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RAYLEIGH-RITZ ANALYSIS OF SANDWICH BEAMS

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The paper discusses the possibility of predicting the dynamic behaviour, in terms of modal frequencies and loss factors, of sandwich beams with a constrained viscoelastic layer configuration. The problem is approached by the Rayleigh–Ritz method so that virtually any boundary condition can be dealt with by applying a single procedure. Simple polynomials are used as admissible functions and evidence of their good performance is given. The method is also suitable for the analysis of those arrangements where not only the viscoelastic material but also the external layers of the sandwich are damped.

1. INTRODUCTION

Flexural vibrations of both beams and plates, with various damping treatments, have been the subject of research in the past decades, starting with the early works of Oberst and Kerwin, Ross and Ungar [1–3]. Also by limiting the attention to beams with a constrained layer configuration, i.e., a single layer of damping material sandwiched between two non-dissipating faces, it is possible to find a number of detailed papers and many different approaches to the topic in the literature. In reference [4], a sixth order differential equation of motion is formulated for the free oscillations of sandwich beams, and is demonstrated that, for each mode, the loss factor is independent of the boundary conditions, given a certain vibration frequency. The problem of computing both frequencies and loss factors is explicitly solved in references [5–7], for both beams and plates, when simply supported end conditions are assumed. In references [8–10], analytical–numerical procedures are proposed to solve the problem when different boundary conditions are assumed. The finite element procedure has also been adopted [11, 12], on the basis of the considerations expressed in reference [13].

In the present investigation, mode shapes, frequencies and loss factors are sought by means of the Rayleigh–Ritz method. Polynomials are used as admissible functions [14], properly chosen so as to satisfy the geometrical boundary conditions.

2. KINETIC AND STRAIN ENERGIES

The first step required by the Rayleigh–Ritz method is to express the kinetic and strain energies of the structure. The following hypotheses, indeed common to many authors, are assumed [7]:

(a) the constitutive materials of the beam are homogeneous and isotropic,
(b) all displacements, both in-plane and transverse as well as rotations, are small,

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(c) the three layers undergo the same transverse deflection,
(d) no slipping occurs at the interface between the three layers,
(e) plane transverse to the middle plane remains plane after bending,
(f) longitudinal displacements vary linearly (Figure 1),
(g) extension and stress σ_x in the core are ignored.

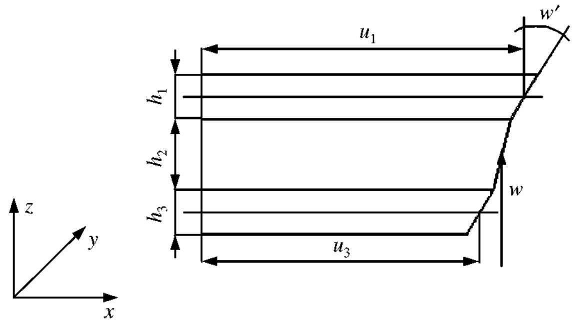

Figure 1. Sketch of the deformations in the sandwich.

Choosing u_1 and u_3 (the in-plane displacements in the x direction) and w (the transverse displacement) as independent variables (subscripts 1 and 3 denoting the external faces), the stress and strain components may be written as (see Appendix A for a list of symbols):

Core $\quad \gamma_{xz} = \dfrac{u_1 - u_3}{h_2} - \dfrac{\partial w}{\partial x}\dfrac{C}{h_2}, \quad \tau_{xz} = G_2\left(\dfrac{u_1 - u_3}{h_2} - \dfrac{\partial w}{\partial x}\dfrac{C}{h_2}\right),$

Face 1 $\quad \varepsilon_{xx} = \dfrac{\partial u_1}{\partial x} - z\dfrac{\partial^2 w}{\partial x^2}, \quad \sigma_{xx} = E_1\left(\dfrac{\partial u_1}{\partial x} - z\dfrac{\partial^2 w}{\partial x^2}\right),$

Face 3 $\quad \varepsilon_{xx} = \dfrac{\partial u_3}{\partial x} - z\dfrac{\partial^2 w}{\partial x^2}, \quad \sigma_{xx} = E_3\left(\dfrac{\partial u_3}{\partial x} - z\dfrac{\partial^2 w}{\partial x^2}\right),$

with z as in Figure 1.

By choosing $\xi = x/L$, $\xi \in [0, 1]$, and letting

$$w' = \frac{\partial w}{\partial \xi}, \quad w'' = \frac{\partial^2 w}{\partial \xi^2}, \quad u_1' = \frac{\partial u_1}{\partial \xi}, \quad u_3' = \frac{\partial u_3}{\partial \xi},$$

the strain energy U is expressed as [7]

$$U = \frac{G_2 C^2}{2Lh_2}\int_0^1 w'^2\,\mathrm{d}\xi + \frac{E_1 h_1^3 + E_3 h_3^3}{24L^3}\int_0^1 w''^2\,\mathrm{d}\xi + \frac{G_2 L}{2h_2}\int_0^1 (u_1^2 + u_3^2)\,\mathrm{d}\xi$$
$$+ \frac{E_1 h_1}{2L}\int_0^1 u_1'^2\,\mathrm{d}\xi + \frac{E_3 h_3}{2L}\int_0^1 u_3'^2\,\mathrm{d}\xi - \frac{G_2 L}{h_2}\int_0^1 u_1 u_3\,\mathrm{d}\xi + \frac{G_2 C}{h_2}\int_0^1 (u_3 w' - u_1 w')\,\mathrm{d}\xi. \tag{1}$$

The kinetic energy T, taking into account transverse and in-plane displacements as well as rotary effects, is [7]

$$2T/L\omega^2 = \rho \int_0^1 w^2\,\mathrm{d}\xi + \left[\frac{\rho_1 h_1^3 + \rho_3 h_3^3}{12L^2} + \frac{\rho_2 h_2}{L^2}\left(\varepsilon_1^2 + \frac{\varepsilon_2^2}{12}\right)\right]\int_0^1 w'^2\,\mathrm{d}\xi + \left(\rho_1 h_1 + \frac{\rho_2 h_2}{3}\right)$$
$$\times \int_0^1 u_1^2\,\mathrm{d}\xi + \left(\rho_3 h_3 + \frac{\rho_2 h_2}{3}\right)\int_0^1 u_3^2\,\mathrm{d}\xi + \frac{\rho_2 h_2}{3}\int_0^1 u_1 u_3\,\mathrm{d}\xi$$
$$+ \frac{\rho_2 h_2}{L}\left(\varepsilon_1 - \frac{\varepsilon_2}{6}\right)\int_0^1 u_1 w'\,\mathrm{d}\xi + \frac{\rho_2 h_2}{L}\left(\varepsilon_1 + \frac{\varepsilon_2}{6}\right)\int_0^1 u_3 w'\,\mathrm{d}\xi. \quad (2)$$

By properly choosing the expressions for the three displacements, w, u_1, and u_3, all the integrals in equations (1) and (2) can be evaluated in order to obtain the circular frequency ω.

It should be noted that different hypotheses, leading to different expressions for the kinetic and strain energies, would not affect the procedure described here.

3. FORMULATION OF THE PROBLEM

The second step required by the Rayleigh–Ritz method is to express the displacements as a sum of admissible functions. With

$$w(\xi) = \sum_{i=1}^{I} a_i\phi_i(\xi), \quad u_1(\xi) = \sum_{i=1}^{I} b_i\psi_i(\xi), \quad u_3(\xi) = \sum_{i=1}^{I} c_i\psi_i(\xi) \quad (3)$$

and minimizing Rayleigh's quotient with respect to the constant coefficients a_i, b_i and c_i, it is possible to assemble a system of equations in matrix form as

$$\mathbf{Ax} = \lambda\mathbf{Bx}. \quad (4)$$

As an example, consider the derivative $\partial U/\partial a_i$:

$$\frac{\partial U}{\partial a_i} = 2\frac{G_2 C^2}{2Lh_2}\sum_{k=1}^{I} a_k \int_0^1 \phi'_k\phi'_i\,\mathrm{d}\xi + 2\frac{E_1 h_1^3 + E_3 h_3^3}{24L^3}\sum_{k=1}^{I} a_k \int_0^1 \phi''_k\phi''_i\,\mathrm{d}\xi$$
$$- \frac{G_2 C}{h_2}\sum_{k=1}^{I} b_k \int_0^1 \psi_k\phi'_i\,\mathrm{d}\xi + \frac{G_2 C}{h_2}\sum_{k=1}^{I} c_k \int_0^1 \psi_k\phi'_i\,\mathrm{d}\xi \quad (5)$$

and $i = 1, \ldots, I$.

With the position

$$E_{ki}^{rs} = \int_0^1 \frac{\mathrm{d}^r\phi_k}{\mathrm{d}\xi^r}\frac{\mathrm{d}^s\phi_i}{\mathrm{d}\xi^s}\,\mathrm{d}\xi, \quad F_{ki}^{rs} = \int_0^1 \frac{\mathrm{d}^r\psi_k}{\mathrm{d}\xi^r}\frac{\mathrm{d}^s\psi_i}{\mathrm{d}\xi^s}\,\mathrm{d}\xi, \quad G_{ki}^{rs} = \int_0^1 \frac{\mathrm{d}^r\psi_k}{\mathrm{d}\xi^r}\frac{\mathrm{d}^s\phi_i}{\mathrm{d}\xi^s}\,\mathrm{d}\xi \quad (6)$$

and

$$m_1 = 2\frac{G_2 C^2}{2Lh_2}, \quad m_2 = 2\frac{E_1 h_1^3 + E_3 h_3^3}{24L^3}, \quad m_3 = \frac{G_2 C}{h_2},$$

equation (5) can be rewritten as

$$\frac{\partial U}{\partial a_i} = m_1\sum_{k=1}^{I} a_k E_{ki}^{11} + m_2\sum_{k=1}^{I} a_k E_{ki}^{22} - m_3\sum_{k=1}^{I} b_k G_{ki}^{01} + m_3\sum_{k=1}^{I} c_k G_{ki}^{01} \quad (i = 1, \ldots, I)$$

so that the first I row and $3I$ columns of matrix $\mathbf{A}$ will be

$$\begin{bmatrix} m_1E_{11}^{11}+m_2E_{11}^{22} & \cdots & m_1E_{I1}^{11}+m_2E_{I1}^{22} & -m_3G_{11}^{01} & \cdots & -m_3G_{I1}^{01} & m_3G_{11}^{01} & \cdots & m_3G_{I1}^{01} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ m_1E_{1I}^{11}+m_2E_{1I}^{22} & \cdots & m_1E_{II}^{11}+m_2E_{II}^{22} & -m_3G_{1I}^{01} & \cdots & -m_3G_{II}^{01} & m_3G_{1I}^{01} & \cdots & m_3G_{II}^{01} \end{bmatrix}.$$

As the $3I \times 3I$ matrices $\mathbf{A}$ and $\mathbf{B}$ depend on the values assumed by the integrals (6), λ is equal to $L\omega^2/2$, the vector $\mathbf{x} = \{a_1 \cdots a_I \; b_1 \cdots b_I \; c_1 \cdots c_I\}^{\mathrm{T}}$ contains the unknown coefficients, the solution of the eigenvector problem (4) yields $3I$ natural frequencies, in terms of eigenvalues λ, and $3I$ mode shapes, in terms of eigenvectors $\mathbf{x}$, i.e., a_i, b_i and c_i. An example of mode shapes is given in Figure 2 where transverse and longitudinal displacements of

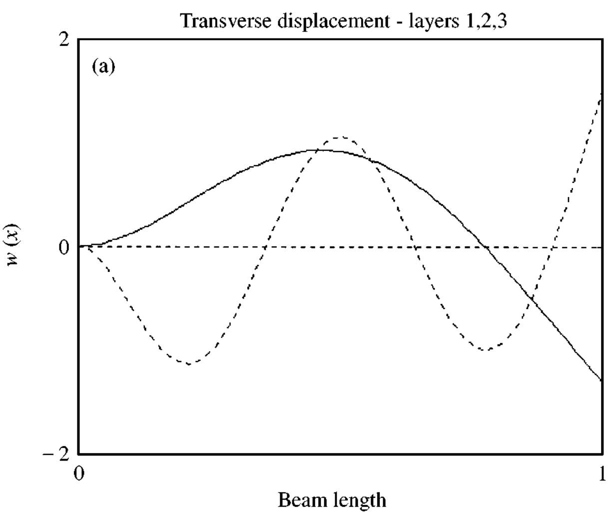

Figure 2. (a) Transverse displacement (——: mode 2, ...: mode 4) of a clamped-free beam.

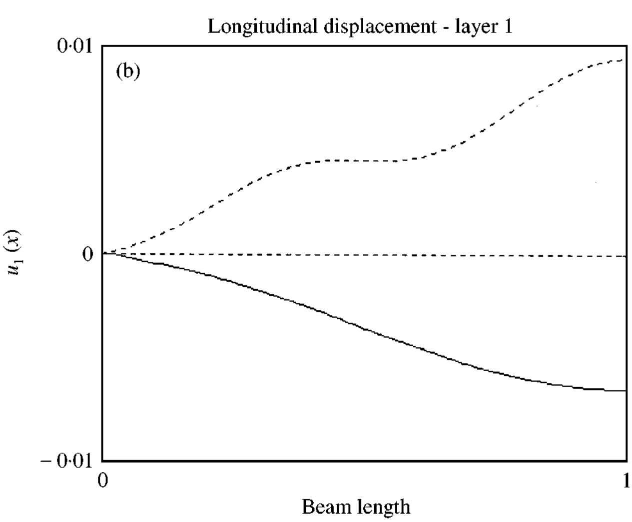

Figure 2. (b) Longitudinal displacement (——: mode 2, ...: mode 4) of layer 1 of a clamped-free beam.

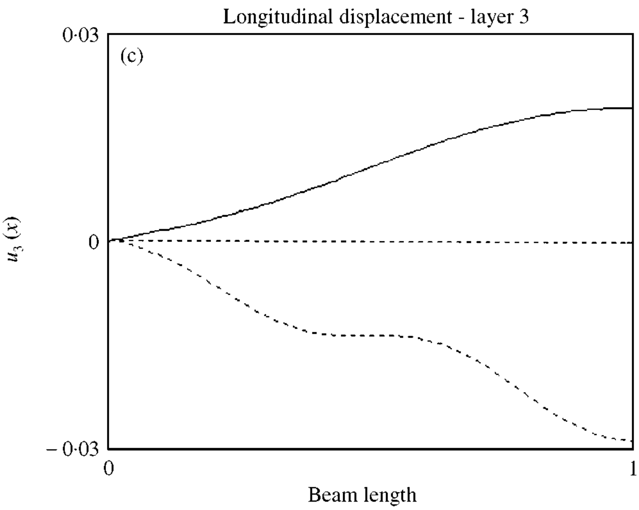

Figure 2. (c) Longitudinal displacement (——: mode 2, ...: mode 4) of layer 3 of a clamped-free beam.

a clamped—free beam are represented, as obtained with the admissible functions defined in the following paragraph.

To compute the loss factor of the beam, it is sufficient to substitute the shear modulus of the core G_2 with the complex form $G^* = G_2(1 + i\eta_v)$; the ratio of the imaginary part to the real part of each eigenvalue λ will give the loss factor of the beam at each vibration frequency. It is also possible to impose complex Young's moduli for the constraining layers, which are usually considered perfectly elastic and unable to dissipate energy.

4. ADMISSIBLE FUNCTIONS

For a correct evaluation of integrals (6), and the following calculation of modes, some care has to be taken in the choice of the admissible functions ϕ_i and ψ_i. Polynomials, as proposed in reference [14], have been used in this work, as summarized in Table 1.

Only for simply supported beams do ϕ_i and ψ_i have to differ, in order to allow in-plane displacements at both ends where transverse deflection is null. Whilst all these admissible functions satisfy the geometrical boundary conditions, the physical boundary conditions (forces and moments) are only satisfied by their sum [14].

TABLE 1

Admissible functions

Boundary conditions	ϕ_i	ψ_i
Clamped–free	ξ^{i+1}	ξ^{i+1}
Free–free	$(\xi - 1/2)^{i-1}$	$(\xi - 1/2)^{i-1}$
Clamped–clamped	$(\xi - 1/2)^{i-1}\xi^2(1 - \xi^2)$	$(\xi - 1/2)^{i-1}\xi^2(1 - \xi^2)$
Simply supported at both ends	$(\xi - 1/2)^{i-1}\xi(1 - \xi)$	$(\xi - 1/2)^{i-1}$

On the basis of these admissible polynomials, all the integrals E_{ki}^{rs}, F_{ki}^{rs} and G_{ki}^{rs} (6), appearing in the expressions for T and U, may be reduced to a couple of typical representations.

The first is valid for the functions selected for clamped–free beams and has been derived in reference [14]. It is a recurrence relationship stating that

$$G(\alpha, \beta + 1) = \frac{\alpha}{\alpha + \beta} G(\alpha, \beta) \tag{7}$$

$$\text{with } G(\alpha, \beta) = \int_0^1 \xi^{\alpha - 1}(1 - \xi)^{\beta - 1}\,\mathrm{d}\xi, \ (\alpha, \beta = 1, 2, 3, \dots) \text{ and } G(\alpha, 1) = 1/\alpha.$$

The second holds true for all the other conditions and states that, after repeated solutions by parts, the integral

$$G(\gamma, \delta) = \int_0^1 \left(\xi - \frac{1}{2}\right)^{\gamma} \xi^{\delta}\,\mathrm{d}\xi$$

assumes the value

$$G(\gamma, \delta) = \left(\frac{1}{2}\right)^{\gamma + 1} \left\{\frac{1}{\gamma + 1} + \sum_{d=1}^{\delta} \left[\frac{\prod_{i=0}^{d-1}(\delta - i)}{\prod_{i=0}^{d}(\gamma + 1 + i)} \left(-\frac{1}{2}\right)^{\delta}\right]\right\}$$
$$+ (-1)^{\delta - 1}\left(-\frac{1}{2}\right)^{\gamma + \delta + 1} \frac{\prod_{i=0}^{\delta - 1}(\delta - i)}{\prod_{i=0}^{\delta}(\gamma + 1 + i)}$$

with γ, $\delta = 1, 2, 3, \dots$.

Of course, the powers α, β, γ and δ depend on the coefficients k, i, r and s in equation (6).

The implementation of the second formula is much slower than the first, but shown to be necessary when numerical simulations were carried out, for various boundary conditions, for admissible functions in the form [14]

$$\phi_i = \xi^{m+i}(1 - \xi)^l.$$

These polynomials lead to the recursive equation (7), but were found not to converge towards satisfactory results. In fact these functions, with m and l as in Table 2, cannot even satisfy, for any i, the geometrical boundary conditions.

TABLE 2

Coefficients m and l

Index	Free	Simply supported	Clamped
$m(\xi = 0)$	−1	0	1
$l(\xi = 1)$	0	1	2

5. NUMERICAL RESULTS

The results obtained by the technique described above have been compared with some of those reported in literature.

TABLE 3

Simply supported beam—comparison with the results collected in reference [6]

	Mode number	1	2	3	4
Markus method	ω_n (rad/s)	—	—	—	—
	η_n (%)	3·42	1·07	0·50	0·28
Rayleigh–Ritz method	ω_n (rad/s)	1188	4573	10207	18094
	η_n (%)	3·43	1·07	0·50	0·28
Cum-search method	ω_n (rad/s)	1188	—	—	—
	η_n (%)	3·38	—	—	—
Conventional method	ω_n (rad/s)	1187	4573	10207	18094
	η_n (%)	3·43	1·07	0·50	0·28
Present study	ω_n (rad/s)	1204	4631	10328	18278
	η_n (%)	3·43	1·07	0·50	0·28

TABLE 4

Simply supported beam—comparison with the results collected in reference [9]

		Mode number	1	2	3	4
Sakiyama Matsuda Morita	$\eta_v = 0{\cdot}1$	η_n (%)	0·28	1·02	2·00	3·08
	$\eta_v = 0{\cdot}4$	η_n (%)	0·96	3·58	7·26	11·3
	$\eta_v = 1{\cdot}0$	η_n (%)	1·40	5·40	11·4	18·5
Present study	$\eta_v = 0{\cdot}1$	η_n (%)	0·27	1·01	2·02	3·09
	$\eta_v = 0{\cdot}4$	η_n (%)	0·96	3·57	7·21	11·2
	$\eta_v = 1{\cdot}0$	η_n (%)	1·41	5·38	11·3	18·4

5.1. SIMPLY SUPPORTED BEAMS

Some of the results collected in reference [6], namely, Problem II, and regarding simply supported conditions, are shown in Table 3.

"Markus method" indicates the values obtained by implementing the method proposed in reference [15], "Rayleigh–Ritz method" indicates the values obtained by implementing the method proposed in reference [6] (transverse displacement is modelled with a sum of sinusoidal functions), "Cum-search method" indicates the values obtained by solving a sixth order differential equation [7], "Conventional method" indicates the values obtained by implementing the method proposed in reference [16]. The correspondence is excellent for loss factor estimations, whilst some percent differences exist for frequencies. The authors' results have been computed with 20 terms, i.e., $I = 20$ in equation (3), but no sensible variation was found with 80 terms. Also for Problems I and III in reference [6] similar precision was achieved. The data shown in Table 4 are relative to the results computed by Sakiyama *et al.* [9] and, again, very good agreement is reached.

TABLE 5

Clamped–free beam—comparison with the results collected in reference [17]

		Mode number	1	2	3	4
Sixth	$\eta_v = 0{\cdot}1$	f_n (Hz)	64·08	296·6	743·7	1394
		η_n (%)	2·82	2·42	1·54	0·889
Sixth	$\eta_v = 1{\cdot}5$	f_n (Hz)	69·88	308·9	754·0	1400
		η_n (%)	23·0	29·6	21·9	13·1
JKR	$\eta_v = 0{\cdot}1$	f_n (Hz)	64·2	297·0	747·2	1408
		η_n (%)	2·82	2·53	1·53	0·88
JKR	$\eta_v = 1{\cdot}5$	f_n (Hz)	70·0	315·0	774·0	1433
		η_n (%)	22·8	29·3	21·90	13·0
Macé	$\eta_v = 0{\cdot}1$	f_n (Hz)	60·9	288·8	732·9	1381
		η_n (%)	2·65	2·17	1·33	0·74
Macé	$\eta_v = 1{\cdot}5$	f_n (Hz)	64·4	303·5	755·3	1401
		η_n (%)	26·5	32·6	19·9	11·1
Present study	$\eta_v = 0{\cdot}1$	f_n (Hz)	63·35	292·1	732·8	1373
		η_n (%)	2·86	2·43	1·54	0·89
Present study	$\eta_v = 1{\cdot}5$	f_n (Hz)	69·13	304·3	743·4	1378
		η_n (%)	23·4	29·6	21·9	13·1

TABLE 6

Clamped–free beam—comparison with the results collected in reference [9]

		Mode number	1	2	3	4
Sakiyama Matsuda Morita	$\eta_v = 0{\cdot}1$	η_n (%)	0·12	0·84	1·78	2·86
	$\eta_v = 1{\cdot}0$	η_n (%)	1·40	5·40	11·4	18·5
Sixth	$\eta_v = 0{\cdot}1$	η_n (%)	0·10	0·58	1·48	2·54
Present study	$\eta_v = 0{\cdot}1$	η_n (%)	0·21	0·79	1·62	2·60
Present study	$\eta_v = 1{\cdot}0$	η_n (%)	0·65	4·37	9·83	16·8

5.2. CLAMPED-FREE BEAMS

Table 5 summarizes the comparison of the analysis of an aluminium beam with some of the results in reference [17]; "sixth" indicates the values obtained by implementing the method proposed in reference [18], "JKR" indicates the values obtained by implementing the method proposed in reference [13], "Macé" indicates the values obtained by implementing the method proposed in reference [17].

Table 6 summarizes the comparison with some of the result in reference [9]; "Sakiyama Matsuda Morita" indicates the values obtained by implementing the method proposed in reference [9] and "sixth" indicates the values obtained by implementing the method proposed in reference [4].

For these boundary conditions also, the agreement with the values reported in literature is good, with the exception of the first mode of Table 6.

TABLE 7

Free–free beam with three damped layers—comparison with the results collected in reference [17]

	η_v (%) 65	G_2 (N/m^2) $1{\cdot}6 \times 10^7$	η_v (%) 73	G_2 (N/m^2) $2{\cdot}2 \times 10^7$	η_v (%) 78	G_2 (N/m^2) $2{\cdot}8 \times 10^7$
	η_n (%)	f_n (Hz)	η_n (%)	f_n (Hz)	η_n (%)	f_n (Hz)
Macé	8·8	193	17·5	493	22·3	900
Macé-experiment	8·9	202	18	512	24·7	941
Present study	7·4	205	15·7	516	20·8	938

Other numerical comparisons have been carried out, on the basis of the values in references [9, 10, 17], to test the capability of the proposed method to deal with free–free and clamped–clamped beams. The result are not reported here for brevity but their precision is equivalent to that of the estimations herewith collected.

The chance of using complex Young's moduli for the constraining layers has been useful to verify (Table 7) the values of an experimental test reported in reference [17]. In that test, glass-fibre reinforced plastic is used for the external faces, so that it is not possible to neglect the energy dissipation due to the deformations of layers 1 and 3. In fact, a constant loss factor (0·005) is attributed to the external faces and a frequency varying behaviour of the damping layer is assumed. The values of G_2 and η_v reported in Table 7 have been evinced from a figure in reference [17].

6. CONCLUSION

The Rayleigh–Ritz method is proposed to calculate the modal frequencies of sandwich beams with various boundary conditions. Polynomials are used as admissible functions, leading to simple expressions of the integrals appearing in the kinetic and strain energies of the system. The opportunity to determine the modal loss factors is given by assuming complex definitions of the elastic moduli of the layers of the beam. Numerical results are in good agreement with those reported in literature and computed with various different methods.

Different hypotheses for the deformations of the beam would generate different formulae for the kinetic and strain energies, but would not modify the technique described above. Future investigations will be devoted to the analysis of partially covered beams and sandwich plates.

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APPENDIX A: NOMENCLATURE

L	length of the beam
h_1, h_2, h_3	thickness of layers 1, 2, 3
E_1, E_3	Young's modulus of layers 1 and 3
G_2	shear modulus of layer 2
η_v	loss factor of layer 2
ρ_1, ρ_2, ρ_3	mass density of layers 1, 2, 3
$\rho = \rho_1 h_1 + \rho_2 h_2 + \rho_3 h_3$	
f	frequency
ω	circular frequency $(2\pi f)$
$\lambda = L\omega^2/2$	eigenvalue
u_1, u_3	longitudinal deformations of layers 1 and 3
w	transverse deformation of layers 1, 2 and 3
γ_{xz}	strain (distortion angle) in layer 2
τ_{xz}	stress in layer 2
ε_{xx}	strain (extension) in layers 1 and 3
$\varepsilon_1 = (h_3 - h_1)/4$	geometrical parameter
$\varepsilon_2 = (h_3 + h_1)/2$	geometrical parameter
$C = (h_3 + h_1)/2 + h_2$	geometrical parameter