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APPLICATION OF THE SPLINE ELEMENT METHOD TO ANALYZE VIBRATION OF SKEW MINDLIN PLATES WITH VARYING THICKNESS IN ONE DIRECTION

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This paper presents an application of the spline element method based on the Mindlin plate theory to analyze the vibration of thick skew plates with varying thickness in the longitudinal direction. To demonstrate the convergence and accuracy of the present method, several examples are solved, and results are compared with those obtained by other numerical methods. Good accuracy is obtained. Frequencies of skew Mindlin plates with varying cross-sections having arbitrary boundary conditions are analyzed for different thickness ratios, aspect ratios, ratios of the width to thickness and skew angles.

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1. INTRODUCTION

Skew plates are employed in several practical structures, such as skew bridge decks, aircraft wings, vehicle bodies and ship fulls. Much research has been carried out on the free vibration of thin skew plates [1, 2]. However, thick skew plates with variables thickness have received little attention. The limitation of thin plate theory is that the effect of transverse shear deformation and rotary inertia is not considered. To allow the effect of shear deformation and rotary inertia, Mindlin [3] proposed the so-called first order shear deformation plate theory.

For free vibration analysis of thick skew plates, several researchers have considered the effects of shear deformation and rotary inertia. Kanaka Raju and Hinton [4] presented results for rhombic Mindlin plates with arbitrary boundary conditions using the finite element method. Ganesan and Nagaraja Rao [5] analyzed the vibration of thick skew plates by a variational approach. Liew *et al.* [6] presented the frequencies of thick skew plates using the pb-2 Rayleigh–Ritz method. McGee *et al.* [7, 8] analyzed natural frequencies of thick skew plates by the finite element methods based on both the Mindlin plate theory and the higher order shear deformation plate theory. They compared the results with those obtained by the three-dimensional finite method [9].

Although most of the works are restricted and related to skew plates of uniform thickness, several authors attacked problems on plates of variable thickness. Chopra and Durvasula [10] analyzed the vibration of simply supported skew thin plates having a linear variation in thickness in one direction using the Ritz method with the double Fourier sine series. Dokainish and Kumar [11] analyzed frequencies of clamped skew thin plate with linearly tapered thickness by the Galerkin method. Banerjee [12] presented frequencies of skew thin plates of variable thickness using the Galerkin method. However, published vibration results for thick skew plates having variable thickness are few. Matsuda and Sakiyama [13]

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analyzed frequencies of skew Mindlin plates with linearly tapered thickness in the longitudinal direction by using the discrete integral equation method.

Recently, some types of spline element methods have been presented to analyze bending and vibrating plates. One of them is the spline finite strip method developed by Cheung and Fan [14] in which the mid-plane deflections are expressed as the sum of a finite series of products of B-spline functions and polynomial interpolation functions. The spline element method presented by Mizusawa *et al.* [2] using B-spline functions was regarded as an alternative form of the displacement-based finite element procedure to analyze vibration of skew thin plates. A new spline element model which employs a set of B-spline shape functions for interpolation of displacements and adopts a subparametric approach for general geometry of plates was proposed by Fan and Luah [15, 16] to analyze bending and vibration of plates.

In the work described in this paper, the spline element method has been used to analyze the vibration of isotropic, skew Mindlin plates of varying thickness. The convergence and accuracy of the present method are demonstrated by comparisons with results obtained by other numerical method. The effects of varying cross-sections, thickness ratios, taper thickness parameters and skew angles on frequencies of skew Mindlin plates with arbitrary boundary conditions are shown.

2. SPLINE ELEMENT METHOD

The solution procedure of vibration of skew thick plates with varying thickness in the x direction is based on the Mindlin plate theory [3] and the spline element method [2], which can be regarded as an alternative form of the finite element method.

The plate is idealized by the discrete skew elements as shown in Figure 1. It is convenient to introduce the non-dimensional skew co-ordinate systems (ξ, η)

$$\xi = x/a, \quad \eta = y/b, \quad W' = W/b, \tag{1}$$

in which a and b are the length and width of the plate respectively.

The displacement functions (ϕ_x, ϕ_y, W') in an element are expressed in terms of two-way spline functions as

$$\phi_x = \sum_{m=1}^{i_x} \sum_{n=1}^{i_y} A_{mn} N_{m,k}(\xi) N_{n,k}(\eta) = [N]_{mn}\{\delta_A\}_{mn},$$

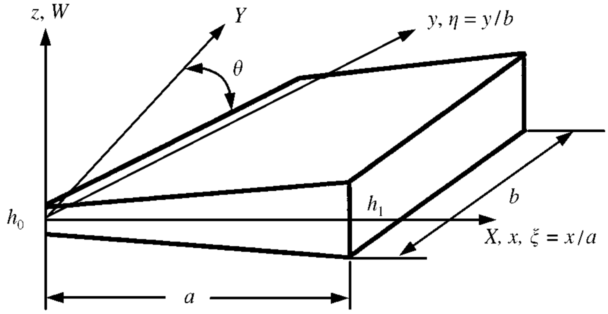

Figure 1. Skew Mindlin plate with varying thickness in the x direction and co-ordinate systems.

$$\phi_y = \sum_{m=1}^{i_x} \sum_{n=1}^{i_y} B_{mn} N_{m,k}(\xi) N_{n,k}(\eta) = [N]_{mn} \{\delta_B\}_{mn},$$

$$W' = \sum_{m=1}^{i_x} \sum_{n=1}^{i_y} C_{mn} N_{m,k}(\xi) N_{n,k}(\eta) = [N]_{mn} \{\delta_C\}_{mn}, \tag{2}$$

where

$$[N]_{mn} = [N_{1,k}(\xi) N_{1,k}(\eta), \ldots, N_{i_x,k}(\xi) N_{i_y,k}(\eta)], \tag{3}$$

$$\{\delta_A\}_{mn} = \{A_{11} A_{12} \ldots A i_x i_y\}^{\mathrm{T}},$$

$$\{\delta_B\}_{mn} = \{B_{11} B_{12} \ldots B i_x i_y\}^{\mathrm{T}}, \tag{4}$$

$$\{\delta_C\}_{mn} = \{C_{11} C_{12} \ldots C i_x i_y\}^{\mathrm{T}},$$

in which $i_x = k - 1 + M_x$, $i_y = k - 1 + M_y$, and $N_{m,k}(\xi)$ and $N_{n,k}(\eta)$ are the normalized B-spline functions, $k - 1$ is the degree of B-spline functions, M_x and M_y are the number of elements in the x and y directions respectively. $\{\delta_A\}_{mn}$, $\{\delta_B\}_{mn}$ and $\{\delta_C\}_{mn}$ are the unknown parameters which can be determined by the principle of minimum total potential energy.

Equations (2) can be expressed in matrix form as follows:

$$\{d\} = [S]_{mn} \{\Delta\}_{mn}, \tag{5}$$

where

$$\{d\} = \{\phi_x, \phi_y, W'\}, \tag{6}$$

$$\{\Delta\}_{mn} = \{\{\delta_A\}_{mn}, \{\delta_B\}_{mn}, \{\delta_C\}_{mn}\}^{\mathrm{T}} \tag{7}$$

and

$$[S]_{mn} = \sum_{m=1}^{i_x} \sum_{n=1}^{i_x} \begin{bmatrix} N_{m,k}(\xi) N_{n,k}(\eta) & 0 & 0 \\ 0 & N_{m,k}(\xi) N_{n,k}(\eta) & 0 \\ 0 & 0 & N_{m,k}(\xi) N_{n,k}(\eta) \end{bmatrix}. \tag{8}$$

In the Mindlin plate theory, the generalized strains comprise the curvature changes $(\varepsilon_x, \varepsilon_y, \varepsilon_{xy})$ and the transverse shear strains $(\gamma_{xz}, \gamma_{yz})$ which are defined as follows:

$$\{\varepsilon\}_b = \left\{ \begin{matrix} (z/a)\partial\phi_x/\partial\xi \\ (z/b)\sec\theta\partial\phi_y/\partial\eta - (z/a)\tan\theta\partial\phi_y/\partial\xi \\ (z/a)\partial\phi_y/\partial\xi + (z/b)\sec\theta\partial\phi_x/\partial\eta - (z/a)\tan\theta\partial\phi_x/\partial\xi \end{matrix} \right\} \tag{9}$$

and

$$\{\varepsilon\}_s = \left\{ \begin{matrix} \phi_x + (b/a)\partial W'/\partial\xi \\ \phi_y + \sec\theta\partial W'/\partial\eta - (b/a)\tan\theta\partial W'/\partial\xi \end{matrix} \right\}, \tag{10}$$

where θ is the angle of the skew plate.

Substituting equations (2) into equations (9) and (10), and performing the differentiations, the strain vector $\{\chi\}$ is obtained by

$$\{\chi\} = \begin{Bmatrix} \{\varepsilon\}_b \\ \{\varepsilon\}_s \end{Bmatrix} = [T][S]_{mn}\{\Delta\}_{mn} = [B]_{mn}\{\Delta\}_{mn}. \tag{11}$$

The differential matrix operator $[T]$ and the strain matrix $[B]_{mn}$ of an element are defined as follows:

$$[T] = \begin{bmatrix} (1/a)\partial/\partial\xi & 0 & 0 \\ 0 & \begin{matrix}(1/b)\sec\theta\partial/\partial\eta \\ -(1/a)\tan\theta\partial/\partial\xi\end{matrix} & 0 \\ \begin{matrix}(1/b)\sec\theta\partial/\partial\eta \\ -(1/a)\tan\theta\partial/\partial\xi\end{matrix} & (1/a)\partial/\partial\xi & 0 \\ 1 & 0 & (b/a)\partial/\partial\xi \\ 0 & 1 & \begin{matrix}\sec\theta\partial/\partial\eta \\ -(b/a)\tan\theta\partial/\partial\xi\end{matrix} \end{bmatrix} \tag{12}$$

and

$$[B]_{mn} = [T][S]_{mn} = \sum_{m=1}^{i_x}\sum_{n=1}^{i_y} \begin{bmatrix} (1/a)\dot{N}_m N_n & 0 & 0 \\ 0 & \begin{matrix}(1/b)\sec\theta N_m\dot{N}_n \\ -(1/a)\tan\theta\dot{N}_m N_n\end{matrix} & 0 \\ \begin{matrix}(1/b)\sec\theta N_m\dot{N}_n \\ -(1/a)\tan\theta\dot{N}_m N_n\end{matrix} & (1/a)N_m N_n & 0 \\ N_m N_n & 0 & (b/a)\dot{N}_m N_n \\ 0 & N_m N_n & \begin{matrix}\sec\theta N_m\dot{N}_n \\ -(b/a)\tan\theta\dot{N}_m N_n\end{matrix} \end{bmatrix}$$

$$= \sum_{m=1}^{i_x}\sum_{n=1}^{i_y} \begin{bmatrix} [B_b]_{mn} \\ [B_s]_{mn} \end{bmatrix}, \tag{13}$$

where $N_m = N_{m,k}(\xi)$, $N_n = N_{n,k}(\eta)$, $\dot{N}_m = \partial N_{m,k}(\xi)/\partial\xi$, $\dot{N}_n = \partial N_{n,k}(\eta)/\partial\eta$.

The geometry of varying cross-section of a skew plate is shown in Figure 2. The thickness of the plate varies in the x direction in arbitrary fashion as follows:
(a) Convex cross-section:

$$h(\xi) = h_0\{\lambda - (\lambda - 1)(1 - \xi)^p\} = h_0 H(\xi). \tag{14}$$

(b) Concave cross-section:

$$h(\xi) = h_0\{(\lambda - 1)\xi^p + 1\} = h_0 H(\xi), \tag{15}$$

where $\lambda = h_1/h_0$, h_0 and h_1 are thicknesses at $\xi = 0$ and 1 respectively. p is the order of polynomials of cross-section. If p is unit, it means linearly tapered cross-section.

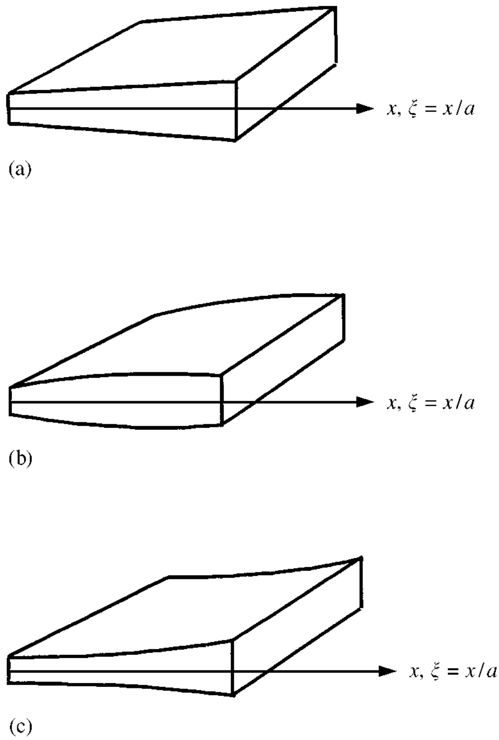

Figure 2. Variation of varying thickness in the x direction: (a) Type 1: $h(\xi) = h_0\{(\lambda - 1)\xi + 1\}$; (b) Type 2 ($p = 2$) and Type 3 ($p = 3$): $h(\xi) = h_0\{\lambda - (\lambda - 1)(\lambda - 1)^p\}$; (c) Type 4 ($p = 2$) and Type 5 ($p = 3$): $h(\xi) = h_0\{(\lambda - 1)\xi^p + 1\}$.

The flexural rigidity of the plate is given by

$$D(\xi) = Eh(\xi)^3/[12(1 - v^2)] = D_0 H(\xi)^3, \tag{16}$$

where $D_0 = Eh_0^3/12(1 - v^2)$, E is the Young modulus and v is the Poisson ratio.

For an isotropic material, the matrices of the flexural and shear rigidities are written as

$$[D]_b = Eh(\xi)^3/12(1 - v^2)\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & (1 - v)/2 \end{bmatrix}, \tag{17}$$

$$[D]_s = Gh(\xi)\kappa\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{18}$$

where $G = E/2(1 + v)$ and κ is a coefficient to take into account the warping of the section.

The strain energy due to bending and transverse shear deformation, U of the skew Mindlin plates with varying thickness is given as

$$U = (ab\cos\theta/2)\int_0^1\int_0^1 [\{\varepsilon\}_b^{\mathrm{T}}[D]_b\{\varepsilon\}_b + \{\varepsilon\}_s^{\mathrm{T}}[D]_s\{\varepsilon\}_s]\,\mathrm{d}\xi\,\mathrm{d}\eta$$

$$= (D_0\,ab\cos\theta/2a^2)\int_0^1\int_0^1 H(\xi)^3[\sec^2\theta\{\partial\phi_x/\partial\xi$$

$$- (a/b) \sin\theta \partial\phi_x/\partial\eta - \sin\theta\partial\phi_y/\partial\xi + (a/b)\partial\phi_y/\partial\eta\}^2$$

$$- (1-\nu)\{2(a/b)(\partial\phi_x/\partial\xi)(\partial\phi_y/\partial\eta) - (1/2)\{(a/b)\partial\phi_x/\partial\eta + \partial\phi_y/\partial\xi\}^2\}]$$

$$+ 6(1-\nu)\kappa(b/h_0)^2(a/b)^2 H(\xi)[\{\cos\theta\phi_x + (b/a)\partial W'/\partial\xi\}^2$$

$$+ \{-\sin\theta\phi_x + \phi_y - (b/a)\tan\theta\partial W'/\partial\xi + \sec\theta\partial W'/\partial\eta\}^2]\, \mathrm{d}\xi\, \mathrm{d}\eta \qquad (19)$$

and the kinetic energy, T is also given by

$$T = (\rho h_0/2)\omega^2 ab^3 \cos\theta \int_0^1 \int_0^1 \{H(\xi)W'^2$$

$$+ (1/12)(h_0/b)^2 H(\xi)^3(\phi_x^2 - 2\sin\theta\phi_x\phi_y + \phi_y^2)\}\, \mathrm{d}\xi\, \mathrm{d}\eta, \qquad (20)$$

where ρ is the mass density of the material, and ω is the circular frequency (rad/s).

To deal with arbitrary boundary conditions along the edges of a plate, the method of artificial springs [2] is applied. Three types of artificial springs corresponding to the deflection W' and the two angles of rotation ϕ_x, ϕ_y are introduced at each edge of the plate.

The functional of the plate, Π, is expressed as follows:

$$\Pi = U - T. \qquad (21)$$

By substituting equations (2) into equation (21) and using the principle of minimum potential energy, the coefficients $\{\Delta\}$ are determined as follows:

$$\partial\Pi/\partial\{\Delta\}_{rs} = 0, \qquad (22)$$

which may be expressed in matrix form as

$$\sum_{m=1}^{i_x}\sum_{n=1}^{i_y}\sum_{r=1}^{i_x}\sum_{s=1}^{i_y} ([K]_{mnrs}\{\Delta\}_{mn} - n^{*2}[M]_{mnrs}\{\Delta\}_{mn}) = 0. \qquad (23)$$

Here n^* is a frequency parameter and is defined by $\omega b^2\sqrt{\rho h_0/D_0}$.

The matrices of $[K]_{mnrs}$ and $[M]_{mnrs}$ are given by

$$[K]_{mnrs} = D_0\, ab\cos\theta/a^2 \begin{bmatrix} [K\phi_x\phi_x] & [K\phi_x\phi_y] & [K\phi_x W'] \\ [K\phi_y\phi_x] & [K\phi_y\phi_y] & [K\phi_y W'] \\ [KW'\phi_x] & [KW'\phi_y] & [KW'W'] \end{bmatrix} \qquad (24)$$

and

$$[M]_{mnrs} = \rho h_0\omega^2 ab^3\cos\theta \begin{bmatrix} [M\phi_x\phi_x] & [M\phi_x\phi_y] & 0 \\ [M\phi_y\phi_x] & [M\phi_y\phi_y] & 0 \\ 0 & 0 & [MW'W'] \end{bmatrix}. \qquad (25)$$

In equations (24) and (25), the sub-matrices of $[K]_{mnrs}$ and $[M]_{mnrs}$ are defined in Appendix A. The order of these matrices is expressed by $3(k + M_x - 1)(k + M_y - 1)$, where $k - 1$ is the degree of the B-spline functions, and M_x and M_y are the number of elements in the x and y directions respectively.

3. NUMERICAL EXAMPLES AND DISCUSSIONS

Natural frequencies of isotropic, skew Mindlin plates with varying thickness in the longitudinal direction are solved to illustrate the convergence and accuracy of the spline element method that is an alternative to the finite element method. For the definition of the boundary conditions of a skew plate, the symbolism SC–SF, for example, identifies a skew plate with the edges $\xi = 0, 1, \eta = 0$ and 1 having simply supported, clamped, simply supported and free boundary condition respectively. In the calculations, the degree of the B-spline functions, $k - 1 = 3$, the Poisson ratio of $\nu = 0{\cdot}3$ and $\kappa = 5/6$ were used.

Table 1 shows the convergence study of the first seven frequency parameters, $n^* = \omega b^2\sqrt{\rho h_0/D_0}$ of skew Mindlin plates (CC–CC, $\theta = 45°$, $a/b = 1{\cdot}0$ and $h_1/h_0 = 2{\cdot}0$) with linearly variable thickness in the x direction for the different number of elements. The number of mesh divisions, $M_x = M_y$ changes from 4 to 16, and b/h_0 varies from 5 to 100. It is found that the solution rapidly converges with an increase in the number of elements.

Table 2 shows the accuracy of the present method for simply supported skew thin plates (SS–SS, $a/b = 1{\cdot}0$, $b/h_0 = 250$, $M_x = M_y = 14$) with linearly varying thickness in the x direction. The values of $h_1/h_0 = 1{\cdot}2$ and $2{\cdot}0$ were used, and the angle of skew, θ varies from 0 to 45°. The results are compared with those obtained by Chopra and Durvasula [10] using the Ritz method with the 36 terms of double sine series based on the thin plate theory. Good agreement is observed in the comparison, but with increasing of the skew angle, the discrepancy between the results becomes larger.

TABLE 1

Convergence study of frequency parameters, $n^* = \omega b^2\sqrt{\rho h_0/D_0}$ of skew plates with linearly varying thickness in the longitudinal direction: CC–CC, $h_1/h_0 = 2{\cdot}0$, $\theta = 45°$ and $a/b = 1{\cdot}0$

		Modes						
b/h_0	$M_x = M_y$	1st	2nd	3rd	4th	5th	6th	7th
	4	43·99	64·75	73·29	79·45	93·42	97·50	106·6
	6	45·77	64·59	79·57	81·45	99·32	107·6	111·7
	8	46·24	64·60	80·97	82·21	98·71	107·3	114·3
5·0	10	46·43	64·62	81·25	82·73	98·64	107·2	114·7
	12	46·53	64·62	81·35	83·00	98·64	107·3	114·8
	14	46·58	64·63	81·39	83·14	98·64	107·3	114·9
	16	46·61	64·63	81·42	83·22	98·64	107·3	114·9
	4	66·67	105·3	125·8	135·2	165·8	175·8	193·9
	6	69·16	103·3	131·9	135·8	168·1	185·4	194·4
	8	69·88	103·2	132·6	137·6	165·2	183·6	195·4
10·0	10	70·18	103·2	132·8	138·6	164·9	183·5	195·4
	12	70·34	103·2	132·9	139·0	164·8	183·5	195·4
	14	70·43	103·2	133·0	139·3	164·8	183·5	195·5
	16	70·46	103·2	133·0	139·4	164·8	183·5	195·6
	4	106·3	226·3	292·8	368·9	780·0	802·3	819·1
	6	97·99	169·7	243·0	266·9	404·3	428·1	460·9
	8	96·07	157·6	224·3	235·4	319·1	357·9	409·9
100·0	10	95·62	155·4	215·7	230·7	292·0	339·0	378·0
	12	95·49	154·3	213·3	229·7	284·4	333·6	362·5
	14	95·45	154·1	212·5	229·4	282·1	332·0	357·3
	16	95·44	154·1	212·2	229·3	281·4	331·4	355·6

TABLE 2

Comparison of frequency parameters, $n^* = \omega b^2\sqrt{\rho h_0/D_0}$ *of simply supported thin skew plates with linearly varying thickness*: $a/b = 1{\cdot}0$, $b/h_0 = 250$ *and* $M_x = M_y = 14$

Skew angle $\theta^{(\circ)}$	h_1/h_0	Methods	Modes 1st	2nd	3rd	4th	5th
	1·2	Present method	2·198	5·492	5·496	8·793	11·03
0		Chopra and Durvasula [10]	2·198	5·488	5·492	8·790	10·95
	2·0	Present method	2·959	7·272	7·354	11·81	14·21
		Chopra and Durvasula [10]	2·960	7·271	7·351	11·81	14·16
	1·2	Present method	2·783	5·864	8·011	9·380	13·65
30		Chopra and Durvasula [10]	2·889	5·993	8·211	9·604	13·83
	2·0	Present method	3·737	7·817	10·59	12·60	17·42
		Chopra and Durvasula [10]	3·886	8·001	10·87	12·92	17·80
45	1·0	Present method	3·953	7·395	11·28	12·14	16·07
		Chopra and Durvasula [10]	4·314	7·786	11·95	12·83	17·05
	2·0	Present method	5·291	9·857	14·78	16·37	20·99
		Chopra and Durvasula [10]	5·791	10·42	15·72	17·36	22·55

Table 3 shows the accuracy comparison of natural frequency parameters. $n^* = \omega b^2\sqrt{\rho h_0/D_0}$ of clamped, skew Mindlin plates (CC–CC and $\theta = 45^\circ$) with linearly varying thickness. Here, $a/b = 2{\cdot}0$, 1·0 and 0·5, $b/h_0 = 10$ and 5 are used. The values of h_1/h_0 are 1·0, 1·5 and 2·0. The results for the plate with uniform thickness ($h_1/h_0 = 1{\cdot}0$) are compared with those obtained by Liew *et al.* [6] using the Ritz method with orthogonal polynomials as the displacement functions. It is seen that excellent agreement is obtained from the comparison.

Table 4 shows the accuracy comparison of frequency parameters, $n^* = \omega b^2\sqrt{\rho h_0/D_0}$ of square Mindlin plates ($a/b = 1{\cdot}0$, $b/h_0 = 10{\cdot}0$ and $\theta = 0^\circ$) with linearly varying thickness in the x direction, having several boundary conditions. The values of h_1/h_0 are 1·0 and 2·0. The results are compared with those obtained by Liew *et al.* [6] using the Ritz method, by Mikami and Yoshimura [17] using the collocation method with shifted Legendre polynomials and by Mizusawa [18] using the spline strip method. It is found that good agreement is observed for the different boundary conditions and the values of h_1/h_0.

Table 5 shows the effects of the skew angles and h_1/h_0 on frequency parameters, $n^* = \omega b^2\sqrt{\rho h_0/D_0}$ of skew Mindlin plates ($a/b = 1{\cdot}0$ and $b/h_0 = 10{\cdot}0$) with linearly varying thickness, having clamped edges (CC–CC) and simply supported edges (SS–SS) respectively. The skew angle of θ varies from 0 to 60°, and the values of h_1/h_0 are 1·0, 1·5 and 2·0. Figure 3 also depicts the effects of b/h_0 and h_1/h_0 on the fundamental frequency parameter, n_1^* of clamped, skew Mindlin plates ($a/b = 1{\cdot}0$ and $\theta = 45^\circ$) with linearly varying thickness.

It is seen that the frequency parameters increase with the increment of skew angle and h_1/h_0. In Figure 3, the frequency parameters show linearly incremental behavior for large value of b/h_0. While decreasing the value of b/h_0, the frequency parameters become non-linear in fashion due to the effects of the transverse shear deformation and rotary inertia.

Table 6 shows the effects of the location of a clamped edge, b/h_0 and h_1/h_0 on the first eight frequency parameters, $n^* = \omega b^2\sqrt{\rho h_0/D_0}$ of cantilevered, skew Mindlin plates ($a/b = 1{\cdot}0$ and $\theta = 45^\circ$) with linearly varying thickness. The value of h_1/h_0 varies from 1·0 to

TABLE 3

The frequency parameters, $n^* = \omega b^2 \sqrt{\rho h_0/D_0}$ *of clamped skew Mindlin plates with linearly varying thickness*: $\theta = 45^\circ$, *CC–CC and* $M_x = M_y = 16$

a/b	b/h_0	h_1/h_0	Modes							
			1st	2nd	3rd	4th	5th	6th	7th	8th
2·0	1 0	1·0	41·26	47·79	59·06	73·80	89·54	97·07	104·4	104·8
		Liew *et al.* [6]	41·24	47·84	59·15	73·74	89·44	96·98	104·5	105·3
		1·5	47·86	55·77	68·42	84·66	101·5	108·6	117·3	118·6
		2·0	52·82	62·09	75·65	92·74	109·8	116·6	126·7	128·5
	5	1·0	31·73	36·44	44·25	53·89	63·56	65·76	70·80	72·59
		Liew *et al.* [6]	31·67	36·42	44·22	53·75	63·36	65·57	70·72	72·59
		1·5	34·60	39·79	48·09	58·12	67·86	69·35	74·89	77·19
		2·0	36·49	42·03	50·64	60·88	70·44	71·69	77·47	80·06
1·0	1 0	1·0	55·26	83·84	110·8	116·2	139·6	157·1	168·0	187·8
		Liew *et al.* [6]	55·31	83·67	110·6	116·3	139·2	156·7	167·7	188·1
		1·5	63·88	95·01	123·9	129·8	154·5	172·7	184·5	204·3
		2·0	70·49	103·2	133·0	139·4	164·8	183·5	195·6	215·2
	5	1·0	41·09	58·47	74·71	76·97	91·34	100·0	107·4	117·1
		Liew *et al.* [6]	41·05	58·25	74·44	76·89	90·96	99·61	107·0	117·1
		1·5	44·46	62·25	78·88	80·83	95·87	104·4	112·1	121·7
		2·0	46·61	64·63	81·42	83·22	98·64	107·3	114·9	124·4
0·5	1 0	1·0	126·9	145·7	177·0	215·6	254·2	263·0	283·2	290·4
		1·5	138·6	159·1	192·5	232·8	272·3	277·1	299·3	309·0
		2·0	146·0	167·9	202·8	244·2	283·9	285·6	309·3	320·9
	5	1·0	79·08	90·96	109·4	130·8	149·0	150·7	162·3	169·4
		1·5	81·84	94·38	113·5	135·1	151·8	154·8	165·9	169·4
		2·0	83·31	96·35	115·9	137·5	153·5	155·3	157·2	168·0

2·5, and b/h_0 of 10 and 100 are used. Figure 4 also depicts the effect of the location of a clamped edge on the fundamental frequency parameter, n_1^* of cantilevered skew Mindlin plates ($a/b = 1{\cdot}0$, $b/h_0 = 10$ and $\theta = 45^\circ$) with linearly varying thickness in the x direction.

It is found that the frequency parameters of cantilevered skew Mindlin plates with varying thickness are influenced by the location of a clamped edge and h_1/h_0. The frequency parameters show linear increment for h_1/h_0, except for the case of CF–$\bar{\text{H}}$F.

Table 7 shows the effects of varying cross-sections with five types depicted in Figure 2 on the frequency parameters, $n^* = \omega b^2 \sqrt{\rho h_0/D_0}$ of skew Mindlin plates ($a/b = 1{\cdot}0$ and $b/h_0 = 10$). The skew angle of θ varies from 15 to 60°, and h_1/h_0 of 2·0, 1·5 and 1·0 are used. Type 1 is in linearly varying fashion, Types 2 and 3 are of varying thickness in convex fashion, and Types 4 and 5 are in concave fashion of varying thickness respectively.

It is found that the frequency parameters of clamped skew plates with convex cross-sections show larger than those of concave cross-sections, and the parameters of cross-section in linearly varying fashion are in the middle of those cross-sections. In view of engineering designs, it is possible to change effectively the frequency properties and flexural rigidities of the plates by varying the thickness.

Table 8 shows the effects of the location of a clamped edge and the variation of cross-sections on the frequency parameters, $n^* = \omega b^2 \sqrt{\rho h_0/D_0}$ of cantilevered skew

Table 4

Comparison of accuracy of frequency parameters, $n^* = \omega b^2\sqrt{\rho h_0/D_0}$ of tapered square Mindlin plates: $\theta = 0°$, $b/h_0 = 10{\cdot}0$, $a/b = 1{\cdot}0$ and $M_x = M_y = 16$

Boundary conditions	h_1/h_0	Modes 1st	2nd	3rd	4th	5th	6th	7th
SS–SS	1·0							
	Present method	19·04	45·44	45·44	69·71	85·04	85·04	106·6
	Liew *et al.* [6]	19·07	45·48	45·48	69·79	85·04	85·04	106·7
	Mikami *et al.* [17]	19·06	45·45	45·45	69·72	84·93		
	Mizusawa [18]	19·06	45·45	45·45	69·72	84·93	84·93	
	2·0							
	Present method	27·10	61·42	61·78	92·09	108·5	110·0	135·3
	Mikami *et al.* [17]	27·11	60·56	61·73	92·02	108·3	109·9	
	Mizusawa [18]	27·12	61·38	61·74	92·01	108·3	109·9	
CC–CC	1·0							
	Present method	32·52	62·03	62·03	86·93	102·4	103·4	123·9
	Liew *et al.* [6]	32·52	62·04	62·04	86·95	102·4	103·4	123·9
	2·0							
	Present method	43·48	79·00	79·16	108·5	124·7	126·3	149·8
FC–FF	1·0							
	Present method	3·423	8·017	20·00	25·49	28·08	47·29	53·98
	Liew *et al.* [6]	3·431	8·061	20·09	25·50	28·25	47·53	54·12
	2·0							
	Present method	7·275	13·62	30·68	34·97	38·69	62·98	70·12

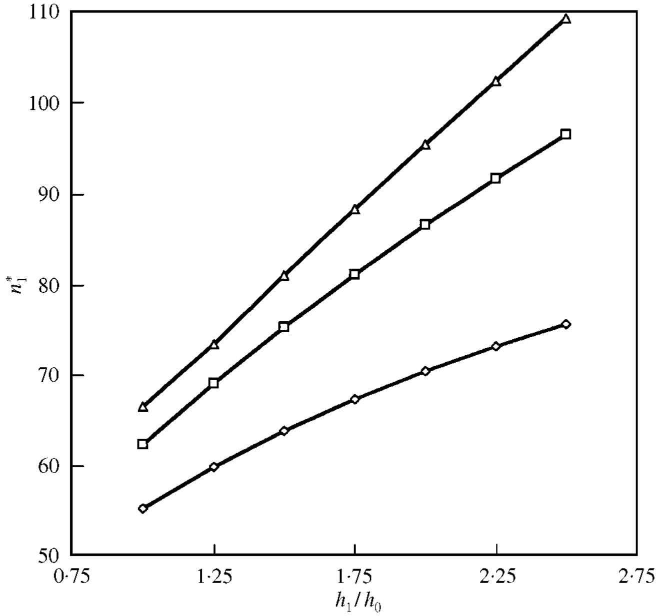

Figure 3. The effect of b/h_0 on the fundamental frequency parameter, n_1^* of skew Mindlin plates with tapered thickness: CC–CC, $a/b = 1{\cdot}0$ and $\theta = 45°$: (—◇—), $b/h_0 = 10$; (—□—), $b/h_0 = 20$; (—△—), $b/h_0 = 100$.

TABLE 5

The effect of skew angle on the frequency parameters, $n^* = \omega b^2\sqrt{\rho h_0/D_0}$ *of tapered skew plates*: $b/h_0 = 10$, $a/b = 1{\cdot}0$ *and* $M_x = M_y = 14$

Boundary conditions	θ^0	h_1/h_0	Modes 1st	2nd	3rd	4th	5th	6th	7th	8th
CC–CC	0	1·0	32·52	62·03	62·03	86·93	102·4	103·4	123·9	123·9
		1·5	38·53	71·66	71·74	99·28	115·5	116·7	139·0	139·2
		2·0	43·47	79·00	79·16	108·5	124·7	126·3	149·8	150·2
	15	1·0	34·32	61·52	68·76	87·69	107·0	111·5	119·1	134·9
		1·5	40·58	71·09	79·08	100·0	120·3	125·3	134·1	150·5
		2·0	45·69	78·40	86·81	109·2	129·6	135·0	145·0	161·5
	30	1·0	40·62	67·52	84·20	93·63	122·6	122·8	135·6	153·7
		1·5	47·68	77·56	95·70	106·3	135·5	138·8	150·8	168·7
		2·0	53·33	85·14	104·0	115·7	144·7	150·0	161·3	179·2
	45	1·0	55·21	83·84	110·8	116·0	139·6	157·1	168·0	187·5
		1·5	63·83	95·01	123·8	129·6	154·5	172·7	184·4	204·0
		2·0	70·43	103·2	132·9	139·0	164·8	183·5	195·5	214·9
	60	1·0	91·16	124·3	153·6	184·0	187·7	213·7	235·5	244·5
		1·5	102·2	137·1	168·1	199·7	202·6	230·6	251·9	262·4
		2·0	109·9	145·9	177·9	210·2	212·1	241·8	262·6	274·1
SS–SS	0	1·0	19·04	45·45	45·45	69·71	84·97	85·04	106·6	106·6
		1·5	23·26	54·16	54·30	82·05	98·42	99·03	122·9	123·1
		2·0	27·10	61·42	61·78	92·09	108·5	110·1	135·3	135·9
	15	1·0	20·08	44·50	51·11	69·81	89·02	92·50	101·1	117·4
		1·5	24·50	53·15	60·73	82·15	103·1	107·0	117·2	134·6
		2·0	28·51	60·42	68·70	92·17	113·7	118·2	129·8	147·5
	30	1·0	23·81	48·28	63·94	73·61	102·8	102·8	115·1	134·6
		1·5	28·95	57·52	75·27	86·49	116·5	120·5	131·9	151·4
		2·0	33·53	65·23	84·39	96·93	127·2	133·8	144·7	163·9
	45	1·0	32·99	59·61	86·26	91·61	115·4	133·9	144·8	165·6
		1·5	39·76	70·49	100·1	106·7	132·2	152·7	163·8	185·6
		2·0	45·61	79·37	110·7	118·8	144·5	166·7	177·4	199·8
	60	1·0	58·16	89·70	120·5	151·9	159·6	183·7	208·5	216·1
		1·5	68·64	104·2	137·8	171·6	179·6	205·1	231·3	239·1
		2·0	77·06	115·4	150·6	185·6	194·2	220·1	247·2	254·8

Mindlin plates ($a/b = 1{\cdot}0$, $\theta = 45^\circ$ and $b/h_0 = 10$) with varying thickness in the x direction. Here h_1/h_0 of 2·0, 1·5 and 1·0 are used.

It is found that the frequency parameters of cantilevered skew Mindlin plates are influenced by the variation of cross-sections compared with those of clamped skew Mindlin plate, and are also dependent on the location of a clamped edge.

4. CONCLUSIONS

The spline element method has been used to predict the natural frequencies of vibration of skew Mindlin plates with varying thickness in which account is taken of the effects of both transverse shear deformation and rotary inertia.

TABLE 6

The effect of b/h_0 and h_1/h_0 on the frequency parameters, $n^* = \omega b^2\sqrt{\rho h_0/D_0}$ of cantilevered skew plates with linearly varying thickness: $\theta = 45°$, $a/b = 1{\cdot}0$ and $M_x = M_y = 14$

Boundary conditions	b/h_0	h_1/h_0	Modes							
			1st	2nd	3rd	4th	5th	6th	7th	8th
CF–FF	1 0	1·0	4·336	10·53	24·56	28·14	44·97	51·50	66·20	71·58
		1·25	4·360	10·90	26·75	30·32	49·46	55·90	72·08	77·93
		1·5	4·387	11·25	28·65	32·38	53·45	59·76	77·21	83·36
		1·75	4·413	11·58	30·28	34·35	57·02	63·14	81·76	87·98
		2·0	4·437	11·89	31·67	36·24	60·21	66·10	85·80	91·87
		2·25	4·458	12·17	32·33	38·04	63·05	68·69	89·40	95·10
		2·5	4·478	12·43	33·81	39·76	65·56	70·95	92·60	97·74
	1 0 0	1·0	4·461	11·23	26·83	31·31	50·55	58·81	76·99	87·56
		1·25	4·501	11·74	29·88	34·19	57·16	65·78	86·53	99·11
		1·5	4·544	12·26	32·81	37·00	63·53	72·54	95·64	110·3
		1·75	4·585	12·76	35·63	39·77	69·70	79·14	104·4	121·2
		2·0	4·625	13·25	38·34	42·51	75·69	85·61	113·0	131·8
		2·25	4·663	13·73	40·95	45·23	81·53	91·98	121·3	142·2
		2·5	4·699	14·21	43·45	47·96	87·23	98·27	129·4	152·4
FC–FF	1 0	1·0	4·336	10·53	24·56	28·14	44·97	51·50	66·20	71·58
		1·25	5·353	12·54	27·39	31·68	49·17	56·61	72·13	77·03
		1·5	6·366	14·52	30·05	34·97	53·05	61·23	77·37	81·99
		1·75	7·375	16·46	32·56	38·05	56·68	65·45	82·04	86·56
		2·0	8·380	18·36	34·95	40·93	60·07	69·32	86·23	90·79
		2·25	9·378	20·22	37·23	43·63	63·26	72·89	90·03	94·73
		2·5	10·37	22·04	39·40	46·16	66·26	76·19	93·49	98·38
	1 0 0	1·0	4·461	11·23	26·83	31·31	50·55	58·81	76·99	87·56
		1·25	5·539	13·51	30·32	36·10	56·26	66·17	86·00	97·27
		1·5	6·629	15·81	33·74	40·83	61·80	73·31	94·57	106·7
		1·75	7·730	18·12	37·10	45·52	67·24	80·28	102·8	116·0
		2·0	8·843	20·45	40·44	50·18	72·60	87·12	110·9	125·1
		2·25	9·968	22·80	43·75	54·82	77·90	93·85	118·7	134·0
		2·5	11·11	25·16	47·05	59·46	83·16	100·5	126·5	142·8
FF–CF	1 0	1·0	4·336	10·53	24·56	28·14	44·97	51·50	66·20	71·58
		1·25	4·991	11·94	28·03	31·20	48·89	57·57	71·96	78·85
		1·5	5·655	13·35	31·22	34·09	52·49	63·09	76·88	85·20
		1·75	6·318	14·74	34·20	36·74	55·86	68·08	81·26	90·55
		2·0	6·974	16·09	36·99	39·14	59·03	72·61	85·24	94·97
		2·25	7·621	17·41	39·61	41·28	62·02	76·71	88·86	98·54
		2·5	8·255	18·67	42·04	43·23	64·84	80·44	92·14	101·4
	1 0 0	1·0	4·461	11·23	26·83	31·31	50·55	58·81	76·99	87·56
		1·25	5·173	12·95	30·98	36·07	56·08	67·60	85·58	101·5
		1·5	5·906	14·76	34·90	41·15	61·39	76·33	93·45	115·4
		1·75	6·654	16·63	38·70	46·39	66·59	84·95	100·9	129·1
		2·0	7·411	18·54	42·45	51·71	71·75	93·47	108·2	142·6
		2·25	8·175	20·48	46·18	57·07	76·88	101·9	115·3	115·8
		2·5	8·943	22·45	49·92	62·43	82·00	110·2	122·3	168·7

TABLE 7

The effect of variation of thickness and h_1/h_0 on the frequency parameters, $n^* = \omega b^2\sqrt{\rho h_0/D_0}$ of skew Mindlin plates: $b/h_0 = 10{\cdot}0$, $v = 0{\cdot}3$, $CC–CC$, $a/b = 1{\cdot}0$ *and* $M_x = M_y = 14$

			Modes							
θ	h_1/h_0	Type	1st	2nd	3rd	4th	5th	6th	7th	8th
15°	2·0	1	45·70	78·40	86·83	109·2	129·6	135·0	145·0	161·5
		2	48·46	82·69	91·36	114·4	135·7	141·3	151·1	167·9
		3	49·79	84·63	93·51	116·8	138·5	144·4	153·8	170·9
		4	42·58	73·02	81·50	102·8	121·6	127·7	137·4	153·4
		5	40·98	70·11	78·78	99·28	117·3	124·1	133·0	149·0
	1·5	1	40·58	71·09	79·09	100·0	120·3	125·3	134·1	150·5
		2	42·25	73·86	82·03	103·5	124·3	129·5	138·3	154·9
		3	43·08	75·18	83·48	105·1	126·3	131·6	140·2	157·0
		4	38·81	67·98	75·91	96·19	115·6	120·8	129·4	145·6
		5	37·92	66·37	74·32	94·18	113·1	118·6	126·9	143·0
	1·0		34·32	61·52	68·77	87·70	107·0	111·5	119·1	134·9
30°	2·0	1	53·35	85·14	104·1	115·7	144·7	150·0	161·5	179·2
		2	56·19	89·44	108·6	120·9	151·0	155·8	167·7	186·2
		3	57·54	91·38	110·8	123·3	154·2	158·2	170·6	189·6
		4	50·11	79·79	98·40	109·2	136·6	142·6	153·7	170·2
		5	48·41	76·89	95·46	105·5	132·7	138·2	149·6	165·8
	1·5	1	47·69	77·56	95·75	106·3	135·5	138·8	150·9	168·7
		2	49·46	80·39	98·83	109·8	139·7	142·8	155·2	173·5
		3	50·34	81·74	100·30	111·5	142·0	144·6	157·3	175·9
		4	45·81	74·40	92·32	102·4	130·7	134·2	146·2	163·3
		5	44·85	72·76	90·59	100·3	128·4	131·6	143·7	160·7
	1·0		40·63	67·52	84·24	93·64	122·6	122·8	135·7	153·7
45°	2·0	1	70·49	103·2	133·0	139·4	164·8	183·5	195·6	215·2
		2	73·31	107·3	138·1	143·8	170·9	189·1	202·1	221·8
		3	74·62	109·2	140·5	145·6	173·6	191·5	205·1	224·8
		4	67·18	97·88	126·5	133·8	157·1	176·0	187·2	206·7
		5	65·39	94·93	122·9	130·5	153·0	171·6	182·8	202·3
	1·5	1	63·88	95·01	123·9	129·8	154·5	172·7	184·5	204·3
		2	65·76	97·89	127·4	133·0	158·7	176·9	189·0	208·9
		3	66·68	99·26	129·2	134·4	160·7	178·8	191·2	211·1
		4	61·85	91·76	119·8	126·2	149·8	167·9	179·3	199·0
		5	60·80	90·05	117·7	124·5	147·3	165·4	176·6	196·4
	1·0		55·26	83·84	110·8	116·2	139·6	157·1	168·0	187·8
60°	2·0	1	110·1	145·9	177·9	210·2	212·7	241·8	262·6	274·0
		2	112·3	149·3	182·3	215·4	215·7	247·5	267·2	280·1
		3	113·2	150·7	184·2	216·7	217·7	250·0	269·1	282·8
		4	107·5	141·5	172·1	203·3	208·2	234·2	256·1	265·8
		5	105·9	138·8	168·7	199·3	205·4	229·8	252·2	261·1
	1·5	1	102·4	137·2	168·1	199·7	203·2	230·1	252·0	262·3
		2	104·1	139·8	171·4	203·5	205·9	234·8	255·7	266·8
		3	104·9	141·0	173·0	205·3	207·1	236·8	257·3	269·0
		4	100·5	134·1	164·3	195·2	200·0	225·7	247·5	256·9
		5	99·47	132·5	162·3	192·8	198·2	223·0	245·1	254·0
	1·0		91·35	124·3	153·7	183·9	188·3	213·7	235·5	244·3

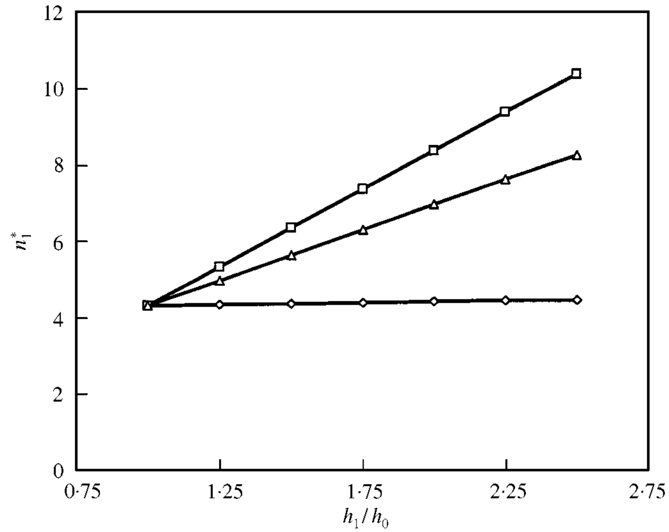

Figure 4. The effects of the position of a clamped edge and h_1/h_0 on the frequency parameter, n_1^* of cantilevered skew Mindlin plates with linearly varying thickness: $\theta = 45°$, $a/b = 1{\cdot}0$ and $b/h_0 = 10$: (—◇—), CF–FF; (—□—), FC–FF; (—△—), FF–CF.

It is found that stable convergence and high accuracy in computed solutions are obtained. The frequency parameters of a skew Mindlin plates with varying thickness are dependent on the tapered thickness parameter of h_1/h_0, the aspect ratio, the ratio of width to thickness and boundary conditions. The frequency parameters show linearly incremental behavior for the large value of b/h_0. While decreasing the value of b/h_0, the frequency parameters become non-linear in fashion due to the effects of the transverse shear deformation and rotary inertia. The frequency parameters of skew plates with convex cross-sections show larger than those of concave cross-sections, and the parameters of cross-section in linear varying fashion are in the middle of those cross-sections.

In view of engineering designs, it is possible to change effectively the frequency properties and flexural rigidities of the skew Mindlin plates by varying the thickness.

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TABLE 8

The effect of variation of thickness on the frequency parameters, $n^* = \omega b^2\sqrt{\rho h_0/D_0}$ of skew Mindlin plates: $\theta = 45^\circ$, $a/b = 1{\cdot}0$, $b/h_0 = 10$, $\nu = 0{\cdot}3$ and $M_x = M_y = 14$

Boundary conditions	h_1/h_0	Type	Modes 1st	2nd	3rd	4th	5th	6th	7th	8th
FC–FF	2·0	1	8·380	18·36	34·95	40·93	60·07	69·32	86·32	90·79
		2	9·099	19·89	38·02	44·37	64·61	75·07	91·94	97·77
		3	9·216	20·23	39·62	45·93	67·23	78·06	95·34	101·5
		4	7·541	16·57	31·64	37·09	55·21	62·87	79·96	82·82
		5	6·892	15·27	30·13	35·16	53·26	60·11	77·55	79·51
	1·5	1	6·366	14·52	30·05	34·97	53·05	61·23	77·37	81·99
		2	6·722	15·31	31·68	36·87	55·52	64·52	80·62	86·09
		3	6·793	15·51	32·57	37·78	56·99	66·30	82·57	88·43
		4	5·982	13·67	28·36	32·99	50·52	57·77	73·95	77·65
		5	5·688	13·07	27·59	32·00	49·51	56·26	72·55	75·93
	1·0	1	4·336	10·53	24·56	28·14	44·96	51·50	66·20	71·57
	Liew *et al.* [6]		4·387	10·54	24·77	28·26	44·95	51·56	66·22	71·71
FF–CF	2·0	1	6·974	16·03	36·99	39·14	59·02	72·61	85·24	94·97
		2	7·536	17·86	39·85	42·99	65·41	78·49	92·75	100·6
		3	7·764	18·57	40·68	45·04	68·70	80·76	96·75	103·2
		4	6·440	14·30	32·71	36·38	52·12	65·77	77·02	89·24
		5	6·117	13·19	30·35	34·88	49·08	61·99	73·49	85·94
	1·5	1	5·655	13·35	31·22	34·09	52·88	62·09	76·88	85·20
		2	5·959	14·31	32·96	36·14	55·94	66·65	81·04	88·80
		3	6·084	14·72	33·60	37·20	57·80	68·15	83·28	90·40
		4	5·359	12·38	29·21	32·29	48·92	59·30	72·54	81·56
		5	5·183	11·80	28·06	31·24	47·34	57·20	70·63	79·45
CF–FF	2·0	1	4·437	11·89	31·67	36·24	60·21	66·10	85·80	91·87
		2	4·876	12·95	34·27	38·88	63·82	70·85	91·13	97·15
		3	5·226	13·69	35·58	40·20	65·26	73·08	93·21	99·26
		4	3·939	10·71	28·82	33·51	56·35	60·84	79·59	86·02
		5	3·817	10·33	27·41	32·14	54·01	58·01	75·75	82·47
	1·5	1	4·387	11·25	28·65	32·38	53·45	59·76	77·21	83·36
		2	4·650	11·87	30·13	33·99	55·62	62·71	80·40	86·89
		3	4·856	12·31	30·85	34·80	56·50	64·09	81·71	88·32
		4	4·105	10·59	27·11	30·74	51·23	56·66	73·77	79·68
		5	4·037	10·38	26·32	29·92	49·89	54·98	71·74	77·46

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APPENDIX A

$$K\phi_x\phi_x = \sec^2\theta[I_{mr}^{113}J_{ns}^{00} - (a/b)\sin\theta I_{mr}^{103}J_{ns}^{01} - (a/b)\sin\theta I_{mr}^{013}J_{ns}^{10} + (a/b)^2\sin^2\theta I_{mr}^{003}J_{ns}^{11}]$$
$$+ 0{\cdot}5(1-\nu)(a/b)^2 I_{mr}^{003}J_{ns}^{11} + 6(1-\nu)(b/h_0)\kappa I_{mr}^{001}J_{ns}^{00},$$

$$K\phi_x\phi_y = \sec^2\theta[-\sin\theta I_{mr}^{113}J_{ns}^{00} + (a/b)I_{mr}^{103}J_{ns}^{01} + (a/b)\sin^2\theta I_{mr}^{013}J_{ns}^{10}$$
$$- (a/b)^2\sin\theta I_{mr}^{003}J_{ns}^{11}] + 0{\cdot}5(1-\nu)(a/b)I_{mr}^{013}J_{ns}^{10} - (1-\nu)(a/b)I_{mr}^{103}J_{ns}^{01}]$$
$$- 6(1-\nu)(b/h_0)\kappa\sin\theta I_{mr}^{001}J_{ns}^{00},$$

$$K\phi_x W' = 6(1-\nu)(b/a)(b/h_0)^2\kappa[(b/a)\cos\theta I_{mr}^{011}J_{ns}^{00} + \sin\theta\tan\theta(b/a)I_{mr}^{011}J_{ns}^{00}$$
$$- \sin\theta\sec\theta I_{mr}^{001}J_{ns}^{01}],$$

$$K\phi_y\phi_x = \sec^2\theta[-\sin\theta I_{mr}^{113}J_{ns}^{00} + (a/b)I_{mr}^{013}J_{ns}^{10} + (a/b)\sin^2\theta I_{mr}^{103}J_{ns}^{01} - (a/b)^2\sin\theta I_{mr}^{003}J_{ns}^{11}]$$
$$+ 0{\cdot}5(1-\nu)(a/b)I_{mr}^{103}J_{ns}^{01} - (1-\nu)(a/b)I_{mr}^{013}J_{ns}^{10}$$
$$- 6(1-\nu)(b/h_0)\kappa\sin\theta I_{mr}^{001}J_{ns}^{00},$$

$$K\phi_y\phi_y = \sec^2\theta[\sin^2\theta I_{mr}^{113}J_{ns}^{00} - (a/b)\sin\theta I_{mr}^{103}J_{ns}^{01} - (a/b)\sin\theta I_{mr}^{013}J_{ns}^{10} + (a/b)^2 I_{mr}^{003}J_{ns}^{11}]$$
$$+ 0{\cdot}5(1-\nu)I_{mr}^{113}J_{ns}^{00} + 6(1-\nu)(b/h_0)^2\kappa I_{mr}^{001}J_{ns}^{00}$$

$$K\phi_y W' = 6(1-\nu)(b/h_0)^2\kappa[\sec\theta I_{mr}^{001}J_{ns}^{01} - (b/a)\tan\theta I_{mr}^{011}J_{ns}^{00}],$$

$$KW'\phi_x = 6(1-\nu)(b/a)(b/h_0)^2\kappa[(b/a)\cos\theta I_{mr}^{101}J_{ns}^{00} + (b/a)\sin\theta\tan\theta I_{mr}^{101}J_{ns}^{00}$$
$$- \sin\theta\sec\theta I_{mr}^{001}J_{ns}^{10}],$$

$$KW'\phi_y = 6(1-\nu)(b/h_0)^2\kappa[\sec\theta I_{mr}^{001}J_{ns}^{10} - (b/a)\tan\theta I_{mr}^{101}J_{ns}^{00}],$$

$$KW'W' = 6(1-\nu)\,(b/h_0)^2\kappa[(b/a)^2 I_{mr}^{111}J_{ns}^{00} + (b/a)^2\tan^2\theta I_{mr}^{111}J_{ns}^{00}$$
$$- (b/a)\tan\theta\sec\theta I_{mr}^{101}J_{ns}^{01} - (b/a)\sec\theta\tan\theta I_{mr}^{011}J_{ns}^{10} + \sec^2\theta I_{mr}^{001}J_{ns}^{11}]$$

and

$$M\phi_x\phi_x = (\tfrac{1}{12})(h_0/b)^2 I_{mr}^{003}J_{ns}^{00},$$

$$M\phi_x\phi_y = -\sin\theta(\tfrac{1}{12})(h_0/b)^2 I_{mr}^{003}J_{ns}^{00},$$

$$M\phi_y\phi_y = (\tfrac{1}{12})(h_0/b)^2 I_{mr}^{003} J_{ns}^{00},$$

$$M\phi_y\phi_x = -\sin\theta(\tfrac{1}{12})(h_0/b)^2 I_{mr}^{003} J_{ns}^{00},$$

$$MW'W' = I_{mr}^{001} J_{ns}^{00},$$

where the integrals I_{mr}^{ijk} and J_{ns}^{ij} are defined by

$$I_{mr}^{ijk} = \int_0^1 [[\mathrm{d}^i N_{m,k}(\xi)/\mathrm{d}\xi^i \times \mathrm{d}^j N_{r,k}(\xi)/\mathrm{d}\xi^j] H(\xi)^k] \, \mathrm{d}\xi,$$

$$J_{ns}^{ij} = \int_0^1 [\mathrm{d}^i N_{n,k}(\eta)/\mathrm{d}\eta^i \times \mathrm{d}^j N_{s,k}(\eta)/\mathrm{d}\eta^j] \, \mathrm{d}\eta.$$