$$\begin{aligned}\frac{Eh}{1-v^2}\int_0^{2\pi}\mathrm{d}\varphi\int_{\vartheta_a}^{\vartheta_b}\sin\theta\,\mathrm{d}\theta\Bigg\{\delta u\operatorname{cotg}\theta\left[u\operatorname{cotg}\theta + vu_{,\theta} + \frac{v_{,\varphi}}{\sin\theta} + (1+v)w\right] \\ - \delta v\,\frac{1-v}{2}\operatorname{cotg}\theta\left(\frac{u_{,\varphi}}{\sin\theta} + v\operatorname{cotg}\theta + v_{,\theta}\right) + (1+v)\,\delta w\left(u\operatorname{cotg}\theta + u_{,\theta} + \frac{v_{,\varphi}}{\sin\theta} + 2w\right) \\ + \delta u_{,\varphi}\frac{1-v}{2\sin\theta}\left(\frac{u_{,\varphi}}{\sin\theta} - v\operatorname{cotg}\theta + v_{,\theta}\right) + \frac{\delta v_{,\varphi}}{\sin\theta}\left[u\operatorname{cotg}\theta + vu_{,\theta} + \frac{v_{,\varphi}}{\sin\theta} + (1+v)w\right) \\ + \delta w_{,\varphi}\frac{h^2}{12R^2}2(1-v)\frac{\operatorname{cotg}\theta}{\sin^2\theta}(w_{,\varphi}\operatorname{cotg}\theta - w_{,\varphi\theta}) \\ + \delta u_{,\theta}\left[vu\operatorname{cotg}\theta + u_{,\theta} + v\frac{v_{,\varphi}}{\sin\theta} + (1+v)w\right] + \frac{1-v}{2}\delta v_{,\theta}\left(\frac{u_{,\varphi}}{\sin\theta} - v\operatorname{cotg}\theta + v_{,\theta}\right) \\ + \delta w_{,\theta}\frac{h^2}{12R^2}\operatorname{cotg}\theta\left(w_{,\theta}\operatorname{cotg}\theta + vw_{,\theta\theta} + \frac{w_{,\varphi\varphi}}{\sin^2\theta}\right) \\ + \delta w_{,\varphi\varphi}\frac{h^2}{12R^2}\frac{1}{\sin^2\theta}\left(w_{,\theta}\operatorname{cotg}\theta + vw_{,\theta\theta} + \frac{w_{,\varphi\varphi}}{\sin^2\theta}\right) \\ + \delta w_{,\theta\theta}\frac{h^2}{12R^2}\left(vw_{,\theta}\operatorname{cotg}\theta + w_{,\theta\theta} + v\frac{w_{,\varphi\varphi}}{\sin^2\theta}\right)\end{aligned}$$

$$- \delta w_{,\varphi\theta} \frac{h^2}{12R^2} \frac{(1-\nu)}{\sin^2\theta} (w_{,\varphi} \operatorname{cotg}\theta - w_{,\varphi\theta}) - \delta w_{,\theta\varphi} \frac{h^2}{12R^2} \frac{(1-\nu)}{\sin^2\theta} (w_{,\varphi} \operatorname{cotg}\theta - w_{,\theta\varphi}) \Big\}$$

$$+ \varrho_s h \int_0^{2\pi} \mathrm{d}\varphi \int_{\vartheta_a}^{\vartheta_b} \sin\theta \, \mathrm{d}\theta \, (\ddot{u}\, \delta u + \ddot{v}\, \delta u + \ddot{w}\, \delta w)$$

$$= \int_0^{2\pi} \mathrm{d}\varphi \int_{\vartheta_a}^{\vartheta_b} \sin\theta \, \mathrm{d}\theta \, (f_\theta \, \delta u + f_\varphi \, \delta v + f_r \, \delta w). \tag{10}$$

Integration by parts are then performed and boundary integrals appear:

$$\frac{Eh}{(1-\nu^2)} \int_0^{2\pi} R \, \mathrm{d}\varphi \int_{\vartheta_a}^{\vartheta_b} R \sin\theta \, \mathrm{d}\theta \, \{ \delta u [\mathscr{M}^s_{\theta\theta} u + \mathscr{M}^s_{\theta\varphi} v + \mathscr{M}^s_{\theta r} w]$$

$$+ \delta v [\mathscr{M}^s_{\varphi\theta} u + \mathscr{M}^s_{\varphi\varphi} v + \mathscr{M}^s_{\varphi r} w] + \delta w [\mathscr{M}^s_{r\theta} u + \mathscr{M}^s_{r\varphi} v + \mathscr{M}^s_{rr} w] \}$$

$$+ \int_0^{2\pi} R \, \mathrm{d}\varphi \left\{ \sin\theta \left[\delta u \, F_{\theta\theta} + \delta v \, F_{\theta\varphi} + \delta w \, F_{\theta r} + \frac{\delta w_{,\varphi}}{R \sin\theta} M_{\theta\varphi} + \frac{\delta w_{,\theta}}{R} M_{\theta\theta} \right] \right\}_{\theta = \vartheta_a}^{\theta = \vartheta_b}$$

$$+ \varrho_s h \int_0^{2\pi} \mathrm{d}\varphi \int_{\vartheta_a}^{\vartheta_b} \sin\theta \, \mathrm{d}\theta \, (\ddot{u}\, \delta u + \ddot{v}\, \delta u + \ddot{w}\, \delta w)$$

$$= \int_0^{2\pi} \mathrm{d}\varphi \int_{\vartheta_a}^{\vartheta_b} \sin\theta \, \mathrm{d}\theta \, (f_\theta \, \delta u + f_\varphi \, \delta v + f_r \, \delta w). \tag{11}$$

The various quantities appearing in this expression are defined as follows:

$$\mathscr{M}^s_{\theta\theta} = -\frac{1}{R^2} \left[\frac{\partial^2}{\partial\theta^2} + \frac{1-\nu}{2\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} + \operatorname{cotg}\theta \frac{\partial}{\partial\theta} - (\operatorname{cotg}^2\theta + \nu) \right],$$

$$\mathscr{M}^s_{\theta\varphi} = -\frac{1}{R^2} \left[\frac{1+\nu}{2\sin\theta} \frac{\partial^2}{\partial\theta \, \partial\varphi} - \frac{3-\nu}{2} \frac{\operatorname{cotg}\theta}{\sin\theta} \frac{\partial}{\partial\varphi} \right], \quad \mathscr{M}^s_{\theta r} = -\frac{1+\nu}{R^2} \frac{\partial}{\partial\theta}, \tag{12}$$

$$\mathscr{M}^s_{\varphi\theta} = -\frac{1}{R^2} \left[\frac{1+\nu}{2\sin\theta} \frac{\partial^2}{\partial\theta \, \partial\varphi} + \frac{3-\nu}{2} \frac{\operatorname{cotg}\theta}{\sin\theta} \frac{\partial}{\partial\varphi} \right],$$

$$\mathscr{M}^s_{\varphi\varphi} = -\frac{1}{R^2} \left[\frac{1-\nu}{2} \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} + \frac{1-\nu}{2} \operatorname{cotg}\theta \frac{\partial}{\partial\theta} \frac{1-\nu}{2} (\operatorname{cotg}^2\theta - 1) \right],$$

$$\mathscr{M}^s_{\varphi r} = -\frac{1+\nu}{R^2 \sin\theta} \frac{\partial}{\partial\varphi}, \tag{12'}$$

$$\mathscr{M}^s_{r\theta} = \frac{1+\nu}{R^2} \left(\frac{\partial}{\partial\theta} + \operatorname{cotg}\theta \right), \quad \mathscr{M}^s_{r\varphi} = \frac{1+\nu}{R^2 \sin\theta} \frac{\partial}{\partial\varphi},$$

$$\mathscr{M}^s_{rr} = \frac{2(1+\nu)}{R^2} + \frac{h^2}{12R^4} \left(\frac{\partial^4}{\partial\theta^4} + \frac{2}{\sin^2\theta} \frac{\partial^4}{\partial\theta^2 \, \partial\varphi^2} + \frac{1}{\sin^4\theta} \frac{\partial^4}{\partial\varphi^4} \right.$$

$$+ 2\operatorname{cotg}\theta\frac{\partial^3}{\partial\theta^3} - \frac{2\operatorname{cotg}\theta}{\sin^2\theta}\frac{\partial^3}{\partial\theta\,\partial\varphi^2} - (1 + \nu + \operatorname{cotg}^2\theta)\frac{\partial^2}{\partial\theta^2}$$

$$+ \frac{3 - \nu + 4\operatorname{cotg}^2\theta}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + (2 - \nu + \operatorname{cotg}^2\theta)\operatorname{cotg}\theta\frac{\partial}{\partial\theta}\Bigg), \tag{12''}$$

$$F_{\theta\theta} = \frac{Eh}{(1-\nu^2)R}\left(u_{,\theta} + \nu u\operatorname{cotg}\theta + \nu\frac{v_{,\varphi}}{\sin\theta} + (1+\nu)w\right),$$

$$F_{\theta\varphi} = \frac{Eh}{(1-\nu^2)R}\frac{1-\nu}{2}\left(\frac{u_{,\varphi}}{\sin\theta} + v_{,\theta} - v\operatorname{cotg}\theta\right),$$

$$F_{\theta r} = \frac{Eh^3}{12(1-\nu^2)R^3}\Bigg\{\operatorname{cotg}\theta\left(\frac{w_{,\varphi\varphi}}{\sin^2\theta} + \nu w_{,\theta\theta} + w_{,\theta}\operatorname{cotg}\theta\right)$$

$$- \frac{1}{\sin\theta}\left[\sin\theta\left(\nu\frac{w_{,\varphi\varphi}}{\sin^2\theta} + w_{,\theta\theta} + \nu w_{,\theta}\operatorname{cotg}\theta\right)\right]_{,\theta}$$

$$- \frac{1-\nu}{\sin^2\theta}[w_{,\theta\varphi} - w_{,\varphi}\operatorname{cotg}\theta]_{,\varphi}\Bigg\},$$

$$M_{\theta\theta} = \frac{Eh^3}{12(1-\nu^2)R^3}\left(w_{,\theta\theta} + \nu\frac{w_{,\varphi\varphi}}{\sin^2\theta} + \nu w_{,\theta}\operatorname{cotg}\theta\right),$$

$$M_{\theta\varphi} = \frac{Eh^3}{12(1-\nu^2)R^2}\frac{1-\nu}{\sin\theta}(w_{,\theta\varphi} - w_{,\varphi}\operatorname{cotg}\theta). \tag{13}$$

The functions $F_{\theta\theta}$, $F_{\theta\varphi}$ and $F_{\theta r}$ are line densities of forces; the functions $M_{\theta\theta}$ and $M_{\theta\varphi}$ are line densities of momentums. Equation (11) is the energetic form of the *Donnell and Mushtari approximation for a spherical shell.*

It must be noticed that, in equation (11), the term $-\delta w_{,\varphi}M_{\theta\varphi}$ can be integrated by parts and thus replaced by $+\delta w(\operatorname{Tr} M_{\theta\varphi})_{,\varphi}$, where $(\operatorname{Tr} M_{\theta\varphi})_{,\varphi}$ is the derivative with respect to φ of the value, along the boundary, of the momentum $M_{\theta\varphi}$.

2.4. REGULARITY CONDITIONS AT THE APEXES OF THE SPHERICAL ELEMENTS

Equation (11) can be solved if boundary conditions are given along the two lines $\theta = \vartheta_a$ and ϑ_b. But if one of these two lines does not exist—which is the case if, for example, $\vartheta_a = 0$—the boundary condition to be imposed to the shell displacement must express that this point is not different from any other point. This implies that the virtual work exerted by the forces and momentums along the line $\theta = \vartheta_a$ must tend to 0 when ϑ_a tends to zero.

2.4.1. *Normal shearing force*

The cancellation of the virtual work due to the normal shearing force

$$F_{\theta r} = \frac{Eh^3}{12(1-\nu^2)R^3}\Bigg\{w_{,\theta\theta\theta} - \frac{w_{,\varphi\varphi\theta}}{\sin^2\theta} + w_{,\theta\theta}\operatorname{cotg}\theta$$

$$+ 2\nu w_{,\varphi\varphi}\frac{\operatorname{cotg}\theta}{\sin^2\theta} + \frac{1 + (1-\nu)\cos^2\theta}{\sin^2\theta}w_{,\theta}\Bigg\}$$

is obtained if $F_{\vartheta_a r}\sin\vartheta_a \to 0$ for $\vartheta_a \to 0$. One can seek w of the form

$$w = w_0(\theta) + \theta\bar{w}(\theta,\varphi), \quad \int_0^{2\pi} \bar{w}(\theta,\varphi)\,\mathrm{d}\varphi = 0,$$

where w_0 and $\bar{w}$ are assumed to have Taylor series around $\theta = 0$. The term $w_{,\theta\theta}\,\mathrm{cotg}\,\theta$ has zero contribution if

$$w_{0,\theta\theta}(0) + 2\bar{w}_{,\theta}(0,\varphi) = 0 \quad \forall\varphi \Rightarrow w_{0,\theta\theta}(0) = \bar{w}_{,\theta}(0,\varphi) = 0.$$

The term $w_{,\varphi\varphi}\,\mathrm{cotg}\,\theta/\sin^2\theta$ has no contribution if $\bar{w}_{,\varphi\varphi}(0,\varphi) = 0$. The virtual work of the term

$$-\frac{w_{,\varphi\varphi\theta}}{\sin^2\theta} + \frac{1 + (1-\nu)\cos^2\theta}{\sin^2\theta}\,w_{,\theta}$$

is zero if the following condition is fulfilled:

$$\lim_{\theta\to 0}\left\{\frac{-\bar{w}_{,\varphi\varphi} + (2-\nu)(w_{0,\theta} + \bar{w})}{\theta} - \bar{w}_{,\varphi\varphi\theta} + (2-\nu)\bar{w}_{,\theta}\right\} = 0.$$

As a consequence of the former conditions, one has

$$w_{0,\theta}(0) = \bar{w}(0,\varphi) = 0.$$

This implies that $w_{,\varphi\varphi}$ is zero at $\theta = 0$.

2.4.2. *Tangential forces*

The forces $F_{\theta\theta}$ and $F_{\theta\varphi}$ produce virtual works which tend to zero for $\vartheta_a \to 0$ if

$$\lim_{\vartheta_a\to 0}\{\theta u_{,\theta} + \nu(u + v_{,\varphi}) + \theta(1+\nu)w\} = 0, \quad \lim_{\vartheta_a\to 0}\{u_{,\varphi} + \theta v_{,\theta} - v\} = 0.$$

The results in two conditions:

$$u(0,\varphi) + v_{,\varphi}(0,\varphi) = 0, \quad u_{,\varphi}(0,\varphi) - v(0,\varphi) = 0.$$

These two conditions are not sufficient. Indeed, one can note that the shell displacement, being a periodic function of φ, can be expanded into a Fourier series with respect to this variable. For conditions at the apex are required for each Fourier component: the former two equalities reduce to only one for the components -1 and $+1$. It is thus necessary to find an additional condition.

2.4.3. *Relationship between the tangential forces $F_{\theta\theta}$ and $F_{\varphi\varphi}$*

The force $F_{\theta\theta}(\vartheta_a, \varphi_1)$ is exerted normally to a length element of the circle $\theta = \vartheta_a$ around the point $\varphi = \varphi_1$. But it can also be considered as the force exerted in the same direction on a length element of the circle γ (see Figure 2). It is thus of the same nature as the force $F_{\varphi\varphi}(\vartheta_a, \varphi_1)$ which is exerted on the length element of the circle $\varphi = \varphi_1$ around the point $\theta = \vartheta_a$. This property remains true when $\vartheta_a \to 0$. The angle between the two circles γ and $\varphi = \varphi_1$ being equal to $\pi/2$, one must have

$$F_{\theta\theta}(0,\varphi) = F_{\varphi\varphi}(0,\varphi - \pi/2)$$

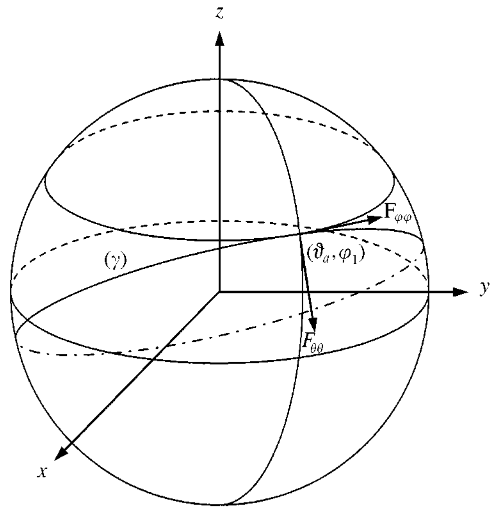

Figure 2. Regularity problems at the apex: relationship between the tangential force $F_{\theta\theta}$ and $F_{\varphi\varphi}$ at a point with co-ordinates $\vartheta_a \to 0$.

(at the other apex, one has $F_{\theta\theta}(\pi, \varphi) = F_{\varphi\varphi}(\pi, \varphi + \pi/2)$.) The force $F_{\varphi\varphi}$ has the following expression:

$$F_{\varphi\varphi} = \frac{Eh}{(1-\nu^2)R}\left(\nu u_{,\theta} + u \operatorname{cotg}\theta + \frac{v_{,\varphi}}{\sin\theta} + (1+\nu)w\right).$$

One can seek u and v of the form

$$u(\theta, \varphi) = u_1(\theta)e^{i\varphi} + u_{-1}(\theta)e^{-i\varphi} + \theta\bar{u}(\theta, \varphi),$$

$$v(\theta, \varphi) = v_0(\theta) + v_1(\theta)e^{i\varphi} + v_{-1}(\theta)e^{-i\varphi} + \theta\bar{v}(\theta, \varphi),$$

where the function $\bar{u}(\theta, \varphi)$ is orthogonal to $e^{i\varphi}$ and $e^{-i\varphi}$, and $\bar{v}(\theta, \varphi)$ is orthogonal to 1, $e^{i\varphi}$ and $e^{-i\varphi}$. One immediately has

$$v_0(0) = 0, \quad u_1(0) + iv_1(0) = 0, \quad u_{-1}(0) - iv_{-1}(0) = 0.$$

The relationship between the two tangential forces is expressed by

$$u_{1,\theta}(0)(1 - i\nu)e^{i\varphi} + u_{-1,\theta}(0)(1 - i\nu)e^{-i\varphi} + (1+\nu)[\bar{u}(0,\varphi) - \bar{u}(0, \varphi - \pi/2)] = 0 \quad \forall\varphi,$$

$$\lim_{\vartheta_a \to 0} \frac{1}{\vartheta_a}(\nu\bar{v}_{,\varphi}(0,\varphi) - \bar{v}_{,\varphi}(0, \varphi - \pi/2)) = 0 \quad \forall\varphi.$$

This provides the missing equalities

$$u_{1,\theta}(0) = 0, \quad u_{-1,\theta}(0) = 0, \quad \bar{u}(0,\varphi) = 0, \quad \bar{v}_{,\varphi}(0,\varphi) = 0.$$

2.4.4. *The boundary conditions at the apex*

One can now gather all the results. First, the shell displacement has the form

$$u(\theta, \varphi) = u_1(\theta)e^{i\varphi} + u_{-1}(\theta)e^{-i\varphi} + \theta\bar{u}(\theta, \varphi),$$

$$v(\theta, \varphi) = v_0(\theta) + v_1(\theta)e^{i\varphi} + v_{-1}(\theta)e^{-i\varphi} + \theta\bar{v}(\theta, \varphi),$$

$$w(\theta, \varphi) = w_0(\theta) + \theta\bar{w}(\theta, \varphi) \qquad (14)$$

with

$$\int_0^{2\pi} \bar{w}(\theta, \varphi)\,d\varphi = \int_0^{2\pi} \bar{v}(\theta, \varphi)\,d\varphi = \int_0^{2\pi} \bar{u}(\theta, \varphi)e^{\pm i\varphi}\,d\varphi = \int_0^{2\pi} \bar{v}(\theta, \varphi)e^{\pm i\varphi}\,d\varphi = 0.$$

The various functions which appear in these formulas satisfy the boundary conditions

$$v_0(0) = 0, \quad u_1(0) + iv_1(0) = 0, \quad u_{1,\theta}(0) = 0,$$

$$u_{-1}(0) - iv_{-1}(0) = 0, \quad u_{-1,\theta}(0) = 0, \quad \bar{u}(0, \varphi) = \bar{v}_{,\varphi}(0, \varphi) = 0,$$

$$w_{0,\theta}(0) = w_{0,\theta\theta}(0) = \bar{w}(0, \varphi) = \bar{w}_{,\theta}(0, \varphi) = 0.$$

If $\vartheta_a > 0$ but $\vartheta_b = \pi$, the boundary conditions which are required at this last point are easily deduced from the previous ones: the second and fourth conditions in equation (14′) are replaced by $u_1(\pi) - iv_1(\pi) = 0$ and $u_{-1}(\pi) + iv_{-1}(\pi) = 0$ respectively.

This result is very similar to that of reference [10] but not totally identical. The proof presented in this paper is very condensed. Thus, it has not been possible to clarify the reasons which lead to authors to slightly different continuity conditions. Moreover, the conditions proposed in reference [10] are not compatible with the energetic form (11) of the Donnell and Mushtari approximation for a spherical shell and so, one cannot examine what are the potential implications when using the continuity conditions from reference [10]. However, according to reference [11], it seems that the eigenfrequencies corresponding to the free vibration problem for deep spherical shell elements are weakly dependent on the boundary conditions imposed at the apex.

3. RESPONSE OF THE *LINE* 2′ SHELL TO A TRANSIENT EXCITATION

Consider a *Line* 2′ shell as defined at the beginning of section 2. It is immersed in a homogeneous and isotropic fluid extending to infinity and characterized by a density ϱ_0 and a sound speed c_0.

One starts with the variational form (energetic form) of the equations which govern the transient response of the fluid-loaded shell. The aim of this study is to express the response of the system by a series of resonance modes, that is of its free oscillations. Two methods are proposed: (1) the transient response of the shell being sought as a series of the resonance modes, the coefficients are solution of an infinite system of linear algebraic equations; (2) the time Fourier transform of the equations are solved in terms of the eigenmodes and, then, by an inverse time Fourier transform, an analytical expression of the coefficients of the resonance modes series is obtained.

3.1. VARIATIONAL FORM OF THE GOVERNING EQUATIONS

It is convenient to define a unique co-ordinate system on the shell surface. The location of a point M is defined by the angular variable φ and a curvilinear abscissa s which varies from $-R\pi/2 - L$ to $+R\pi/2 + L$ (see Figure 3). The displacement of M is the vector $u\mathbf{e}_s + v\mathbf{e}_\varphi + w\mathbf{e}_r$ where $\mathbf{e}_s$ is the unit vector tangent to the shell and parallel to the z-axis, $\mathbf{e}_\varphi$ is the unit vector tangent to the shell and orthogonal to the z-axis, and $\mathbf{e}_r$ is the unit vector orthogonal to the shell surface and pointing out to its exterior; these vectors form a direct trihedron.

Without loss of any generality in the method developed here, it is assumed that the system shell/fluid is excited by an incident acoustic field p^i, which is zero for $t < 0$ and is assumed to be a square integrable function on any finite space domain and any finite time interval: this corresponds to an incident field of finite power. The diffracted field is denoted by p: this function satisfies a homogeneous wave equation and an outgoing wave condition to ensure the uniqueness of the solution. The values on the shell surface of p^i and p are, respectively, denoted by $\mathrm{Tr}\, p^i$ and $\mathrm{Tr}\, p$; the values of their normal derivatives are denoted by $\mathrm{Tr}\, \partial_r p^i$ and $\mathrm{Tr}\, \partial_r p$. The variational form of the governing equations is written as

$$\frac{Eh}{1-v^2}\int_0^{+\infty}\left(\mathscr{H}(u, v, w; \delta u, \delta v, \delta w)(t) + \varrho_s h \int_\Sigma [\ddot{u}\, \delta u^* + \ddot{v}\, \delta v^* + \ddot{w}\, \delta w^*](M, t)\, \mathrm{d}M\right)\mathrm{d}t$$

$$+ \int_0^{+\infty}\int_\Sigma [\mathrm{Tr}\, p\, \delta w^*](M, t)\, \mathrm{d}M\, \mathrm{d}t = -\int_0^{+\infty}\int_\Sigma [\mathrm{Tr}\, p^i\, \delta w^*](M, t)\, \mathrm{d}M\, \mathrm{d}t$$

$$\varrho_0 \int_0^{+\infty}\int_\Sigma [\ddot{w}\, \delta\psi^*](M, t)\, \mathrm{d}M\, \mathrm{d}t + \int_0^{+\infty}\int_\Sigma [\mathrm{Tr}\, \partial_r p\, \delta\psi^*](M, t)\, \mathrm{d}M\, \mathrm{d}t$$

$$= -\int_0^{+\infty}\int_\Sigma [\mathrm{Tr}\, \partial_r p^i\, \delta\psi^*](M, t)\, \mathrm{d}M\, \mathrm{d}t, \quad \forall\, \delta u, \delta v, \delta w, \delta\psi. \tag{15}$$

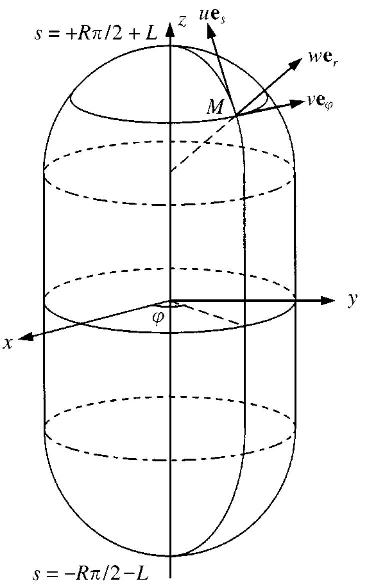

Figure 3. Components (u, v, w) of the shell displacement at a point M with co-ordinates (φ, s).

The first equality is the energy balance of the shell, while the second expresses the continuity between the normal velocity of the shell and that of the fluid particles. The functional $\mathscr{H}$ is the bilinear form associated with the potential energy. It has different expressions on the various shell elements, which are defined as follows:

On the cylindrical element Σ_2, it is given by

$$\mathscr{H}_2(u, v, w; \delta u, \delta v, \delta w) = \int_0^{2\pi} R\,\mathrm{d}\varphi \int_{-L}^{L} \mathrm{d}s \left\{ u_{,s}\, \delta u^*_{,s} + \frac{1}{R^2}(v_{,\varphi} + w)(\delta v^*_{,\varphi} + \delta w^*) \right.$$

$$+ \frac{\nu}{R}(v_{,\varphi} + w)\,\delta u^*_{,s} + \frac{\nu}{R} u_{,s}(\delta v^*_{,\varphi} + \delta w^*) + \frac{1-\nu}{2}\left(\frac{u_{,\varphi}}{R} + v_{,s}\right)\left(\frac{\delta u^*_{,\varphi}}{R} + \delta v^*_{,s}\right)$$

$$+ \frac{h^2}{12}\Big[w_{,ss}\, \delta w^*_{,ss} + \frac{1}{R^4} w_{,\varphi\varphi}\, \delta w^*_{,\varphi\varphi} + \frac{\nu}{R^2} w_{,\varphi\varphi}\, \delta w^*_{,ss}$$

$$\left. + \frac{\nu}{R^2} w_{,ss}\, \delta w^*_{,\varphi\varphi} + \frac{1-\nu}{R^2} w_{,s\varphi}\, \delta w^*_{,s\varphi} + \frac{1-\nu}{R^2} w_{,\varphi s}\, \delta w^*_{,\varphi s} \Big] \right\}, \tag{16}$$

On the spherical elements $\Sigma_i (i = 1, 3)$, $\mathscr{H}$ is given by

$$\mathscr{H}_i(u, v, w; \delta u, \delta v, \delta w) = \int_0^{2\pi} R\,\mathrm{d}\varphi \int_{a_i}^{b_i} \sin\theta_i\, \mathrm{d}s \left\{ \left(u_{,s} + \frac{w}{R}\right)\left(\delta u^*_{,s} + \frac{\delta w^*}{R}\right) \right.$$

$$+ \frac{1}{R^2}\left(- u \operatorname{cotg}\theta_i + \frac{v_{,\varphi}}{\sin\theta_i} + w\right)\left(- \delta u^* \operatorname{cotg}\theta_i + \frac{\delta v^*_{,\varphi}}{\sin\theta_i} + \delta w^*\right)$$

$$+ \frac{\nu}{R}\left(- u \operatorname{cotg}\theta_i + \frac{v_{,\varphi}}{\sin\theta_i} + w\right)\left(\delta u^*_{,s} + \frac{\delta w^*}{R}\right)$$

$$+ \frac{\nu}{R}\left(u_{,s} + \frac{w}{R}\right)\left(- \delta u^* \operatorname{cotg}\theta_i + \frac{\delta v^*_{,\varphi}}{\sin\theta_i} + \delta w^*\right)$$

$$+ \frac{1-\nu}{2}\left(\frac{u_{,\varphi}}{R\sin\theta_i} + \frac{v \operatorname{cotg}\theta_i}{R} + v_{,s}\right)\left(\frac{\delta u^*_{,\varphi}}{R\sin\theta_i} + \frac{\delta v^* \operatorname{cotg}\theta_i}{R} + \delta v^*_{,s}\right)$$

$$+ \frac{h^2}{12}\left[w_{,ss}\, \delta w^*_{,ss} + \frac{1}{R^4}\left(\frac{w_{,\varphi\varphi}}{\sin^2\theta_i} - R w_{,s} \operatorname{cotg}\theta_i\right)\left(\frac{\delta w^*_{,\varphi\varphi}}{\sin^2\theta_i} - R\,\delta w^*_{,s} \operatorname{cotg}\theta_i\right)\right.$$

$$+ \frac{\nu}{R^2}\left(- R w_{,s} \operatorname{cotg}\theta_i + \frac{w_{,\varphi\varphi}}{\sin^2\theta_i}\right)\delta w^*_{,ss} + \frac{\nu}{R^2} w_{,ss}\left(- R\,\delta w^*_{,s} \operatorname{cotg}\theta_i + \frac{\delta w^*_{,\varphi\varphi}}{\sin^2\theta_i}\right)$$

$$+ \frac{1-\nu}{R^4 \sin^2\theta_i}(R w_{,s\varphi} + w_{,\varphi} \operatorname{cotg}\theta_i)(R\,\delta w^*_{,s\varphi} + \delta_i w^*_{,\varphi} \operatorname{cotg}\theta_i)$$

$$\left.\left. + \frac{1-\nu}{R^4 \sin^2\theta_i}(R w_{,\varphi s} + w_{,\varphi} \operatorname{cotg}\theta_i)(R\,\delta w^*_{,\varphi s} + \delta w^*_{,\varphi} \operatorname{cotg}\theta_i)\right]\right\}. \tag{17}$$

In these equations, the unknown functions u, v, w and $\mathrm{Tr}\,p$ (together with the test functions $\delta u, \delta v, \delta w$ and $\delta\psi$) belong to convenient functional spaces corresponding to the following properties:

(1) u, v, w and $\mathrm{Tr}\, p$ are square integrable on Σ and on any finite time interval;

(2) the first order derivatives of u and v, and the derivatives of orders 1 and 2 of w with respect to the space variables s and φ are square integrable on Σ and on any finite time interval;

(3) u, v, and w satisfy the conditions at the apexes: by choosing the shell displacement of the form

$$u(s,\varphi) = u_1(s)\mathrm{e}^{\mathrm{i}\varphi} + u_{-1}(s)\mathrm{e}^{-\mathrm{i}\varphi} + (s - L - R\pi/2)(s + L + R\pi/2)\,U(s,\varphi),$$

$$v(s,\varphi) = v_0(s) + v_1(s)\mathrm{e}^{\mathrm{i}\varphi} + v_{-1}(s)\mathrm{e}^{-\mathrm{i}\varphi} + (s - L - R\pi/2)(s + L + R\pi/2)\,V(s,\varphi),$$

$$w(s,\varphi) = w_0(s) + (s - L - R\pi/2)(s + L + R\pi/2)\,W(s,\varphi) \tag{18}$$

with

$$\int_0^{2\pi} U(s,\varphi)\,\mathrm{e}^{\pm\mathrm{i}\varphi}\,\mathrm{d}\varphi = \int_0^{2\pi} V(s,\varphi)\,\mathrm{d}\varphi = \int_0^{2\pi} V(s,\varphi)\mathrm{e}^{\pm\mathrm{i}\varphi}\,\mathrm{d}\varphi = \int_0^{2\pi} W(s,\varphi)\,\mathrm{d}\varphi = 0,$$

the regularity conditions are

$$\left.\begin{array}{c} v_0(s) = 0,\\ -u_1(s) + \mathrm{sgn}(s)\mathrm{i}v_1(s) = 0, \quad u_{1,s}(s) = 0,\\ -u_{-1}(s) - \mathrm{sgn}(s)\mathrm{i}v_{-1}(s) = 0, \quad u_{-1,s}(s) = 0,\\ U(s,\varphi) = V_{,\varphi}(s,\varphi) = 0,\\ w_{0,s}(s) = w_{0,ss}(s) = W(s,\varphi) = W_{,s}(s,\varphi) = 0, \end{array}\right\} \quad \text{at} \quad s = \pm(L + R\pi/2) \tag{18'}$$

(the time variable has been omitted)

(4) along the lines $s = \pm L$, the shell displacement components and the efforts densities—as defined by equations (6), (6′), (6″) and (13)—must satisfy the following continuity conditions:

$$\lim_{\varepsilon\to 0}\{u(\pm L + \varepsilon,\varphi) - u(\pm L - \varepsilon,\varphi)\} = 0,$$

$$\lim_{\varepsilon\to 0}\{v(\pm L + \varepsilon,\varphi) - v(\pm L - \varepsilon,\varphi)\} = 0,$$

$$\lim_{\varepsilon\to 0}\{w(\pm L + \varepsilon,\varphi) - w(\pm L - \varepsilon,\varphi)\} = 0,$$

$$\lim_{\varepsilon\to 0}\{\partial_s w(\pm L + \varepsilon,\varphi) - \partial_s w(\pm L - \varepsilon,\varphi)\} = 0, \tag{19}$$

$$\lim_{\varepsilon\to 0}\left[u_{,s} + \frac{v}{R}(v_{,\varphi} + w)\right](-L + \varepsilon,\varphi) = \lim_{\varepsilon\to 0}\left[u_{,s} + \frac{v}{R}(v_{,\varphi} + w) + \frac{w}{R}\right](-L - \varepsilon,\varphi),$$

$$\lim_{\varepsilon\to 0}\left[u_{,s} + \frac{v}{R}(v_{,\varphi} + w)\right](+L - \varepsilon,\varphi) = \lim_{\varepsilon\to 0}\left[u_{,s} + \frac{v}{R}(v_{,\varphi} + w) + \frac{w}{R}\right](+L + \varepsilon,\varphi),$$

$$\lim_{\varepsilon\to 0}\left(\frac{u_{,\varphi}}{R}+v_{,s}\right)(-L+\varepsilon,\varphi)=\lim_{\varepsilon\to 0}\left(\frac{u_{,\varphi}}{R}+v_{,s}\right)(-L-\varepsilon,\varphi),$$

$$\lim_{\varepsilon\to 0}\left(\frac{u_{,\varphi}}{R}+v_{,s}\right)(+L-\varepsilon,\varphi)=\lim_{\varepsilon\to 0}\left(\frac{u_{,\varphi}}{R}+v_{,s}\right)(+L+-\varepsilon,\varphi),$$

$$\lim_{\varepsilon\to 0}\left(w_{,ss}+\frac{v}{R^2}w_{,\varphi\varphi}\right)(-L+\varepsilon,\varphi)=\lim_{\varepsilon\to 0}\left(w_{,ss}+\frac{v}{R^2}w_{,\varphi\varphi}\right)(-L-\varepsilon,\varphi),$$

$$\lim_{\varepsilon\to 0}\left(w_{,ss}+\frac{v}{R^2}w_{,\varphi\varphi}\right)(+L-\varepsilon,\varphi)=\lim_{\varepsilon\to 0}\left(w_{,ss}+\frac{v}{R^2}w_{,\varphi\varphi}\right)(+L+\varepsilon,\varphi),\qquad(19')$$

$$\lim_{\varepsilon\to 0}\left\{\left(w_{,ss}+\frac{w_{,\varphi\varphi s}}{R^2}\right)+\frac{1-v}{R^2}w_{,\varphi\varphi s}\right\}(-L+\varepsilon,\varphi)$$

$$=\lim_{\varepsilon\to 0}\left\{\left(w_{,sss}+\frac{w_{,\varphi\varphi s}}{R^2}+v\frac{w_{,s}}{R^2}\right)+\frac{1-v}{R^2}w_{,\varphi\varphi s}\right\}(-L-\varepsilon,\varphi),$$

$$\lim_{\varepsilon\to 0}\left\{\left(w_{,sss}+\frac{w_{,\varphi\varphi s}}{R^2}\right)+\frac{1-v}{R^2}w_{,\varphi\varphi s}\right\}(+L-\varepsilon,\varphi)$$

$$=\lim_{\varepsilon\to 0}\left\{\left(w_{,sss}+\frac{w_{,\varphi\varphi s}}{R^2}+v\frac{w_{,s}}{R^2}\right)+\frac{1-v}{R^2}w_{,\varphi\varphi s}\right\}(+L+\varepsilon,\varphi),\qquad(19'')$$

It is interesting, in particular for numerical purposes, to be left with equations along Σ only. For this purpose, a boundary integral representation of the diffracted pressure is introduced: use of a hybrid layer potential representation is preferred to the Green representation; its interest appears when harmonic regimes are considered:

$$p(M,t)=\int_0^{+\infty}\mathrm{d}t'\int_\Sigma \mu(M',t')(\mathscr{G}(M,\ M';t-t')+\varepsilon\,\partial_{r'}\mathscr{G}(M,M';t-t'))\,\mathrm{d}M'\qquad(20)$$

$$\text{with } \mathscr{G}(M,M';t-t')=-\frac{\delta(t-t'-r(M,M')/c_0)}{4\pi r(M,M')},$$

where $\mathscr{G}(M,M';t-t')$ is the freefield Green's kernel of the wave equation satisfying the outgoing wave condition; $r(M,M')$ is the distance between M and M'; $\mu(M',t')$ is an unknown square integrable layer density; ε is an arbitrary constant. The Green kernel involving a Dirac measure, the above integrals must be understood as duality products. One can introduce the following boundary operators:

$$\kappa_1(\mu)=\mathrm{Tr}\left\{\int_0^{+\infty}\mathrm{d}t'\int_\Sigma \mu(M',t')(\mathscr{G}(M,\ M';t-t')+\varepsilon\,\partial_{r'}\mathscr{G}(M,M';t-t'))\,\mathrm{d}M'\right\},$$

$$\kappa_2(\mu)=\mathrm{Tr}\,\partial_r\left\{\int_0^{+\infty}\mathrm{d}t'\int_\Sigma \mu(M',t')(\mathscr{G}(M,\ M';t-t')+\varepsilon\,\partial_{r'}\mathscr{G}(M,M';t-t'))\,\mathrm{d}M'\right\}.$$

The variational equations (15) thus become

$$\frac{Eh}{1-v^2}\int_0^{+\infty}\left(\mathscr{H}(u,v,w;\delta u,\delta v,\delta w)(t)+\varrho_s h\int_\Sigma[\ddot{u}\,\delta u^*+\ddot{v}\,\delta v^*+\ddot{w}\,\delta w^*](M,t)\,\mathrm{d}M\right)\mathrm{d}t$$

$$+\int_0^{+\infty}\int_\Sigma [\kappa_1(\mu)\,\delta w^*](M,t)\,\mathrm{d}M\,\mathrm{d}t = -\int_0^{+\infty}\int_\Sigma [\mathrm{Tr}\, p^i\,\delta w^*](M,t)\,\mathrm{d}M\,\mathrm{d}t,$$

$$\varrho_0\int_0^{+\infty}\int_\Sigma [\ddot{w}\,\delta\psi^*](M,t)\,\mathrm{d}M\,\mathrm{d}t + \int_0^{+\infty}\int_\Sigma [\kappa_2(\mu)\,\delta\psi^*](M,t)\,\mathrm{d}M\,\mathrm{d}t$$

$$= -\int_0^{+\infty}\int_\Sigma [\mathrm{Tr}\,\partial_r p^i\,\delta\psi^*](M,t)\,\mathrm{d}M\,\mathrm{d}t, \quad \forall\,\delta u, \delta v, \delta w, \delta\psi. \tag{21}$$

Finally, all the functions involved in equations (21) being 2π-periodic with respect to the variable φ, this equation can be replaced by a sequence of variational equations for the Fourier components of the unknown functions. Let $f(s, \varphi)$ be any function defined on Σ; its Fourier series is written as

$$f(s,\varphi) = \sum_{n=-\infty}^{+\infty} f_n(s)\mathrm{e}^{in\varphi}.$$

The Fourier components of the operators $\mathscr{H}$ are, respectively, denoted by $\mathscr{H}^n$: they are obtained by replacing the derivation operator ∂_φ by in. The Fourier components of κ_1 and κ_2 are denoted κ_1^n and κ_2^n: they are deduced from the Fourier components of the freefield Green's kernel which are known in spherical and in cylindrical co-ordinates as inverse time Fourier transforms (see, for example, reference [12]). The variational equations are replaced by the following set of unidimensional variational equations:

$$\frac{Eh}{1-\nu^2}\int_0^{+\infty}\left(\mathscr{H}^n(u_n, v_n, w_n; \delta u_n, \delta v_n, \delta w_n)(t) + \varrho_s h\int_{\mathscr{L}} [\ddot{u}_n\,\delta u_n^* + \ddot{v}_n\,\delta v_n^* + \ddot{w}_n\,\delta w_n^*](s,t)\,\mathrm{d}s\right)\mathrm{d}t$$

$$+\int_0^{+\infty}\int_{\mathscr{L}} [\kappa_1^n(\mu_n)\,\delta w_n^*](s,t)\,\mathrm{d}s\,\mathrm{d}t = -\int_0^{+\infty}\int_{\mathscr{L}} [\mathrm{Tr}\,p_n^i\,\delta w_n^*](s,t)\,\mathrm{d}s\,\mathrm{d}t,$$

$$\varrho_0\int_0^{+\infty}\int_{\mathscr{L}} [\ddot{w}_n\,\delta\psi_n^*](s,t)\,\mathrm{d}s\,\mathrm{d}t + \int_0^{+\infty}\int_{\mathscr{L}} [\kappa_2^n(\mu_n)\,\delta\psi_n^*](s,t)\,\mathrm{d}s\,\mathrm{d}t$$

$$= -\int_0^{+\infty}\int_{\mathscr{L}} [\mathrm{Tr}\,\partial_r p_n^i\,\delta\psi_n^*](s,t)\,\mathrm{d}s\,\mathrm{d}t, \quad \forall\,\delta u_n, \delta v_n, \delta w_n, \delta\psi_n \quad \text{and} \quad -\infty < n < +\infty, \tag{22}$$

where $\mathscr{L}$ is the interval $[-L - R\pi/2, +L + R\pi/2]$ of variation of s. The functions u_n, v_n, w_n, δu_n, δv_n and δw_n satisfy the continuity conditions and the regularity conditions at the apexes deduced from equations (18), (18′), (19), (19′) and (19″).

3.2. EIGENMODES AND RESONANCE MODES

Consider now a harmonic time dependence ($\mathrm{e}^{-\mathrm{i}\omega t}$). The time Fourier transform of a function f is denoted by $\hat{f}$. The set of equations (22) becomes

$$\frac{Eh}{1-\nu^2}\mathscr{H}^n(\hat{u}_n, \hat{v}_n, \hat{w}_n; \delta\hat{u}_n, \delta\hat{v}_n, \delta\hat{w}_n) - \varrho_s h\omega^2\int_{\mathscr{L}} [\hat{u}_n\,\delta\hat{u}_n^* + \hat{v}_n\,\delta\hat{v}_n^* + \hat{w}_n\,\delta\hat{w}_n^*](s)\,\mathrm{d}s$$

$$+\int_{\mathscr{L}} [\hat{\kappa}_{1\omega}^n(\hat{\mu})\,\delta w_n^*](s)\,\mathrm{d}s = -\int_{\mathscr{L}} [\mathrm{Tr}\,\hat{p}_n^i\,\delta\hat{w}_n^*](s)\,\mathrm{d}s,$$

$$- \varrho_0 \omega^2 \int_{\mathscr{L}} [\hat{w}_n \, \delta\psi_n^*](s)\,\mathrm{d}s + \int_{\mathscr{L}} [\hat{\kappa}^n_{2\omega}(\hat{\mu}_n)\,\delta\psi_n^*](s)\,\mathrm{d}s$$

$$= - \int_{\mathscr{L}} [\mathrm{Tr}\,\partial_r \hat{p}^i_n \,\delta\hat{\psi}_n^*](s)\,\mathrm{d}s, \quad \forall\, \delta\hat{u}_n, \delta\hat{v}_n, \delta\hat{w}_n, \delta\hat{\psi}_n \quad \text{and} \quad -\infty < n < +\infty. \tag{23}$$

The boundary operators $\hat{\kappa}^n_{1\omega}$ and $\hat{\kappa}^n_{2\omega}$ are defined by

$$\hat{\kappa}^n_{\omega}(\hat{\mu}_n)(M) = -2\pi \int_{\mathscr{L}} \hat{\mu}_n(s')[G^n(M, M') + \varepsilon\,\partial_{r'} G^n(M, M')]\,\mathrm{d}s',$$

$$G_n(M, M') = - \int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}\omega r(M;\rho',z',\theta')/co}}{4\pi r(M;\rho',z',\theta')}\,\mathrm{e}^{\mathrm{i}n\theta'}\,\mathrm{d}\theta,$$

$$\hat{\kappa}^n_{1\omega}(\hat{\mu}_n) = \mathrm{Tr}\,\hat{\kappa}^n_{\omega}(\mu_n), \quad \hat{\kappa}^n_{2\omega}(\hat{\mu}_n) = \mathrm{Tr}\,\partial_r \hat{\kappa}^n_{\omega}(\mu_n), \tag{23'}$$

where the point M' is defined by its cylindrical co-ordinates (ρ', z', θ'); let s' be its curvilinear abscissa on the shell surface.

The eigenmodes $(\tilde{u}^m_n, \tilde{v}^m_n, \tilde{w}^m_n, \tilde{\mu}^m_n)$ of the fluid-loaded shell and the eigenvalues Λ^m_n are the solutions of the following homogeneous equations:

$$\frac{Eh}{1-\nu^2}\mathscr{H}^n(\tilde{u}^m_n, \tilde{v}^m_n, \tilde{w}^m_n; \delta\hat{u}_n, \delta\hat{v}_n, \delta\hat{w}_n) - \Lambda^m_v \left\{ \int_{\mathscr{L}} [\tilde{u}^m_n\,\delta\hat{u}^*_n + \tilde{v}^m_n\,\delta\hat{v}^*_n + \tilde{w}^m_n\,\delta\hat{w}^*_n](s)\,\mathrm{d}s \right.$$

$$\left. - \frac{1}{\omega^2 \varrho_s h} \int_{\mathscr{L}} [\hat{\kappa}^n_{1\omega}(\tilde{\mu}^m_n)\,\delta w^*_n](s)\,\mathrm{d}s \right\} = 0.$$

$$- \varrho_0\omega^2 \int_{\mathscr{L}} [\tilde{w}^m_n\,\delta\hat{\psi}^*_n](s)\,\mathrm{d}s + \int_{\mathscr{L}} [\hat{\kappa}^n_{2\omega}(\tilde{\mu}^m_n)\,\delta\hat{\psi}^*_n](s)\,\mathrm{d}s = 0,$$

$$\forall \delta\hat{u}_n, \delta\hat{v}_n, \delta\hat{w}_n, \delta\hat{\psi}_n \quad \text{and} \quad -\infty < n < +\infty. \tag{24}$$

They depend on the angular frequency through the boundary operator $\hat{\kappa}^n_{1\omega}$ and $\hat{\kappa}^n_{2\omega}$. The eigenmodes $(\tilde{u}^{-m}_n, \tilde{v}^{-m}_n, \tilde{w}^{-m}_n, \tilde{\mu}^{-m}_n)$ and eigenvalues Λ^{-m}_n which correspond to a negative angular frequency $-\omega$ are given by

$$(\tilde{u}^{-m}_n, \tilde{v}^{-m}_n, \tilde{w}^{-m}_n, \tilde{\mu}^{-m}_n) = (\tilde{u}^{m*}_n, \tilde{v}^{m*}_n, \tilde{w}^{m*}_n, \tilde{\mu}^{m*}_n), \quad \Lambda^{-m}_n = \Lambda^{m*}_n.$$

The eigenmodes satisfy the orthogonality relationship

$$\frac{Eh}{1-\nu^2}\mathscr{H}^n(\tilde{u}^m_n, \tilde{v}^m_n, \tilde{w}^m_n; \tilde{u}^{q*}_n, \tilde{v}^{q*}_n, \tilde{w}^{q*}_n) = \mathscr{N}^q_n \delta^q_m$$

or

$$\int_{\mathscr{L}} [\tilde{u}^m_n \tilde{u}^q_n + \tilde{v}^m_n \tilde{v}^q_n + \tilde{w}^m_n \tilde{w}^q_n](s)\,\mathrm{d}s - \frac{1}{\omega^2 \varrho_s h} \int_{\mathscr{L}} [\hat{\kappa}^n_{1\omega}(\tilde{\mu}^m_n)\, w^q_n](s)\,\mathrm{d}s = \frac{\mathscr{N}^q_n \delta^q_m}{\Lambda^q_n}, \tag{25}$$

where δ^q_m is the Kronecker symbol. These equalities are obtained by (1) replacing $(\delta\hat{u}_n, \delta\hat{v}_n, \delta\hat{w}_n)$ by $(\tilde{u}^{q*}_n, \tilde{v}^{q*}_n, \tilde{w}^{q*}_n)$ in equation (24); and (2) applying the Green formula to $\hat{\kappa}_n(\tilde{\mu}^m_n)$ and $\hat{\kappa}_n(\tilde{\mu}^q_n)$.

The resonance modes $(u_n^m, v_n^m, w_n^m, \mu_n^m)$ and the resonance angular frequencies ω_n^m are defined as the solutions of the homogeneous equations

$$\frac{Eh}{1-\nu^2}\mathscr{H}^n(u_n^m, v_n^m, w_n^m; \delta\hat{u}_n, \delta\hat{v}_n, \delta\hat{w}_n) - \varrho_s h\omega_n^{m2}\left\{\int_{\mathscr{L}}[u_n^m\,\delta\hat{u}_n^* + v_n^m\,\delta\hat{v}_n^* + w_n^m\,\delta\hat{w}_n^*](s)\,\mathrm{d}s\right.$$

$$\left. - \frac{1}{\omega_n^{m2}\varrho_s h}\int_{\mathscr{L}}[\hat{\kappa}^n_{1\omega_n^m}(\mu_n^m)\,\delta\hat{w}_n^*](s)\,\mathrm{d}s\right\} = 0,$$

$$-\varrho_0\omega_n^{m2}\int_{\mathscr{L}}[w_n^m\,\delta\hat{\psi}_n^*](s)\,\mathrm{d}s + \int_{\mathscr{L}}[\hat{\kappa}^n_{2\omega_n^m}(\mu_n^m)\,\delta\hat{\psi}_n^*](s)\,\mathrm{d}s = 0,$$

$$\forall\delta\hat{u}_n, \delta\hat{v}_n, \delta\hat{w}_n, \delta\hat{\psi}_n \quad \text{and} \quad -\infty < n < +\infty. \tag{26}$$

The resonance modes are the free oscillations of the fluid-loaded shell. It has been shown [6] that the resonance modes have a positive damping: that is,

$$\omega_n^m = \Omega_n^m - \mathrm{i}\tau_n^m, \quad \omega_n^{-m} = -\Omega_n^m - \mathrm{i}\tau_n^m \quad \text{with } \tau_n^m > 0.$$

This result is in accordance with the general theory presented in reference [13].

3.3. RESONANCE MODES SERIES OF THE RESPONSE OF THE FLUID-LOADED SHELL TO A TRANSIENT EXCITATION

Let U_n denote the nth Fourier component of the shell displacement vector with components (u_n, v_n, w_n) and $\mathscr{U}_n^m$ the resonance displacement mode vector with components (u_n^m, v_n^m, w_n^m). The solution (u_n, v_n, w_n, μ_n) of equations (22) is sought as a resonance modes series of the form

$$\begin{pmatrix} U_n \\ \mu_n \end{pmatrix} = Y(t)\sum_{m=1}^{+\infty}\left\{\alpha_n^m\begin{pmatrix}\mathscr{U}_n^m \\ \mu_n^m\end{pmatrix}\mathrm{e}^{-\mathrm{i}\omega_n^m t} + \alpha_n^{-m}\begin{pmatrix}\mathscr{U}_n^{-m} \\ \mu_n^{-m}\end{pmatrix}\mathrm{e}^{-\mathrm{i}\omega_n^{-m} t}\right\}, \tag{27}$$

where $Y(t)$ denotes the Heaviside step function.

3.3.1. *Direct solution*

This expression can be introduced into equations (22) with $(u_n^{q*}, v_n^{q*}, w_n^{q*}, \mu_n^{q*})$ as test functions. Thus, an infinite system of linear algebraic equations is obtained to determine the coefficients of the series expansion. From a numerical point of view, an approximation of the solution of this system is obtained by a truncation procedure. Nevertheless, an analytical expression of the coefficients can be established.

3.3.2. *Inverse Fourier transform method*

The solution of equations (23) is sought as a series of the eigenmodes, that is

$$\begin{pmatrix} \hat{U}_n \\ \hat{\mu}_n \end{pmatrix} = \sum_{m=1}^{\infty}\hat{\alpha}_n^m\begin{pmatrix}\tilde{U}_n^m \\ \tilde{\mu}_n^m\end{pmatrix}, \tag{28}$$

where $\hat{U}_n$ is the shell displacement vector with components $(\hat{u}_n, \hat{v}_n, \hat{w}_n)$ and $\tilde{U}_n^m$ is the eigenmode displacement vector with components $(\tilde{u}_n^m, \tilde{v}_n^m, \tilde{w}_n^m)$. By replacing the test functions by $(\tilde{u}_n^{q*}, \tilde{v}_n^{q*}, \tilde{w}_n^{q*}, \tilde{\mu}_n^{q*})$ and using the orthogonality relationship between the eigenmodes, the coefficients are obtained as

$$\hat{\alpha}_n^m = \frac{\mathcal{N}_n^m(\omega)\Lambda_n^m(\omega)}{\varrho_s h\omega^2 - \Lambda_n^m(\omega)}\left[\int_{\mathcal{L}} \mathrm{Tr}\,\hat{p}_i^n \tilde{w}_n^m\right](\omega). \tag{29}$$

This expression is introduced into equation (28) and the inverse Fourier transform of the corresponding series is evaluated by the residue method. It is easily seen that the coefficients α_n^m and α_n^{-m} are given by

$$\alpha_n^m = -\,\mathrm{i}\,\frac{\mathcal{N}_n^m(\omega_n^m)\Lambda_n^m(\omega_n^m)}{2\varrho_s h\omega_n^m - \Lambda_n^{m'}(\omega_n^m)}\int_{\mathcal{L}} \mathrm{Tr}\,\hat{p}_i^n(\omega_n^m)w_n^m,$$

$$\alpha_n^{-m} = -\,\mathrm{i}\,\frac{\mathcal{N}_n^m(\omega_n^{-m})\Lambda_n^m(\omega_n^{-m})}{2\varrho_s h\omega_n^{-m} - \Lambda_n^{m'}(\omega_n^{-m})}\int_{\mathcal{L}} \mathrm{Tr}\,\hat{p}_i^n(\omega_n^{-m})w_n^{-m}, \tag{30}$$

where $\Lambda_n^{m'}(\omega) = \partial\Lambda_n^m(\omega)/\partial\omega$. The derivatives of the eigenvalues with respect to ω are not known explicitly. This restricts the use of the analytical expressions (30) to simple cases as, for example, when the fluid is a gas and that a light-fluid approximation can be used.

It must be recalled that, if the excitation term is real, then the solution given by equation (29) with the expansion coefficients given by equation (15) is also real.

4. NUMERICAL METHOD AND RESULTS

The main problem is to compute the resonance frequencies and modes of the fluid-loaded structure, that is to solve the homogeneous system of equations (26). The unknown functions can be approximated by various methods; the most popular is the finite element method. A polynomial approximation method is used here, however.

4.1. POLYNOMIAL APPROXIMATION OF THE RESONANCE MODES

The advantage of such an approximation is to provide approximate resonance modes which are regular functions (indefinitely derivable) and thus have the regularity required.

The layer density μ_n^m is approximated by a truncated series of Legendre polynomials of the variable s.

The shell displacement components are approximated by a truncated series of polynomial functions built with a finite number of Legendre polynomials. Each polynomial function is chosen so that it satisfies the regularity conditions at the apexes. Thus, two different approximations are adopted: (1) for the tangential displacement components u_n^m and v_n^m, the approximation functions involve a linear combination of two Legendre polynomials $P_q(s)$ and $P_{q+2}(s)$, with $q = 0, 1, \dots$; and (2) for the normal displacement components w_n^m, the approximation functions involve a linear combination of three Legendre polynomials $P_q(s)$, $P_{q+2}(s)$ and $P_{q+4}(s)$, with $q = 0, 1, \dots$. In both cases, the coefficients of the linear combination depend on the angular harmonic index n and are calculated analytically. The truncated series are introduced into the resonance mode equations, in which the approximation functions are used as test functions. A linear system of equations is thus obtained for the coefficients of the expansions of the resonance modes. The resonance angular frequencies ω_n^m are approximated by the values of ω for which the determinant of the system equals to zero. All the details can be found in reference [6] and will be presented in a forthcoming paper devoted to comparisons of the predicted transient response of the fluid-loaded shell with experiments.

4.2. COMPARISON OF COMPUTED RESONANCE FREQUENCIES WITH MEASURED ONES

The first 10 resonance frequencies of a *Line* 2′ shell have been computed for the following data: $R = L = 27$ mm, $h = 0{\cdot}81$ mm, $\varrho_s = 7900\ \mathrm{kg/m^2}$, $E = 199{\cdot}73$ GPa, $\nu = 0{\cdot}314$, $\varrho_0 = 1000\ \mathrm{kg/m^3}$, $c_0 = 1470$ m/s.

Table 1 presents a comparison between the experimental results published in reference [14] with the values predicted by the method proposed here. For information, the *in vacuo* resonance frequencies have been calculated. It appears that the agreement is excellent: the relative error on the real part of the resonance frequencies is 1·8% on the first one and 0·12% on the tenth one. It must be recalled that the physical data, and, in particular the Young's, modulus and the Poisson ratio, have been measured very accurately by the authors of the experimental study. This explains that the numerical prediction can agree so well with the experiments.

The numerical method used by the authors in references [1–3] couples the finite element method and a boundary element method to predict the acoustic diffraction by elastic structures. To the authors' knowledge, the method has been applied to the diffraction problem by spherical shells for $0 \leqslant kR \leqslant 15$ and by a *Line*-2′ shell for $0 \leqslant kR \leqslant 10$ ($k = \omega/c_0$ is the acoustic wavenumber in the fluid). As will be detailed in a forthcoming paper, the resonance modes series method enables one to predict the diffracted field over a frequency band $0 \leqslant kR \leqslant 50$ for spherical shells and $0 \leqslant kR \leqslant 30$ for a *Line*-2′ shell while keeping a reasonable computational time. Thus, it seems that, for a similar numerical cost, the method based on the resonance modes expansion enables one to reach higher frequencies than the direct method described in reference [3]. Indeed, the numerical cost of the resonance modes method mainly depends on the number of modes required to predict with a sufficient accuracy the transient response of a fluid-loaded structure whereas the cost for a direct resolution of the problem in the frequency domain is closely related to the frequency step required by the analysis.

TABLE 1

Comparison between computed and measured resonance frequencies for a steel-made Line 2′ shell with radius $R = 27$ mm, total length $4R$ and thickness $h = 0{\cdot}81$ mm

Resonance frequencies of *Line* 2′ (kHz)				
Measured fluid loaded	Computed fluid loaded	Relative error (%)	Computed *in vacuo*	Mode no.
68·0	$69{\cdot}3 - i3{\cdot}03 10^{-6}$	1·8	72·6	3
88·0	$88{\cdot}6 - i2{\cdot}2$	0·7	89·9	4
107·1	$107{\cdot}9 - i0{\cdot}0059$	0·74	108·0	5
124·0	$121{\cdot}4 - i5{\cdot}15$	2	127·1	6
143·5	$143{\cdot}4 - i1{\cdot}2$	0·07	146·1	7
163·8	$164{\cdot}4 - i7{\cdot}1$	0·36	167·9	8
183·6	$183{\cdot}3 - i2{\cdot}8$	0·14	184·7	9
198·6	$198{\cdot}9 - i0{\cdot}6$	0·15	199·0	10
239·0	$239{\cdot}9 - i8{\cdot}2$	0·37	240·2	11
257·0	$256{\cdot}7 - i6{\cdot}3$	0·12	259·5	12

5. CONCLUSION

The first important results of this work concerns the regularity conditions of a thin spherical shell at the apexes. As has been mentioned, the only result which looks correctly established [10] is slightly different from that presented here. But the proof does not include enough details to understand the reason for this difference. The proof here is based on the finiteness of local efforts and energy density together with the hypothesis that the simplest equations—The Donnell and Mushtari approximation—are valid.

The method developed for the calculation of the shell response to a transient excitation does not seem to be very often used. In general, the response of a fluid-loaded structure is expanded in terms of the *in vacuo* resonance modes. It has been proposed here to use the fluid-loaded resonance modes of the structure, instead. In the authors' opinion they are much better adapted.

The approximation of the resonance modes by polynomial functions is not quite new. Other techniques, as, for example, finite elements approximations, could be used. Nevertheless, the authors think that it is efficient to account for the simple geometry of the *Line*-2′ structure which authorizes the use of orthogonal polynomials. The main advantage is that the Ritz–Galerkin equations which approximate the variational equations can be reduced to collocation equations [6]: the computing time is thus reduced while keeping the same accuracy of the result. In a forthcoming paper, the numerical method will be presented with much more detail and comparisons between numerical predictions and experiments will prove the efficiency of the fluid-loaded modes expansion for the representation of the response of a thin structure—sphere or *Line*-2′—to a transient incident acoustic wave.

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