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TORSIONAL FREE VIBRATION OF A CYLINDER WITH VARYING CROSS-SECTION AND ADHESIVE MASSES

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The natural vibration frequency of a torsion cylinder with varying cross section and an adhesive mass is investigated in this paper. From the numerical solution of the governing equation under the relevant boundary conditions we can define a function which is called the target function in this paper. It is proved that the problem for evaluating the natural vibration frequencies is equivalent to finding the zeros of the target function. An improvement formulation of the target function is suggested in this paper. Without regard to the number of the adhesive masses, the target function is obtained from a solution of one particular initial boundary value problem of the ordinary differential equation. The zeros of target function can be easily evaluated by the well known half-division technique. The suggested method depends significantly on the computer computation. It is proved that the target function approach provides one more effective method in this field. Many numerical results are carried out in this paper. The given numerical examples generally show that the adhesive masses influence by lowering the vibration frequency of torsion cylinder in general.

1. INTRODUCTION

Free vibration analysis of a torsion cylinder with varying cross-section and an adhesive mass is carried out in the paper. The governing equation for the problem was formulated in reference [1]. Since the computer was not available at the age of Timoshenko, the previously obtained solutions were limited to a very elementary scale. For example, for the case of two adhesive masses which are huge compared to the mass of cylinder, only the fundamental vibration frequency was obtained approximately [1]. Evaluation of the bucking loading of beam by using numerical solution of the differential equation has been introduced in an earlier paper [2]. Exact solutions for the longitudinal vibration of non-uniform rods were obtained in reference [3]. Recently, the target function method for evaluating the buckling loading and the vibration frequency was suggested [4]. It is found that, the idea of target function method is a general one, which can also be used in the present analysis.

The idea of target function can be described as follows. In this paper, the vibration frequencies are equal to the zeros of a target function $T(\omega)$, where ω is the vibration frequency. Alternatively, the natural vibration frequencies can also be defined as the roots of the following equation:

$$T(\omega) = 0. \tag{1}$$

In the previous investigation [4], if there are three adhesive masses, one should solve three particular boundary value problems so as to obtain the target function. Probably, this is inefficient in the case of large number of adhesive masses. In this paper, we suggest a new approach in which the target function can be obtained from the solution of one boundary value problem for any number of adhesive masses.

The studied target functions $T(\omega)$ are generally obtained from the results of numerical solution of relevant ordinary differential equation. In fact, it is necessary to compute these functions for many given values of ω, until equation (1) is satisfied. Therefore, the suggested method depends on the computer computation intensively. Comparatively speaking, the numerical solution of the ordinary differential equation is easy by using the computer. Therefore, the target function method is successful in solving the present problem.

2. ANALYSIS

The vibration problem of a torsion cylinder with varying cross-section and adhesive masses is analyzed below. The governing equation for the twist angle $\phi(x, t)$ is as follows (Figure 1):

$$G\frac{\partial}{\partial x}\left(I(x)\frac{\partial \phi}{\partial x}\right) = \rho I(x)\frac{\partial^2 \phi}{\partial t^2} \quad (0 \leqslant x \leqslant L), \tag{2}$$

where G is the shear modulus of elasticity, ρ is the mass density of materials, $GI(x)$ denotes the torsion rigidity of the cylinder at the position of x, and $\rho I(x)$ is the rotary inertia per unit length of cylinder at the position of x. In the case of the tapered configuration, $I(x)$ has the expression

$$I(x) = I_0 g(x) \quad (0 \leqslant x \leqslant L), \tag{3}$$

where

$$I_0 = I(0) = \frac{\pi a^4}{2}, \quad g(x) = \left(1 + \frac{mx}{L}\right)^4, \tag{4}$$

and m represents the degree of the taper (Figure 1).

It is assumed that there are two masses with the rotary inertia R_A, R_C adhered to the ends of the cylinder, at $x = 0$ and L respectively (Figure 1). In addition, there are many adhesive masses R_j $(j = 1, 2, \ldots, M)$ placed at the intermediate sections $x = b_j$ $(j = 1, 2, \ldots, M)$. In this case, the boundary conditions and the conditions at the section $x = b_j$ $(j = 1, 2, \ldots, M)$ take the form [1]

$$I(0)\left.\frac{\partial \phi}{\partial x}\right|_{x=0} - \frac{R_A}{G}\frac{\partial^2 \phi(0, t)}{\partial t^2} = 0, \tag{5}$$

$$I(b_j)\left(\left.\frac{\partial \phi}{\partial x}\right|_{x=b_j^+} - \left.\frac{\partial \phi}{\partial x}\right|_{x=b_j^-}\right) - \frac{R_j}{G}\frac{\partial^2 \phi(b_j, t)}{\partial t^2} = 0 \quad (j = 1, 2, \ldots, M), \tag{6}$$

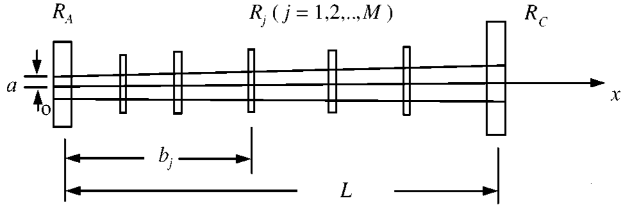

Figure 1. A tapered cylinder with many adhesive masses.

$$I(L)\frac{\partial\phi}{\partial x}\bigg|_{x=L}+\frac{R_C}{G}\frac{\partial^2\phi(L,t)}{\partial t^2}=0. \tag{7}$$

In the following analysis, the adhesive rotary inertia R_A, R_C and R_j are expressed by

$$R_A=r_A(\rho I_0 L),\quad R_C=r_C(\rho I_0 L),\quad R_j=r_j(\rho I_0 L)\quad (j=1,2,\ldots,M). \tag{8}$$

Clearly, $r_A, r_C, r_j\,(j=1,2,\ldots,M)$ represents the non-dimensional values.

In the natural vibration problem, it is assumed that

$$\phi(x,t)=\Phi(x)\sin(\omega t+\alpha), \tag{9}$$

where ω is the natural vibration frequency.

Substituting equation (9) into equation (2) yields the governing equation for the function $\Phi(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(I(x)\frac{\mathrm{d}\Phi}{\mathrm{d}x}\right)=-\frac{\omega^2\rho}{G}I(x)\Phi(x)\quad (0\leqslant x\leqslant L). \tag{10}$$

In addition, substituting equation (9) into equations (5)–(7) yields the boundary conditions for the function $\Phi(x)$:

$$I(0)\frac{\mathrm{d}\Phi}{\mathrm{d}x}\bigg|_{x=0}+\frac{\omega^2 R_A}{G}\Phi(0)=0, \tag{11}$$

$$I(b_j)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}x}\bigg|_{x=b_j^+}-\frac{\mathrm{d}\Phi}{\mathrm{d}x}\bigg|_{x=b_j^-}\right)+\frac{\omega^2 R_j}{G}\Phi(b_j)=0\quad (j=1,2,\ldots,M), \tag{12}$$

$$I(L)\frac{\mathrm{d}\Phi}{\mathrm{d}x}\bigg|_{x=L}-\frac{\omega^2 R_C}{G}\Phi(L)=0. \tag{13}$$

Obviously, the natural vibration problem is reduced to find a particular frequency ω and relevant function $\Phi(x)$ so as to satisfy equation (10) and conditions (11)–(13).

Below the target function method is introduced to solve the problem numerically. In fact, for any given ω, for equation (10) we can propose a boundary value problem with the following conditions:

$$\Phi|_{x=0}=1,\quad \frac{\mathrm{d}\Phi}{\mathrm{d}x}\bigg|_{x=0}=-\frac{\omega^2 R_A}{GI(0)},\quad \frac{\mathrm{d}\Phi}{\mathrm{d}x}\bigg|_{x=b_j^+}-\frac{\mathrm{d}\Phi}{\mathrm{d}x}\bigg|_{x=b_j^-}=-\frac{\omega^2 R_j}{GI(b_j)}\Phi(b_j)\quad (j=1,2,\ldots,M). \tag{14a–c}$$

The relevant solution for the governing equation (10) under conditions (14a–c) is denoted by

$$\Phi=p(x,\omega)\quad (0\leqslant x\leqslant L). \tag{15}$$

Here, the function $p(x,\omega)\,(0\leqslant a\leqslant L)$ is obtained in the form of a numerical solution, rather than in the form of an analytical solution. That is to say, from the governing equation (10) and the initial boundary conditions (14a, b) and condition (14c), we can obtain the values of functions $p(x,\omega)$, $\mathrm{d}p(x,\omega)/\mathrm{d}x$ at the discrete points $x=0, L/N, 2L/N, 3L/N,\ldots,L$, where N is the division number used in the integration of ordinary differential equation. The above numerical solution can be easily obtained by using the well-known Runge–Kutta integration rule [5, p. 290].

After comparing conditions (14a–c) with equations (11) and (12), we see that conditions (11) and (12) are satisfied by the function $p(x, \omega)$. Substituting function (15) into condition (13) yields

$$T(\omega) = 0, \tag{16}$$

where

$$T(\omega) = I(L)p'(L, \omega) - \frac{\omega^2 R_C}{G} p(L, \omega). \tag{17}$$

The function $T(\omega)$ is called the target function in this paper.

Clearly, the investigated frequencies are equal to the zeros of the target function. The zeros of the target function $T(\omega)$ can be easily obtained by using half-division method in numerical computation.

3. NUMERICAL EXAMPLES

In the computation, the Runge–Kutta method with N divisions along the interval $(0, L)$ is used [5, p. 290]. Also, the well-known half-division technique is used for finding the zeros of the target function.

There are two groups of numerical examples. In the first group, there are three masses with the rotary inertia R_A, R_B and R_C are placed at the positions $x = 0$, b and L respectively (Figure 2). The adhesive rotary inertia R_A, R_B and R_C are expressed by

$$R_A = r_A(\rho I_0 L), \quad R_B = r_B(\rho I_0 L), \quad R_C = r_C(\rho I_0 L). \tag{18}$$

In computation, $N = 96$ divisions is used in the numerical integration. Finally, the first six natural vibration frequencies are obtainable, and they are expressed as

$$\omega = F\left(r_A, r_B, r_C, m, \frac{b}{L}\right)\sqrt{\frac{G}{\rho}}\frac{\pi}{L}. \tag{19}$$

Five examples are presented to verify the efficiency of the suggested method or to provide some new results.

Example 3.1. *In the first example we assume that*, (a) *the cylinder has a constant section*, i.e., $m = 0$ *in equation* (4) *and* (b) *the intermediate mass vanishes* $r_B = 0$. *In this case, the parameter*

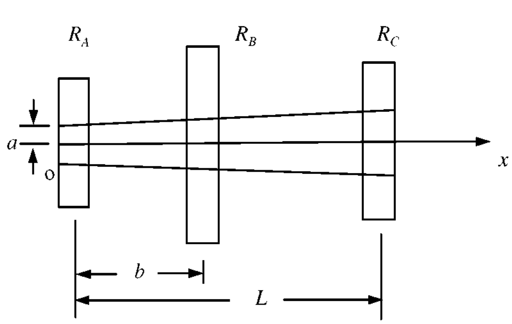

Figure 2. A tapered cylinder with three adhesive masses.

TABLE 1

The first six normalized natural vibration frequency $F_1(r_A, r_C)$ for the cylinder with two adhesive masses (see Figure 2 and equation (20))

		1st	2nd	3rd	4th	5th	6th
$r_A = 0$	$r_C = 0$	1·0000	2·0000	3·0000	4·0000	5·0000	6·0001
$r_A = 0$	$r_C = 0^\dagger$	1·0000	2·0000	3·0000	4·0000	5·0000	6·0000
$r_A = 0$	$r_C = 2$	0·5846	1·5329	2·5201	3·5144	4·5112	5·5092
$r_A = 0$	$r_C = 4$	0·5461	1·5167	2·5101	3·5072	4·5056	5·5046
$r_A = 2$	$r_C = 2$	0·3056	1·0921	2·0493	3·0334	4·0252	5·0202
$r_A = 2$	$r_C = 4$	0·2650	1·0706	2·0372	3·0251	4·0189	5·0152
$r_A = 4$	$r_C = 4$	0·2205	1·0482	2·0250	3·0168	4·0126	5·0101

† From the exact solution.

TABLE 2

The fundamental natural vibration frequency $F_1(r_A, r_C)$ for the cylinder with two adhesive masses (see Figure 2 and equations (20) and (21))

$r_A = r_C =$	5	10	15	20	25	30	35	40
†	0·1980	0·1412	0·1156	0·1002	0·0897	0·0820	0·0759	0·0710
‡	0·2013	0·1424	0·1162	0·1007	0·0900	0·0822	0·0761	0·0712
Error % §	1·66	0·83	0·56	0·42	0·33	0·28	0·24	0·21

† This paper.
‡ From the approximate solution shown by equation (21) [1].
§ Error of the approximate solution.

b/L is of no use. The calculated results are expressed by

$$\omega = F_1(r_A, r_C)\sqrt{\frac{G}{\rho}}\frac{\pi}{L} \quad (r_B = 0, m = 0). \tag{20}$$

The calculated results are listed in Table 1. Timoshenko proposed an approximate equation as follows [1]:

$$F_1(r_A, r_C) = \frac{1}{\pi}\sqrt{\frac{r_A + r_C}{r_A r_C}} \quad (\text{for } r_A \gg 1 \text{ and } r_C \gg 1). \tag{21}$$

The equation is only valid in the case $r_A \gg 1$ and $r_C \gg 1$. For comparison, the calculated results by the present method and by using equation (21) are listed in Table 2. From Table 2 we see that, only if $r_A > 10$ and $r_C > 10$, the relative error for Timoshenko's approximation is less than 1%.

Example 3.2. *In the second example, we assume that, (a) the cylinder has a tapered configuration,* i.e., *$m \neq 0$ in equation (4) and (b) the intermediate mass vanishes $r_B = 0$, and $r_A = r_C$. In this case, the parameter b/L is of no use. The calculated results are expressed by*

$$\omega = F_2(r_A, m)\sqrt{\frac{G}{\rho}}\frac{\pi}{L} \quad (r_B = 0, r_A = r_C). \tag{22}$$

TABLE 3

The first six normalized natural vibration frequency $F_2(r_A, m)$ for the cylinder with two adhesive masses (see Figure 2 and equation (22))

m	1st	2nd	3rd	4th	5th	6th
			$r_A = r_C = 0$			
0·01	1·0000	2·0000	3·0000	4·0000	5·0000	6.0001
0·25	1·0150	2·0076	3·0051	4·0038	5·0031	6·0026
0·50	1·0482	2·0250	3·0168	4·0126	5·0101	6·0085
0·75	1·0890	2·0477	3·0322	4·0243	5·0195	6·0163
1·00	1·1318	2·0732	3·0498	4·0376	5·0302	6·0253
			$r_A = r_C = 2$			
0·00	0·3056	1·0921	2·0493	3·0334	4·0252	5·0202
0·25	0·3680	1·1489	2·0847	3·0582	4·0442	5·0356
0·50	0·4146	1·2243	2·1421	3·1013	4·0780	5·0633
0·75	0·4494	1·2996	2·2142	3·1616	4·1281	5·1055
1·00	0·4777	1·3622	2·2874	3·2318	4·1914	5·1616

TABLE 4

The first six normalized natural vibration frequency $F_3(r_A, r_B)$ for the cylinder with three adhesive masses (see Figure 2 and equation (23))

r_B	1st	2nd	3rd	4th	5th	6th
			$r_A = r_C = 0$			
0	1·0000	2·0000	3·0000	4·0000	5·0000	6·0001
2	1·0000	1·1692	3·0000	3·0659	5·0000	5·0402
4	1·0000	1·0921	3·0000	3·0334	5·0000	5·0202
			$r_A = r_C = 2$			
0	0·3056	1·0921	2·0493	3·0334	4·0252	5·0202
2	0·3056	0·5300	2·0493	2·1412	4·0252	4·0745
4	0·3056	0·4410	2·0493	2·0965	4·0252	4·0500
			$r_A = r_C = 4$			
0	0·2205	1·0482	2·0250	3·0168	4·0126	5·0101
2	0·2205	0·4786	2·0250	2·1188	4·0126	4·0623
4	0·2205	0·3820	2·0250	2·0732	4·0126	4·0376

They are listed in Table 3. From Table 3 we see that, generally, if the taper parameter m is higher, the relevant natural vibration frequency is elevated.

Example 3.3. *In the third example we assume that* (a) *the cylinder has a constant section*, i.e., $m = 0$ *in equation* (4), (b) $r_A = r_C$ *and* (c) *the intermediate mass is placed at the position* $x = b$ *with* $b/L = 0{\cdot}5$. *The calculated results are expressed by*

$$\omega = F_3(r_A, r_B)\sqrt{\frac{G}{\rho}}\frac{\pi}{L} \quad (r_A = r_C,\ b/L = 0{\cdot}5,\ m = 0). \tag{23}$$

They are listed in Table 4. In all investigated cases we see that, generally the adhesive mass can lower the relevant vibration frequencies. However, if the immediate mass is placed

TABLE 5

The first six normalized natural vibration frequency $F_4(r_A, b/L)$ for the cylinder with three adhesive masses (see Figure 2 and equation (24))

$r_A = r_B = r_C$	1st	2nd	3rd	4th	5th	6th
			$b/L = \frac{1}{4}$			
2	0·2905	0·6427	1·4308	2·7156	4·0227	4·1090
4	0·2095	0·4640	1·3830	2·6916	4·0114	4·0553
6	0·1722	0·3815	1·3667	2·6834	4·0076	4·0371
			$b/L = \frac{1}{3}$			
2	0·2981	0·5761	1·5950	3·0318	3·1151	4·5335
4	0·2150	0·4154	1·5490	3·0160	3·0587	4·5168
6	0·1768	0·3415	1·5330	3·0107	3·0394	4·5112
			$b/L = \frac{1}{2}$			
2	0·3056	0·5300	2·0493	2·1412	4·0252	4·0745
4	0·2205	0·3820	2·0250	2·0732	4·0126	4·0376
6	0·1813	0·3140	2·0167	2·0494	4·0084	4·0252

at the node of the vibration deflection, the natural vibration frequency is not changing. For example, in $r_A = r_C = 2$ case, the third natural vibration frequency is always equal to 2·0493.

Example 3.4. *In the fourth example we assume that*, (a) *the cylinder has a constant section*, i.e., $m = 0$ *in equation* (4), (b) $r_A = r_C = r_B$ *and* (c) *the position of the intermediate mass is subjected to change, from* $b/L = \frac{1}{4}, \frac{1}{3}$ *to* $\frac{1}{2}$, *respectively. The calculated results are expressed by*

$$\omega = F_4\left(r_A, \frac{b}{L}\right)\sqrt{\frac{G}{\rho}}\frac{\pi}{L} \quad (r_A = r_C = r_B, m = 0). \tag{24}$$

They are listed in Table 5.

Example 3.5. *In the fifth example we assume that*, (a) *the cylinder has a tapered configuration*, i.e., $m \neq 0$ *in equation* (4), (b) $r_A = r_C = r_B$ *and* (c) *the intermediate mass is placed at the position* $x = b = 0{\cdot}5L$. *The calculated results are expressed by*

$$\omega = F_5(r_A, m)\sqrt{\frac{G}{\rho}}\frac{\pi}{L} \quad (r_A = r_C = r_B, b/L = 0{\cdot}5). \tag{25}$$

The calculated results are listed in Table 6.

In a particular case, $m = 0{\cdot}25$, $r_A = r_C = r_B = 4$, and $b/L = 0{\cdot}5$, we let

$$S(f) = S_1(f)/c, \quad \text{where } S_1(f) = T(\omega) = T\left(f\sqrt{\frac{G}{\rho}}\frac{\pi}{L}\right), \quad c = \text{constant}. \tag{26}$$

The $S(f)$ variation with respect to the augment f is shown in Figure 3. From Figure 3 the relevant non-dimensional eigenvalues $f = 0{\cdot}2669$ and $0{\cdot}4843$ can be found. From the calculated result we find that, if the ω values are varying within a wide range, the $T(\omega)$ values are also changed in a wide range. In this case, it is suitable to display the $T(\omega)$ values in the vicinity of a particular eigenvalue.

TABLE 6

The first six normalized natural vibration frequency $F_5(r_A, m)$ for the cylinder with three adhesive masses (see Figure 2 and equation (25))

m	1st	2nd	3rd	4th	5th	6th
			$r_A = r_C = r_B = 2$			
0·00	0·3056	0·5300	2·0493	2·1412	4·0252	4·0745
0·25	0·3661	0·6635	2·0782	2·2289	4·0405	4·1253
0·50	0·4109	0·8068	2·1155	2·3586	4·0615	4·2082
0·75	0·4457	0·9433	2·1599	2·5183	4·0885	4·3253
1·00	0·4750	1·0627	2·2088	2·6873	4·1217	4·4707
			$r_A = r_C = r_B = 4$			
0·00	0·2205	0·3820	2·0250	2·0732	4·0126	4·0376
0·25	0·2669	0·4843	2·0413	2·1227	4·0210	4·0644
0·50	0·3027	0·6017	2·0647	2·2014	4·0335	4·1095
0·75	0·3313	0·7243	2·0947	2·3081	4·0500	4·1764
1·00	0·3549	0·8438	2·1305	2·4367	4·0709	4·2668

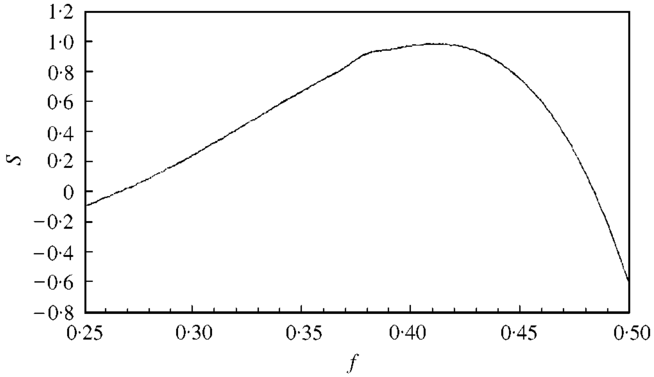

Figure 3. The $S(f)$ (or $T(\omega)$) dependence for a particular adhesive mass case (see equation (26)).

In the second group, there are two masses with the rotary inertia R_A and R_C placed at the positions $x = 0$ and L respectively (Figure 1). The adhesive rotary inertia R_A and R_C are expressed by

$$R_A = r_A(\rho I_0 L), \quad R_C = r_C(\rho I_0 L). \tag{27}$$

In addition, there are 99 adhesive masses with rotary inertia R_j placed at the positions $x = b_j$ $(j = 1, 2, \ldots, 99)$. As before, the adhesive rotary inertia R_j are expressed by

$$R_j = r_j(\rho I_0 L) \quad (j = 1, 2, \ldots, 99). \tag{28}$$

In computation, $N = 400$ divisions is used in the numerical integration. Finally, the first six natural vibration frequencies are obtainable, and they are expressed as

$$\omega = H\sqrt{\frac{G}{\rho}}\frac{\pi}{L}. \tag{29}$$

TABLE 7

The first six normalized natural vibration frequency H for the cylinder with many adhesive masses (see Figure 1 and equations (29) and (30))

	1st	2nd	3rd	4th	5th	6th
			Case (a)			
$r_A = 0$ $r_C = 0$	0·1600	0·3094	0·4584	0·6072	0·7557	0·9039
$r_A = 0$ $r_C = 2$	0·1554	0·3006	0·4454	0·5900	0·7345	0·8789
$r_A = 0$ $r_C = 4$	0·1512	0·2927	0·4343	0·5764	0·7189	0·8619
$r_A = 2$ $r_C = 0$	0·1501	0·2846	0·4169	0·5515	0·6900	0·8315
$r_A = 2$ $r_C = 2$	0·1462	0·2773	0·4063	0·5372	0·6717	0·8092
$r_A = 2$ $r_C = 4$	0·1425	0·2707	0·3971	0·5255	0·6577	0·7935
$r_A = 4$ $r_C = 0$	0·1407	0·2642	0·3935	0·5313	0·6741	0·8191
$r_A = 4$ $r_C = 2$	0·1373	0·2578	0·3834	0·5171	0·6557	0·7966
$r_A = 4$ $r_C = 4$	0·1342	0·2521	0·3746	0·5054	0·6415	0·7807
			Case (b)			
$r_A = 0$ $r_C = 0$	0·1585	0·2794	0·4095	0·5392	0·6687	0·7979
$r_A = 0$ $r_C = 2$	0·1432	0·2664	0·3871	0·5067	0·6267	0·7480
$r_A = 0$ $r_C = 4$	0·1378	0·2536	0·3679	0·4852	0·6065	0·7304
$r_A = 2$ $r_C = 2$	0·1383	0·2550	0·3678	0·4789	0·5902	0·7036
$r_A = 2$ $r_C = 4$	0·1335	0·2436	0·3507	0·4591	0·5708	0·6860
$r_A = 4$ $r_C = 4$	0·1291	0·2335	0·3349	0·4397	0·5509	0·6677

Example 3.6. *In the example, the* r_A *and* r_C *values are varying from* 0, 2 *to* 4, *and two cases are assigned for* b_j, r_j $(j = 1, 2, \ldots, 99)$ *values*

$$r_j = j/100 \quad (\textit{at the position } x = b_j = j(L/100) \quad j = 1, 2, \ldots, 99 \quad (\textit{case (a)}), \tag{30a}$$

$$r_j = \sin(j\pi/100) \quad (\textit{at the position } x = b_j = j(L/100) \quad j = 1, 2, \ldots, 99 \quad (\textit{case (b)}). \tag{30b}$$

The calculated rsults for H values are listed in Table 7.

The validity of the calculated results can be verified by the following ways. We make two cases of computation, for example, under the conditions

(a) $r_A = 2$, $r_j = 0{\cdot}01, 0{\cdot}02, \ldots, 0{\cdot}99$ (at the position $x = b_j = j(L/100)$), $r_C = 4$.

(b) $r_A = 4$, $r_j = 0{\cdot}99, 0{\cdot}98, \ldots, 0{\cdot}01$ (at the position $x = b_j = j(L/100)$), $r_C = 2$.

It is easy to see the final results for the eigenvalue should be the same for two cases. In fact, we have found the same result for the two cases.

4. REMARKS

Previously, when the computer were not available as nowadays, investigators paid attention to the solution which can be performed by hand or very elementary computation. On the contrary, the present study mainly depends on the successful numerical solutions and computer computation. This can be seen from the following facts. It has been shown that, the numerical solution of ordinary differential equation and searching zero of a given function are two key points in the present study. It is well known that the numerical solution of ordinary differential equation by using computer is rather easy, and successful accuracy can be achieved by using Runge–Kutta method. Secondly, it is also easy to find the zeros of the target function, for example, if the half-division technique is used.

It is proved that the target function method is a general one in the field of evaluating the natural vibration frequency in one-dimensional case. Meantime, all the problems for finding the natural vibration frequency of bars in longitudinal vibration, in bending vibration and in torsion vibration, can be easily solved by the target function method.

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