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FREE VIBRATION OF ANNULAR ELLIPTIC PLATES USING BOUNDARY CHARACTERISTIC ORTHOGONAL POLYNOMIALS AS SHAPE FUNCTIONS IN THE RAYLEIGH–RITZ METHOD

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1. INTRODUCTION

In recent years lightweight plate structures have been widely used in many engineering applications and vibration analyses of plates of different shapes have been carried out extensively. Annular elliptic and circular plates are used quite often in aeronautical and ship structures and in several other industrial applications.

Annular circular plates are special cases of annular elliptic plates and are quite simple to analyze using polar co-ordinates. The solution is found to be in the form of Bessel functions for all the nine cases of inner and outer boundary conditions. A survey of literature on the vibration of annular circular plates and results for several cases are provided in a monograph by Leissa [1]. Compared with the amount of information available for circular plates, studies reported on the vibration of elliptic plates are scarce. The main difficulty in studying elliptic plates is the choice of co-ordinates. Elliptic co-ordinates may be used with the exact mode shape in the form of Mathieu functions [1]; however, they are quite cumbersome to handle. Vibration of elliptic plates were studied using modified polar co-ordinates by Rajalingham *et al.* [2–5]. They used a one-dimensional characteristic orthogonal polynomial shape functions suggested originally by Bhat [6] in the Rayleigh–Ritz method. Chakraverty [7] and Singh and Chakraverty [8–10] analyzed the vibration of elliptic plates using two-dimensional boundary characteristic orthogonal polynomials first suggested by Bhat [11] in the Rayleigh–Ritz method.

In view of the difficulty of studying elliptic plates, annular elliptic plates also have not been studied in detail until recently. Chakraverty [7] and Singh and Chakraverty [12] reported the fundamental frequency for different aspect ratios of the annular elliptic plates for various conditions at the inner and outer boundaries. They encountered difficulties in the numerical computations of the higher natural frequencies and the convergency was poor for certain cases of annular elliptic plates.

The present paper provides free vibration natural frequencies of higher modes of annular elliptic plates for all the nine boundary conditions at the inner and outer edges using two-dimensional boundary characteristic orthogonal polynomials in the Rayleigh–Ritz method. Employing a higher computational accuracy than that was used in reference [12],

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the first 12 natural frequencies of annular elliptic and circular plates are presented. These results provide benchmark values that can be used to validate results obtained by other approximate approaches such as the finite element method, the finite difference method and the boundary element method.

2. ANALYSIS

The detailed analysis can be seen in reference [12]. A brief outline is given below.

The outer boundary of the elliptic plate as shown in Figure 1 is defined as

$$R = \left\{ (x,y),\ \frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1,\ x,\ y \in \mathbf{R} \right\}, \tag{1}$$

where a and b are the semi-major and -minor axes respectively. A family of concentric ellipses are defined by introducing a variable C, where

$$x^2 + \frac{y^2}{m^2} = 1 - C, \quad 0 \leqslant C \leqslant C_0, \tag{2}$$

with $m = b/a$ and C_0 defines the inner boundary of the ellipse. The eccentricity of the inner boundary is defined by k, where

$$k = \sqrt{1 - C_0}. \tag{3}$$

When the structure is undergoing simple harmonic motion, the maximum strain energy, V_{max}, and the maximum kinetic energy, T_{max}, of the deformed annular elliptic plate, respectively, are given by

$$V_{max} = \frac{D}{2} \iint_R \left[W_{xx}^2 + 2\nu W_{xx} W_{yy} + W_{yy}^2 + 2(1-\nu) W_{xy}^2 \right] \mathrm{d}y\mathrm{d}x, \tag{4}$$

$$T_{max} = \frac{\rho h \omega^2}{2} \iint_R W^2 \,\mathrm{d}y\ \mathrm{d}x. \tag{5}$$

Equating the maximum strain and kinetic energies we obtain the Rayleigh quotient as

$$\omega^2 = \frac{D \iint_R \left[W_{xx}^2 + 2\nu W_{xx}\, W_{yy} + W_{yy}^2 + 2(1-\nu) W_{xy}^2 \right] \mathrm{d}y\mathrm{d}x}{\rho h \iint_R W^2 \,\mathrm{d}y\mathrm{d}x}, \tag{6}$$

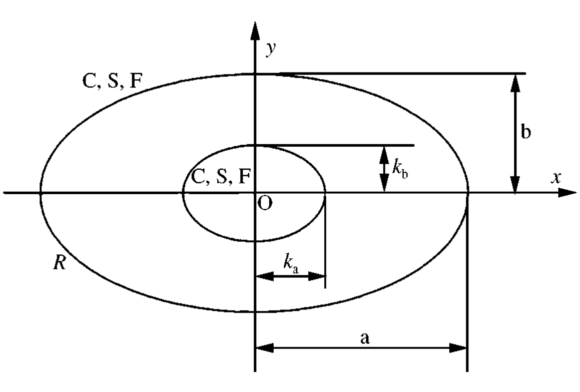

Figure 1. Geometry of the annular elliptic plate.

where $W(x, y)$ is the deflection of the plate; subscripts on W denote differentiation with respect to the subscripted variable.

Substituting a N-term approximation for the deflection,

$$W(x, y) = \sum_{j=1}^{N} c_j \phi_j (x,y), \tag{7}$$

and applying the condition for stationarity of ω^2 with respect to the coefficients c_j results in

$$\sum_{j=1}^{N} (a_{ij} - \lambda^2 b_{ij}) c_j = 0, \quad i = 1,2, \dots, N, \tag{8}$$

where

$$a_{ij} = \iint_{R'} [(\phi_i)_{XX}(\phi_j)_{XX} + (\phi_i)_{YY}(\phi_j)_{YY} + \nu\{(\phi_i)_{XX}(\phi_j)_{YY} + (\phi_i)_{YY}(\phi_j)_{XX}\}$$

$$+ 2(1-\nu)(\phi_i)_{XY}(\phi_j)_{XY}] \, \mathrm{d}Y \mathrm{d}X, \tag{9}$$

$$b_{ij} = \iint_{R'} \phi_i \phi_j \, \mathrm{d}Y \, \mathrm{d}X, \quad \lambda^2 = \frac{a^2 \rho h \omega^2}{D}, \tag{10, 11}$$

and

$$X = x/a, \; Y = y/a.$$

The ϕ_i's are orthogonal polynomials in two variables [11, 12]. Further, $(\phi_i)_{XX}$, $(\phi_i)_{YY}$, etc., are second derivatives of ϕ_i with respect to X and Y. The new domain R' is defined by

$$R' = \left\{(X,Y),\; X^2 + \frac{Y^2}{m^2} \leqslant 1, \quad X, Y \in R\right\}.$$

Since the ϕ_i's are orthogonal, equation (8) reduces to

$$\sum_{j=1}^{N} (a_{ij} - \lambda^2 \delta_{ij}) c_j = 0, \quad i = 1,2, \dots, N, \tag{12}$$

where

$$\delta_{ij} = 0 \quad \text{if } i \neq j$$
$$= 1 \quad \text{if } i = j.$$

Equation (12) is a standard eigenvalue problem and can be solved to obtain the natural frequencies.

3. GENERATION OF ORTHOGONAL POLYNOMIALS

For the generation of the two-dimensional orthogonal polynomials, the following linearly independent set of functions are employed:

$$F_i(X,Y) = g(X,Y)\{f_i(X,Y)\}, \quad i = 1,2, \dots, N, \tag{13}$$

where $g(X,Y)$ satisfies the essential boundary conditions and the $f_i(X,Y)$ are taken as the combinations of terms of the form $X^{l_i} Y^{n_i}$ where l_i and n_i are non-negative positive integers. The function $g(X,Y)$ is defined by

$$g(X,Y) = C^s (C_0 - C)^t, \tag{14}$$

where s takes the value of 0,1 or 2 in order to define free, simply supported or clamped conditions, respectively, at the outer boundary of the annular elliptic plate. Similarly, $t = 0$, 1 or 2 will define the corresponding boundary conditions at the inner edge of the annular elliptic plate.

The inner product of two functions $\phi_i(X,Y)$ and $F_i(X,Y)$ is defined as

$$\langle \phi_i, F_i \rangle = \iint_R \phi_i(X,Y)\, F_i(X,Y)\, \mathrm{d}x\, \mathrm{d}y. \tag{15}$$

The norm of ϕ_i is therefore given by

$$\| \phi_i \| = \langle \phi, \phi_i \rangle^{1/2}, \tag{16}$$

Proceeding as in Singh and Chakraverty [12] and Chakraverty [7], the Gram–Schmidt orthogonalization process can be written as

$$\phi_1 = F_1, \quad \left. \begin{array}{l} \phi_i = F_i - \sum_{j=1}^{i-1} \alpha_{ij}\ \phi_j \\ \alpha_{ij} = \dfrac{\langle F_l, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle},\ j = 1,2, \ldots, (i-1) \end{array} \right\}, \quad i = 2, \ldots, N. \tag{17}$$

where the ϕ_i are the constructed orthogonal polynomials.

4. NUMERICAL RESULTS AND DISCUSSIONS

Numerical results for the first 12 natural frequencies have been computed by taking the aspect ratio of the outer boundary, $b/a = m = 0.5$ and 1.0 with the inner boundary parameter k varying from 0.2 to 0.8 at the interval of 0.2. Results are provided for the nine cases of boundary conditions (CC, CS, CF; SC, SS, SF; FC, FS and FF) at both the inner and outer edges. Here C, S, and F designate clamped, simply supported and free boundary, and first and second letters denote the conditions at the outer and inner edges respectively. The Poisson's ratio, ν, is taken as 1/3 in all the calculations. Computations are simplified by considering the shape functions to represent symmetric–symmetric, symmetric–antisymmetric, antisymmetric–symmetric and antisymmetric–antisymmetric groups. First 12 natural frequencies are arranged in ascending order by choosing the first three frequencies from each of the above mode groups. The convergence of the results is studied by computing the results for different values of N until the first four significant digits converge. It was found that the results converged for N values of 26–30 in all the cases. The convergence of the first 12 natural frequencies is shown in Table 1 taking $m = 0.5$ and $k = 0.2$ for CF and FC boundary conditions. The results of the computations for various values of m and k and for different boundary conditions at the outer and inner edges are presented in Tables 2–10.

Results for the special cases such as for (1) full circular ($k = t = 0$, $m = 1.0$) and elliptic ($k = t = 0$) plates and (2) annular circular plates ($m = 1.0$) were compared with those from earlier studies and were found to be in good agreement. The higher modes for annular circular plates are compared with the axisymmetric results from Leissa [1]. It was pointed out in references [7, 12] that although the results given in reference [1] are the exact solutions, their accuracy is poor which explains the difference between the present results with those in reference [1]. The results of reference [2] have been divided by $b/a\,(=m)$ in order to agree with the frequency parameter employed in this study. It was noted that Rajalingham *et al.* [2] presented the results for axisymmetric modes, and for the free

TABLE 1

Convergence of results ($m = 0{\cdot}5$, $k = 0{\cdot}2$ *and* $\nu = 1/3$, $\lambda = \omega a^2(\rho h/D)^{1/2}$)

BC	N	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}
CF	3	29·12	39·87	63·11	70·28	89·38	87·32	150·7	124·7	178·9	161·0	248·0	293·3
	10	28·76	39·72	57·38	69·91	77·60	86·21	106·3	109·2	139·3	134·9	171·5	210·6
	15	28·52	39·68	57·25	69·78	77·43	86·14	103·8	108·9	134·2	134·4	165·0	200·1
	20	28·39	39·63	57·16	69·69	77·31	86·10	103·6	108·8	133·5	134·3	164·0	198·5
	25	28·21	39·58	57·07	69·51	77·17	86·06	103·5	108·7	133·4	134·2	163·8	198·2
	28	28·03	39·52	57·00	69·18	77·13	86·03	103·5	108·5	133·3	134·2	163·7	198·2
	30	28·03	39·52	56·98	69·18	77·03	86·03	103·5	108·5	133·2	134·2	163·7	198·2
FC	3	21·85	24·48	41·07	40·88	72·97	102·9	121·6	166·8	152·0	373·4	676·9	684·4
	10	11·54	12·97	21·65	21·29	25·15	32·30	36·99	62·33	50·17	85·07	89·24	117·2
	15	9·771	10·67	18·46	18·43	22·53	28·39	32·26	46·63	44·83	65·02	68·61	91·69
	20	9·698	9·750	17·94	18·40	21·69	26·69	30·48	41·72	42·64	59·12	62·39	84·20
	25	8·686	9·299	16·58	16·75	21·36	26·54	30·11	40·19	42·44	56·40	60·94	81·89
	28	8·167	8·445	15·41	15·69	20·78	25·68	28·97	39·40	41·31	55·41	59·78	87·12
	30	8·167	8·448	15·41	15·69	20·78	25·68	28·97	39·36	41·31	55·30	59·78	81·10

Table 2

Frequency parameters for annular elliptic plate with CC conditions ($\nu = \frac{1}{3}$, $\lambda = \omega a^2(\rho h/D)^{1/2}$)

m	k	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}
0·5	0·0	58·67	59·88	99·89	103·7	105·2	106·2	132·9	135·2	153·4	162·3	186·9	218·0
	0·2	66·09	67·31	114·6	115·6	119·8	120·3	148·1	151·7	163·3	172·3	195·4	227·4
	†	71·99											
	0·4	94·54	95·50	152·7	154·3	178·3	179·1	219·2	219·8	232·7	233·5	254·7	272·6
	†	98.94											
	0·6	181·7	182·7	260·7	265·1	326·8	329·2	382·9	386·0	432·1	433·3	476·9	483·4
	†	184·8											
	0·8	632·0	641·7	787·4	796·7	946·3	1019	1105	1138	1261	1427	1428	1487
	†	641·7											
1·0	0·0	27·88	32·31	32·31	40·90	42·04	55·51	55·52	72·29	73·64	89·47	89·47	104·9
	‡	22·7											
	0·2	35·12	37·82	37·91	44·98	46·04	57·96	58·18	74·25	75·71	95·10	95·26	117·2
	†	35·87											
	§	34·61											
	¶	36·23			41·80								
	0·4	61·88	63·04	63·04	66·87	67·17	74·97	75·00	86·88	88·55	105·3	105·7	125·3
	†	61·91											
	§	61·87											
	¶	62·33	62·92		66·41								
	0·6	139·6	140·5	140·5	143·1	143·1	147·9	147·9	154·8	155·0	166·1	166·4	178·3
	†	139·6											
	§	139·6											
	0·8	559·1	559·8	559·8	561·9	561·9	565·3	565·3	570·2	570·2	576·7	576·7	584·8
	†	559·1											
	§	559·1											

†, From references [7, 12], $\nu = \frac{1}{3}$; ‡, from reference [1], $\nu = 0{\cdot}3$; §, exact solution from references [7, 12], $\nu = \frac{1}{3}$; ¶, from reference [1], $\nu = \frac{1}{3}$.

TABLE 3

Frequency parameters for annular elliptic plate with CS conditions ($\nu = \frac{1}{3}$, $\lambda = \omega a^2(\rho h/D)^{1/2}$)

m	k	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}
0·5	0·0	49·70	50·67	85·01	87·96	91·49	94·00	117·5	118·9	141·8	145·0	172·3	205·7
	0·2	56·22	56·67	96·11	97·49	101·2	102·8	125·4	126·3	148·3	151·9	177·3	210·0
	†	58·40											
	0·4	78·41	78·76	129·6	130·0	146·7	147·4	171·6	177·9	181·4	191·6	207·6	236·4
	†	79·91											
	0·6	142·7	143·0	213·8	214·5	261·8	262·6	308·0	308·7	339·2	339·2	370·8	374·5
	†	144·1											
	0·8	467·8	468·6	600·9	601·7	722·9	730·6	835·9	836·7	944·9	968·0	1044	1051
	†	470·8											
1·0	0·0	23·15	26·69	26·80	36·75	37·09	51·92	51·98	63·91	70·10	73·49	7382	91·56
	0·2	26·68	30·29	30·39	39·42	39·77	53·29	53·36	70·74	71·19	85·83	86·00	100·4
	†	26·78											
	‡	26·61											
	§	26·57	29·11		37·54								
	0·4	44·93	46·74	46·74	52·39	52·45	62·60	62·67	77·23	77·90	96·32	96·34	118·0
	†	44·93											
	‡	44·93											
	§	44·89	47·09		51·81								
	0·6	98·79	100·1	100·1	104·0	104·0	110·7	110·7	120·4	120·4	133·3	133·3	149·4
	†	98·79											
	‡	98·79											
	§	99·16			104·5								
	0·8	389·5	390·5	390·5	393·4	393·4	398·2	398·2	405·1	405·1	414·0	414·0	425·1
	†	389·5											
	‡	389·5											

†, From references [7, 12], $\nu = \frac{1}{3}$; ‡, exact solution from references [7, 12], $\nu = \frac{1}{3}$; §, from reference [1], $\nu = \frac{1}{3}$.

TABLE 4

Frequency parameters for annular elliptic plate with CF conditions ($\nu = \frac{1}{3}$, $\lambda = \omega a^2(\rho h/D)^{1/2}$)

m	k	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}
0·5	0·0	27·38	39·50	55·98	69·86	77·00	88·05	102·6	109·9	133·0	135·7	165·5	199·4
	†	27·38	39·50	55·99	69·86								
	‡	27·74						108·0					
	0·2	28·03	39·52	56·98	69·18	77·03	86·03	103·5	108·5	133·2	134·2	163·7	198·2
	§	28·38											
	0·4	36·35	41·04	54·03	62·69	77·09	82·64	103·4	104·6	128·2	133·1	160·6	194·8
	§	36·52											
	0·6	59·75	59·89	86·30	91·71	93·93	106·7	112·6	130·1	135·9	156·5	166·2	199·2
	§	59·99											
	0·8	151·5	151·6	218·5	218·8	262·5	263·0	300·6	301·3	330·6	331·5	350·5	358·3
	§	152·0											
1·0	0·0	10·22	21·26	21·26	34·88	34·88	39·77	51·03	51·03	60·83	60·83	69·67	84·58
	†	10·22	21·26		34·88		39·77						
	‡	10·22					39·77						
	¶	10·24	21·25		34·88								
	0·2	10·46	21·14	21·17	33·91	33·93	43·16	50·59	50·59	60·49	60·68	69·56	82·73
	§	10·56											
	‖	10·34											
	¶	10·34	20·48		33·86								
	0·4	13·50	19·46	19·48	31·74	32·06	47·81	47·89	66·81	66·94	72·00	72·03	86·61
	§	13·51											
	‖	13·50											
	¶	13·54	19·80		31·34								
	0·6	25·24	28·50	28·50	36·42	36·43	48·11	48·12	63·18	63·67	82·56	82·73	105·1
	§	25·24											
	‖	25·24											
	¶	25·60	28·52		36·60								
	0·8	92·81	94·32	94·32	98·78	98·78	106·1	106·1	116·2	116·2	128·9	128·9	144·3
	§	92·81											
	‖	92·81											

†, From reference [8], $\nu = 0{\cdot}3$; ‡, from reference [2]; §, from references [7, 12], $\nu = \frac{1}{3}$; ¶, from reference [1], $\nu = \frac{1}{3}$; ‖, exact solution from references [7, 12], $\nu = \frac{1}{3}$.

TABLE 5

Frequency parameters for annular elliptic plate with SC conditions ($v = \frac{1}{3}$, $\lambda = \omega a^2(\rho h/D)^{1/2}$)

m	k	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}
0·5	0·0	38·52	39·43	73·04	73·43	76·35	81·78	101·0	108·2	122·0	133·1	154·2	186·0
	0·2	43·46	44·52	82·25	82·70	87·07	91·26	110·1	117·8	128·5	140·7	160·0	191·4
	†	48·91											
	0·4	62·90	63·77	110·1	111·5	132·7	133·4	160·8	164·7	169·9	180·2	193·2	219·6
	†	67·21											
	0·6	122·5	123·2	188·9	190·9	241·3	243·8	288·0	290·9	325·7	328·7	360·8	361·0
	†	126·4											
	0·8	436·1	438·9	570·1	571·4	718·3	736·9	823·1	836·7	988·2	1007	1061	1097
	†	443.7											
1·0	0·0	14·75	22·42	22·51	30·40	31·47	43·55	43·71	59·28	60·68	74·43	74·82	90·01
	‡	14·8											
	0·2	23·34	26·05	26·10	33·02	34·07	45·37	45·54	60·56	62·19	80·32	80·57	99·10
	†	24·18											
	§	22·76											
	¶	22·79	24·32	31·08									
	0·4	41·27	42·63	42·63	47·05	47·52	56·04	56·14	68·66	70·64	87·11	87·43	107·1
	†	41·37											
	§	41·26											
	¶	41·23	42·56		46·81								
	0·6	94·26	95·32	95·32	98·57	98·58	104·3	104·4	112·6	113·3	125·5	125·9	140·7
	†	94·26											
	§	94·26											
	¶	95·16	96·67		98·84								
	0·8	381·6	382·5	382·5	385·2	385·2	389·6	389·6	396·0	396·0	404·2	404·3	414·6
	†	381·6											
	§	381·6											

†, From references [7, 12], $v = \frac{1}{3}$; ‡, from reference [1], $v = 0{\cdot}3$; §, exact solution from references [7, 12], $v = \frac{1}{3}$; ¶, from reference [1], $v = \frac{1}{3}$.

TABLE 6

Frequency parameters for annular elliptic plate with SS conditions ($v = \frac{1}{3}$, $\lambda = \omega a^2(\rho h/D)^{1/2}$)

m	k	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}
0·5	0·0	31·60	32·24	60·61	61·36	67·43	69·83	90·16	92·80	112·3	118·9	141·5	173·4
	0·2	35·74	35·86	66·89	67·04	72·15	77·20	94·18	98·49	116·7	123·2	144·9	176·4
	†	37·51											
	0·4	49·93	50·22	89·66	90·30	101·8	105·9	118·7	128·0	135·5	148·0	162·6	192·7
	†	51·33											
	0·6	91·78	92·07	148·8	149·4	185·9	186·4	219·5	220·2	238·5	244·7	255·8	274·7
	†	93·06											
	0·8	302·8	303·4	321·2	414·0	415·3	512·7	516·7	598·5	602·4	608·0	755·4	757·0
	†	305·8											
1·0	0·0	15·15	18·19	18·25	27·12	27·44	40·76	40·82	51·31	57·29	60·14	60·39	76·81
	0·2	16·86	20·47	20·53	28·83	29·17	41·64	41·71	57·73	58·18	69·24	69·41	83·48
	†	17·03											
	‡	16·72											
	§	17·39	19·91		27·55								
	0·4	28·08	30·09	30·09	36·27	36·45	47·15	47·23	61·69	62·40	80·11	80·19	100·8
	†	28·08											
	‡	28·08											
	§	28·25	30·00		36·14								
	0·6	62·12	63·68	63·68	68·35	68·35	76·15	76·15	87·11	87·19	101·5	101·5	119·2
	†	62.12											
	‡	62·12											
	§	62·09	62·41		68·41								
	0·8	247.0	248·3	248·3	252·0	252·0	258·2	258·2	266·8	266·8	277·9	277·9	291·6
	†	247·0											
	‡	247·0											

†, From references [7, 12], $v = \frac{1}{3}$; ‡, exact solution from references [7, 12], $v = \frac{1}{3}$; §, from reference [1], $v = \frac{1}{3}$.

TABLE 7

Frequency parameters for annular elliptic plate with SF conditions ($\nu = \frac{1}{3}$, $\lambda = \omega a^2(\rho h/D)^{1/2}$)

m	k	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}
0·5	0·0	13·27	23·72	38·43	46·19	57·59	62·81	81·31	83·07	107·2	109·6	135·2	167·3
	†	13·21	23·64	38·35	46·15								
	‡	13·46						80·76					
	0·2	12·79	23·71	38·80	45·98	57·63	61·44	81·62	82·37	106·0	109·7	134·0	166·4
	§	13·07											
	0·4	12·26	22·53	31·85	40·94	56·22	59·43	77·61	81·35	101·3	108·7	132·0	163·5
	§	12·40											
	0·6	14·56	24·04	27·15	42·67	48·75	65·65	72·87	91·61	99·11	118·1	130·3	162·4
	§	14·59											
	0·8	24·37	37·27	38·31	59·83	64·61	88·81	92·59	120·3	122·7	153·6	155·7	190·9
	§	24·27											
1·0	0·0	4·984	13·94	13·94	25·65	25·65	29·76	34·00	34·00	48·52	48·52	56·88	70·15
	†	4·935	13·90		25·61		29·72						
	‡	4·935					29·72						
	¶		13·93		25·65								
	0·2	4·851	13·90	13·91	25·00	25·00	31·75	39·74	39·75	48·46	48·55	56·82	68·52
	§	4·930											
	‖	4·732											
	¶	4·726	12·60		24·97								
	0·4	4·748	12·06	12·10	23·56	23·82	37·91	37·95	47·32	53·27	53·34	55·15	69·74
	§	4·777											
	‖	4·743											
	¶	4·752	11·66		23·09								
	0·6	5·663	11·73	11·73	22·16	22·21	34·98	35·02	50·54	51·42	70·17	70·35	91·89
	§	5·664											
	‖	5·663											
	¶	5·664	12·27		22·20								
	0·8	9·455	16·78	16·78	29·65	29·65	44·11	44·11	59·90	56·94	77·27	77·29	96·42
	§	9·455											
	‖	9·455											
	¶	9·431	17·05		29·92								

†, From reference [9], $\nu = 0·3$; ‡, from reference [2]; §, from references [7, 12], $\nu = \frac{1}{3}$; ¶, from reference [1], $\nu = \frac{1}{3}$; ‖, exact solution from references [7, 12], $\nu = \frac{1}{3}$.

Table 8

Frequency parameters for annular elliptic plate with FC conditions ($\nu = \frac{1}{3}$, $\lambda = \omega a^2(\rho h/D)^{1/2}$)

m	k	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}
0·5	0·0	6·982	7·535	13·36	14·17	17·74	23·76	26·98	37·48	39·93	53·41	58·57	80·08
	0·2	8·167	8·448	15·41	15·69	20·78	25·68	28·97	39·36	41·31	55·30	59·78	81·10
	†	9·750											
	0·4	12·19	12·50	22·35	22·78	33·12	34·90	39·92	48·41	48·66	63·25	65·89	86·37
	†	13·60											
	0·6	25·38	25·68	42·44	42·91	63·05	63·43	77·81	79·39	86·66	94·36	96·47	112·3
	†	26·73											
	0·8	96·51	97·02	137·2	138·3	184·8	187·5	225·5	228·0	266·1	270·5	296·5	299·9
	†	98·93											
1·0	0·0	4·524	4·538	4·541	6·627	6·973	12·94	12·96	21·86	22·16	33·47	33·49	44·43
	‡	3·752							20·91				
	0·2	5·384	5·394	5·404	7·265	7·620	13·28	13·31	22·05	22·42	33·63	33·67	47·23
	†	5·593											
	§	5·214											
	‡	5·244	4·814		6·345								
	0·4	9·082	9·176	9·188	10·53	10·74	15·34	15·35	23·33	23·89	34·67	34·73	48·01
	†	9·118											
	§	9·071											
	‡	9·096			10·37								
	0·6	20·60	20·98	20·98	22·43	22·45	25·77	25·78	31·75	32·17	41·31	41·35	53·35
	†	20·60											
	§	20·60											
	‡	20·63	20·93		21·63								
	0·8	84·67	85·29	85·29	87·17	87·17	90·47	90·48	95·39	95·40	102·1	102·2	116·5
	†	84·67											
	§	84·67											

†, From references [7, 12], $\nu = \frac{1}{3}$; ‡, from reference [1], $\nu = \frac{1}{3}$; §, exact solution from references [7, 12], $\nu = \frac{1}{3}$.

Table 9

Frequency parameters for annular elliptic plate with FS conditions ($v = \frac{1}{3}$, $\lambda = \omega a^2(\rho h/D)^{1/2}$)

m	k	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}
0·5	0·0	5·010	5·684	9·556	11·64	15·09	20·79	24·19	33·28	37·56	48·42	55·76	77·56
	0·2	5·895	6·254	10·92	12·74	13·75	22·09	24·92	34·56	38·53	50·15	56·37	78·04
	†	6·823											
	0·4	7·666	7·770	10·68	14·69	15·05	24·82	25·68	38·40	40·47	55·17	59·04	80·41
	†	8·215											
	0·6	10·84	11·89	14·15	20·23	20·32	32·35	32·45	47·93	48·16	66·26	67·32	89·32
	†	11·08											
	0·8	22·22	25·66	28·33	39·56	41·33	44·91	59·24	59·33	80·92	81·68	106·0	133·3
	†	21·24											
1·0	0·0	3·154	3·171	3·802	5·672	5·783	12·36	12·37	21·55	21·60	26·68	26·77	38·21
	0·2	3·466	3·679	3·699	6·081	6·195	12·49	12·50	21·60	21·67	30·07	30·15	40·78
	†	3·592											
	‡	3·313											
	§	3·312			5·513								
	0·4	3·634	4·001	4·009	6·918	7·160	13·26	13·29	22·06	22·22	33·45	33·46	47·03
	†	3·654											
	‡	3·629											
	§	3·610	3·940		6·620								
	0·6	4·809	6·047	6·052	9·721	9·738	15·85	15·86	24·40	24·70	35·67	35·72	48·96
	†	4·809											
	‡	4·809											
	§	4·951	6·027		9·653								
	0·8	8·782	12·48	12·48	20·15	20·15	29·51	29·51	40·28	40·29	52·55	52·55	66·47
	†	8·782											
	‡	8·782											
	§	8·779	12·55		19·95								

†, From references [7, 12], $v = \frac{1}{3}$; ‡, exact solution from references [7, 12], $v = \frac{1}{3}$; §, from references [1], $v = \frac{1}{3}$.

Table 10

Frequency parameters for annular elliptic plate with FF conditions ($\nu = \frac{1}{3}$, $\lambda = \omega a^2(\rho h/D)^{1/2}$)

m	k	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}
0·5	0·0	6·589	10·32	16·74	21·62	27·76	31·19	36·26	42·84	49·78	54·52	70·53	76·48
	†	6·597	10·35	16·76	21·66								
	‡	24·54											
	0·2	6·404	9·910	16·71	21·55	26·34	31·15	35·90	42·79	49·69	54·33	70·25	76·25
	§	6·421											
	0·4	5·876	8·807	15·98	19·91	23·86	31·78	35·06	40·82	49·45	50·89	54·57	75·02
	§	5·886											
	0·6	5·201	7·403	14·73	16·32	25·21	31·33	32·97	40·73	45·96	50·64	51·72	79·00
	§	5·201											
	0·8	5·156	6·962	13·03	13·70	25·40	26·86	41·87	43·35	46·98	64·23	66·87	69·57
	§	4·417											
1·0	0·0	5·251	5·251	9·076	12·22	12·22	20·52	20·52	21·49	21·49	33·01	33·01	35·24
	†	5·262		9·069	12·24		20·51						
	‡	9·003											
	0·2	5·056	5·058	8·812	12·19	12·19	20·48	20·49	21·49	21·49	33·01	33·01	34·36
	§	5·065											
	¶	5·049											
	‖	5·053											
	0·4	4·533	4·533	8·583	11·77	11·77	17·80	17·91	21·27	21·28	32·92	32·92	33·09
	§	4·541											
	¶	4·532											
	‖	4·567											
	0·6	3·865	3·866	10·55	10·57	10·57	18·21	18·21	19·86	19·90	31·64	31·65	32·24
	§	3·871											
	¶	3·864											
	‖	3·865											
	0·8	3·197	3·198	8·873	8·873	16·90	16·91	18·26	27·29	27·29	29·39	29·39	40·05
	§	3·197											
	¶	3·197											
	‖	3·200											

†, From reference [10], $\nu = 0{\cdot}33$; ‡, from reference [2]; §, from references [7, 12], $\nu = \frac{1}{3}$; ¶, exact solution from references [7, 12], $\nu = \frac{1}{3}$; ‖, from reference [1], $\nu = \frac{1}{3}$.

boundary case they missed the fundamental frequency, which may be because of their choice of deflection functions. The present values are computed using higher numerical accuracy than that used in references [7, 12]. Further, the use of symmetry classes enabled the use of more deflection functions when compared to references [7, 12]. Hence, the fundamental frequencies reported in references [7, 12] are poor in comparison with the present results.

It is interesting to note the effects of hole size on the natural frequencies from Tables 2–10. As k increases for a particular value of m ($= b/a$) the frequencies increase, for all the boundary conditions, except for the exceptional case of the FF boundary in Table 10. For the FF boundary condition, frequencies decrease as k is increased for a particular value of m.

In the tables presented below, the results are grouped into three sets corresponding to the supporting condition of the external boundaries of the plates. Set 1 stands for clamped conditions on the external edge of the plate (CC, CS and CF), set 2 stands for supported external boundaries of the elliptic plate (SC, SS and SF) while set 3 stands for free external boundaries of the plate (FC, FS and FF).

It is seen from Tables 2–4 (set 1) that all the frequencies decrease from CC to CF. Similar behaviour is also observed for set (2) boundary conditions as seen from Tables 5–7. Here, frequencies decrease from SC to SF. For set (3) boundary conditions, with smaller hole size, i.e., for $k = 0$ and 0.2, Tables 8–10 show that the FC condition gives higher frequencies and the FS condition gives lower frequencies. But for larger hole size, viz., for k greater than or equal to 0·4, the results are smaller for the FF boundary, as in set (1) and (2). It may also be noted that for set (3) with k less than or equal to 0.2 (smaller hole size), the frequencies (particularly the lower modes) for conditions FC and FF are closer. This behaviour is analogous to that of the beams with CC and FF boundary conditions, where the frequencies are identical.

5. CONCLUDING REMARKS

Two-dimensional boundary characteristic orthogonal polynomials in the Rayleigh–Ritz method have been used to obtain higher modes of annular plate with a curves boundary (especially for elliptical boundaries) in this study. The effect of boundary conditions and hole size on different modes of vibrations has been fully investigated. As the hole size increases, the natural frequencies increase except when the inner and the outer boundaries of the annular elliptic plate are free, for which they decrease with the hole size.

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APPENDIX A: NOMENCLATURE

a, b	semi-major and -minor axes of the outer boundary of the annular elliptic plate (Figure 1)
x, y	Cartesian co-ordinates
$R(x,y)$	domain occupied by the annular elliptic plate
m	aspect ratio of the outer boundary, b/a
k	eccentricity of the inter boundary of the annular elliptic plate
$W(x,y)$	deflection of the plate
h	thickness of the plate
ρ	mass per unit volume
E	Young's modulus of the material of the plate
ν	Poisson's ratio
D	flexural rigidity, $Eh^3/(12(1-\nu^2))$
ω	angular frequency
λ	non-dimensional frequency, parameter, $a^2\omega\sqrt{\rho h/D}$
$R'(X,Y)$	new domain occupied by the plate, $X = x/a$, $Y = y/a$
$\phi_i(x,y)$	orthogonal polynomials
N	number of terms used in the approximation
c_j	the coefficients in the expansion
a_{ij}	defined in equation (9)
b_{ij}	defined in equation (10)
δ_{ij}	Kronecker delta
$F_i(X,Y)$	defined in equation (13)
$g(X,Y)$	defined in equation (14)
$\langle\phi_i,F_i\rangle$	inner product of functions ϕ_i and F_i
$\Vert\phi_i\Vert$	norm of ϕ_i
α_{ij}	coefficients of Gram–Schmidt orthogonalization process
$\hat{\phi}_i$	orthonormal polynomials
C	clamped boundary
S	simply supported boundary
F	free boundary