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FREE VIBRATION OF MULTI-SPAN TIMOSHENKO BEAMS USING STATIC TIMOSHENKO BEAM FUNCTIONS

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I. INTRODUCTION

It is well known that Euler–Bernoulli beam theory neglects the effect of transverse shear strain on the bending solutions because the assumption of plane cross-sections perpendicular to the axis of the beam remaining plane and perpendicular to the axis after deformation. This simple beam theory can give excellent solutions to the vibration analysis of slender beams. However, it cannot present accurate values for the modes of thick beams or sandwich beams.

Timoshenko [1, 2] was evidently the first to study thick beams taking into consideration the influences of transverse shear deformation. In the Timoshenko beam theory, plane cross-sections remain plane but not necessarily normal to the neutral axis after deformation, thus admitting a non-zero transverse shear strain. The study on vibration of multi-span Euler–Bernoulli beams has been carried out by various methods such as graphical network method [3], finite element method [4], integral equation method [5] and U-transformation method [6, 7], etc. Huang [8] derived the exact solutions of eigenfrequencies and modes for a one-span Timoshenko beam under various boundary conditions. He and Huang [9] used the dynamic stiffness method to analyze the free vibration of continuous Timoshenko beam. Moreover, Chen and Cai [10] used the U-transformation method to analyze the static deformation of the Timoshenko beams with period supports.

In this paper, the free vibration of multi-span Timoshenko beams is studied by the Rayleigh–Ritz method. The static Timoshenko beam functions, which are composed of a set of transverse deflection functions and a set of rotational angle functions, are developed as the trial functions. These transverse deflection functions and rotation-angle functions are the complete solutions of a multi-span Timoshenko beam under a series of static sinusoidal loads distributed along the length of the beam. Each of the trial functions is a sine or cosine function plus a polynomial function of no more than the third order. A unified program can easily be provided because the change of boundary conditions of the beam and the number and locations of internal point supports only results in a corresponding change of coefficients of the low order polynomials.

2. EIGENFREQUENCY EQUATION

Consider a straight multi-span beam with the length l, the cross-sectional area A and the area moment of inertia I, as shown in Figure 1. The beam has J internal point supports,

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respectively, at x_j ($j = 1, 2, \ldots, J$), which prevent from transverse deflection of the beam but offer no resistance to normal rotation of the beam.

According to Timoshenko beam theory, two independent variables: transverse deflection y and normal rotational angle ψ due to bending are used to describe the deformation of the beam. The strain energy U and the kinetic energy T of the beam are given as

$$U = \frac{1}{2}\int_0^l \left\{ EI\left[\frac{\partial\psi(x,t)}{\partial x}\right]^2 + \kappa GA\left[\psi(x,t) - \frac{\partial y(x,t)}{\partial x}\right]^2\right\} \mathrm{d}x,$$

$$T = \frac{1}{2}\int_0^l \left\{ \rho A\left[\frac{\partial y(x,t)}{\partial t}\right]^2 + \rho I\left[\frac{\partial\psi(x,t)}{\partial t}\right]^2\right\} \mathrm{d}x, \tag{1}$$

where E is Young's modulus, G is the shear modulus, ρ is the mass per unit volume and κ is the shear correction factor. When the beam makes free vibration, the transverse deflection and the normal rotation can be written as

$$y(x,t) = Y(x)\mathrm{e}^{\mathrm{i}\omega t}, \qquad \psi(x,t) = \Psi(x)\mathrm{e}^{\mathrm{i}\omega t}, \tag{2}$$

where ω is the radian eigenfrequency and $\mathrm{i} = \sqrt{-1}$.

Introducing dimensionless co-ordinate and parameters

$$\xi = x/l, \qquad \Omega^2 = \rho A\omega^2 l^4/(EI\pi^4), \qquad \gamma = EI/(\kappa GAl^2), \qquad \eta = I/(Al^2). \tag{3}$$

The Lagrangian function L can be written as follows:

$$L = U_{max} - T_{max} = \frac{1}{2}\int_0^1 \left\{\left[\frac{\mathrm{d}\Psi(\xi)}{\mathrm{d}\xi}\right]^2 + \gamma\left[\Psi(\xi) - \frac{\mathrm{d}Y(\xi)}{\mathrm{d}\xi}\right]^2\right\}\mathrm{d}\xi$$

$$-\frac{1}{2}\Omega^2\pi^2\int_0^1 [Y^2(\xi) + \eta\Psi^2(\xi)]\,\mathrm{d}\xi. \tag{4}$$

Assuming that $Y(\xi)$ and $\Psi(\xi)$ can be written in the form of infinite series as follows:

$$Y(\xi) = \sum_{n=1}^{\infty} a_n Y_n(\xi), \qquad \Psi(\xi) = \sum_{n=1}^{\infty} b_n \Psi_n(\xi)/l, \tag{5}$$

where both a_n and b_n are unknown coefficients, $Y_n(\xi)$ and $\Psi_n(\xi)$ are the trial functions, which satisfy at least the geometric boundary conditions of the beam and if possible, all the boundary conditions.

Substituting equation (5) into equation (4), then truncating the series up to $N+1$ (for simplicity, the same number of terms are taken for $Y(\xi)$ and $\Psi(\xi)$) and applying the well-known Rayleigh–Ritz approach

$$\frac{\partial L}{\partial a_n} = 0, \quad \frac{\partial L}{\partial b_n} = 0, \qquad n = 1, 2, \ldots, N, \tag{6}$$

one has the eigenfrequency equation

$$\begin{bmatrix} K_{n\bar{n}} & K_{n\bar{m}} \\ K_{m\bar{n}} & K_{m\bar{m}} \end{bmatrix}\begin{bmatrix}\{A\} \\ \{B\}\end{bmatrix} - \Omega^2\pi^4\begin{bmatrix} M_{n\bar{n}} & M_{n\bar{m}} \\ M_{m\bar{n}} & M_{m\bar{m}} \end{bmatrix}\begin{bmatrix}\{A\} \\ \{B\}\end{bmatrix} = \begin{Bmatrix}0 \\ 0\end{Bmatrix}, \tag{7}$$

where

$$K_{n\bar{n}} = \gamma \int_0^1 \frac{dY_n(\xi)}{d\xi}\frac{dY_{\bar{n}}(\xi)}{d\xi}\,d\xi, \quad K_{n\bar{m}} = \gamma \int_0^1 \frac{dY_n(\xi)}{d\xi}\,\Psi_{\bar{m}}(\xi)\,d\xi,$$

$$K_{m\bar{n}} = \gamma \int_0^1 \Psi_m(\xi)\frac{dY_{\bar{n}}(\xi)}{d\xi}\,d\xi, \quad K_{m\bar{m}} = \int_0^1 \left[\frac{d\Psi_m(\xi)}{d\xi}\frac{d\Psi_{\bar{m}}(\xi)}{d\xi} + \gamma\Psi_m(\xi)\,\Psi_{\bar{m}}(\xi)\right]d\xi,$$

$$M_{n\bar{n}} = \int_0^1 Y_n(\xi)\,Y_{\bar{n}}(\xi)\,d\xi, \quad M_{n\bar{m}} = M_{m\bar{n}} = 0,$$

$$M_{m\bar{m}} = \eta \int_0^1 \Psi_m(\xi)\,\Psi_{\bar{m}}(\xi)\,d\xi, \quad n, \bar{n}, m, \bar{m} = 1, 2, \ldots, N,$$

$$\{A\} = [a_1, a_2, \ldots, a_N], \quad \{B\} = [b_1, b_2, \ldots, b_N]. \tag{8}$$

Eigenvalues Ω_j $(j = 1, 2, \ldots, 2N)$ and the $2N$ unknown coefficients a_n and b_n $(n = 1, 2, \ldots, N)$ corresponding to every eigenvalue can be easily given by using the standard eigenvalue program to equation (7).

3. STATIC TIMOSHENKO BEAM FUNCTIONS

Again consider the multi-span Timoshenko beam as shown in Figure 1. Now, the beam is acted by the static $q(\xi)$ along the length of the beam. The stress–displacement relations of the Timoshenko beam are given by

$$M(\xi) = -\frac{EI}{l^2}\frac{d\Psi(\xi)}{d\xi}, \quad V(\xi) = \frac{\kappa GA}{l}\left[\frac{dY(\xi)}{d\xi} - \Psi(\xi)\right], \tag{9}$$

where $M(\xi)$ is the bending moment of the beam and $V(\xi)$ is the transverse shear force. Considering the reaction forces p_j $(j = 1, 2, \ldots, J)$ of the point supports as external forces acted on beam, the equilibrium equations of stress are given by

$$\frac{dM(\xi)}{d\xi} = lV(\xi), \quad \frac{dV(\xi)}{d\xi} = -\frac{EI}{l^3}Q(\xi) - \frac{EI}{l^3}\sum_{j=1}^{J} P_j\delta(\xi - \xi_j), \tag{10}$$

where $Q(\xi) = EIq(\xi)/l^4$ is the dimensionless load, $P_j = p_j l^3/EI$ is the dimensionless reaction force of the jth point support and $\delta(\xi - \xi_j)$ is the Dirac-delta function. At each end of the beam, two boundary conditions can be presented. Taking the end $\xi = 0$ as an example, one has

$$\begin{aligned} &Y(0) = 0, \quad \Psi(0) = 0 \quad \text{for the clamped end } (C),\\ &Y(0) = 0, \quad M(0) = 0 \quad \text{for the simply supported end } (S),\\ &M(0) = 0, \quad V(0) = 0 \quad \text{for the free end } (F). \end{aligned} \tag{11}$$

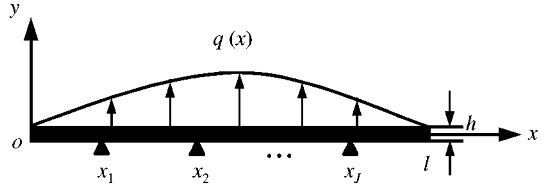

Figure 1. A Timoshenko beam with internal point supports.

Similarly, the boundary conditions at the end $\xi = 1$ can also be presented. The zero-deflection conditions of the beam at point supports can be given as

$$Y(\xi_j) = 0, \quad j = 1, 2, 3, \ldots, J. \tag{12}$$

For an arbitrarily distributed load $Q(\xi)$, one can expand it into a Fourier sinusoidal series in the interval (0, 1) as follows:

$$Q(\xi) = \sum_{n=1}^{\infty} Q_n (n\pi)^4 \sin(n\pi\xi), \tag{13}$$

where Q_n $(n = 1, 2, 3, \ldots)$ are the unknown coefficients, which can be uniquely determined by $Q(\xi)$. Correspondingly, the solutions of $Y(\xi)$ and $\Psi(\xi)$ can be written as follows:

$$Y(\xi) = \sum_{n=1}^{\infty} Q_n Y_n(\xi), \qquad \Psi(\xi) = \sum_{n=1}^{\infty} Q_n \Psi_n(\xi). \tag{14}$$

Substituting equations (10), (13) and (14) into equation (9), one has

$$\frac{d^3\Psi(\xi)}{d\xi^3} = \sum_{n=1}^{3} Q_n (n\pi)^4 \sin(n\pi\xi) + \sum_{j=1}^{J} P_j \delta(\xi - \xi_j). \tag{15}$$

Solving the differential equation gives

$$\Psi_n(\xi) = D_{n1} + D_{n2}\xi + D_{n3}\xi^2/2 + \sum_{j=1}^{J} P_{nj}(\xi - \xi_j)^2 U(\xi - \xi_j)/2 + n\pi \cos(n\pi\xi). \tag{16}$$

Substituting the above equation into equation (9), $Y_n(\xi)$ can be solved as follows:

$$Y_n(\xi) = D_{n0} + D_{n1}\xi + D_{n2}\xi^2/2 + D_{n3}(\xi^3/6 - \gamma\xi)$$
$$+ \sum_{j=1}^{J} P_{nj}[(\xi - \xi_j)^3/6 - \gamma(\xi - \xi_j)]U(\xi - \xi_j) + [\gamma(n\pi)^2 + 1]\sin(n\pi\xi). \tag{17}$$

In equations (16) and (17), both D_{ni} $(i = 0, 1, 2, 3)$ and P_{nj} $(j = 1, 2, \ldots, J)$ are unknown constants. For beams without rigid-body movements, they can be uniquely determined by the boundary conditions and zero-deflection conditions at internal point supports.
Substituting equations (16) and (17) into equations (11) and (12) gives

$$\begin{bmatrix} \bar{D}_n \\ \bar{P}_n \end{bmatrix} = -\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} \begin{bmatrix} S_{n1} \\ S_{n2} \end{bmatrix}, \tag{18}$$

where

$$\bar{D}_n = [D_{n0}, D_{n1}, D_{n2}, D_{n3}], \quad \bar{P}_n = [P_1, P_{n2}, \ldots, P_{nJ}]. \tag{19}$$

In equation (18), A_{11} is a $J \times 4$ matrix, A_{12} is a $J \times J$ matrix and S_{n1} is a $J \times 1$ matrix. They can be determined by equation (12). Without loss of generality, assuming $\xi_i < \xi_j$

if $i < j$, one has

$$A_{11} = \begin{bmatrix} 1 & \xi_1 & \xi_1^2/2 & \xi_1^3/6 - \gamma\xi_1 \\ 1 & \xi_2 & \xi_2^2/2 & \xi_2^3/6 - \gamma\xi_2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \xi_J & \xi_J^2/2 & \xi_J^3/6 - \gamma\xi_J \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ (\xi_2-\xi_1)^3/6 - \gamma(\xi_2-\xi_1) & 0 & 0 & \cdots & 0 \\ (\xi_3-\xi_1)^3/6 - \gamma(\xi_3-\xi_1) & (\xi_3-\xi_2)^3/6 - \gamma(\xi_3-\xi_2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\xi_J-\xi_1)^3/6 - \gamma(\xi_J-\xi_1) & (\xi_J-\xi_2)^3/6 - \gamma(\xi_J-\xi_2) & \cdots & (\xi_J-\xi_{J-1})^3/6 - \gamma(\xi_J-\xi_{J-1}) & 0 \end{bmatrix},$$

$$S_{n1} = \begin{bmatrix} [\gamma(n\pi)^2 + 1]\sin(n\pi\xi_1) \\ [\gamma(n\pi)^2 + 1]\sin(n\pi\xi_2) \\ \vdots \\ [\gamma(n\pi)^2 + 1]\sin(n\pi\xi_J) \end{bmatrix}. \tag{20}$$

While A_{21} is a 4×4 matrix, A_{22} is a $4\times J$ matrix and S_{n2} is a 4×1 matrix. They can be determined by equation (11). For example, if the beam is simply supported at two ends, one has

$$A_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1/2 & 1/6-\gamma \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ (1-\xi_1)^3/6 - \gamma(1-\xi_1) & (1-\xi_2)^3/6 - \gamma(1-\xi_2) & \cdots & (1-\xi_J)^3/6 - \gamma(1-\xi_J) \\ 1-\xi_1 & 1-\xi_2 & \cdots & 1-\xi_J \end{bmatrix},$$

$$S_{n2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{21}$$

and if the beam is clamped at two ends, one has

$$A_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1/2 & 1/6-\gamma \\ 0 & 1 & 1 & 1/2 \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ (1-\xi_1)^3/6 - \gamma(1-\xi_1) & (1-\xi_2)^3/6 - \gamma(1-\xi_2) & \cdots & (1-\xi_J)^3/6 - \gamma(1-\xi_J) \\ (1-\xi_1)^2/2 & (1-\xi_2)^2/2 & \cdots & (1-\xi_J)^2/2 \end{bmatrix},$$

$$S_{n2} = \begin{bmatrix} 0 \\ n\pi \\ 0 \\ n\pi(-1)^n \end{bmatrix}. \quad (22)$$

Similar formulations can be obtained for other boundary conditions. It should be pointed out that for S–F, F–S and F–F beams without internal point supports and for F–F beams only with an internal point support, the unknown coefficients in equations (16) and (17) cannot be determined by equations (11) and (12) because of the presence of rigid-body movements. In such a case, the modes of rigid-body modes should be added into the basic solutions described by equation (14). An effective approach to solving this problem has been presented [11], which is given in Table 1. Observing equations (20)–(22), one can see that the change of boundary conditions of a beam and the number and locations of internal point supports only results in a corresponding change of the coefficients of the low order polynomials. And every element in sub-matrices A_{ij} (i, $j = 1, 2$) are independent of the variable n. Therefore, only one inverse calculation is needed to determine $\bar{D}_n$ and $\bar{P}_n$ for all n. This will result in a very small computational cost. Moreover, the parameter γ in equation (17), which is referred to as shear correction coefficient of Timoshenko beams, represents the effect of shear strain of the beam on the trial functions. It is obvious that for an Euler–Bernoulli beam, γ takes zero value because the effect of shear deformation is neglected. In this case, the static Timoshenko beam functions automatically degenerate into the static Euler–Bernoulli beam functions which have been successfully applied to the vibration analysis of rectangular thin plates with internal line supports [12].

TABLE 1

The static Timoshenko beam functions (STBF) for beams with rigid-body movements

Boundary and internal point-support conditions	The first STBF	The second STBF	The third and higher STBF
F–F beam without internal point supports	$Y_1(\xi) = 1$, $\Psi_1(\xi) = 0$	$Y_2(\xi) = \xi - 1/2$, $\Psi_2(\xi) = 1$	The first and higher STBF for the S–S beam without internal point supports
S–F beam without internal point supports	$Y_1(\xi) = \xi$, $\Psi_1(\xi) = 1$	The first STBF for the S–S beam without internal point supports	The second and higher STBF for the S–S beam without internal point supports
F–S beam without internal point supports	$Y_1(\xi) = 1 - \xi$, $\Psi_1(\xi) = -1$	As above	As above
F–F beam with an internal point support at $\xi = \xi_1$	$Y_1(\xi) = \xi - \xi_1$, $\Psi_1(\xi) = 1$	The first STBF for the S–F (or F–S) beam with the internal point support	The second and higher STBF for the S–F (or F–S) beam with the internal point support

4. CONVERGENCE AND COMPARISON STUDIES

In order to demonstrate the low computational cost and high accuracy of the present method, the convergence and comparison studies are carried out. In all the following analysis, the rectangular cross-sectional beams with shear correction factor $\kappa = 5/6$ and the Poisson ratio $\mu = 0{\cdot}3$ are considered. The first six dimensionless eigenfrequencies of simply–simply supported and clamped–clamped beams, respectively, with two, three and four unequal spans are given in Table 2. The number of terms of the static Timoshenko beam functions steadily increases from 6 to 10. One can see that the convergence is very rapid. In general, 10 terms of the static Timoshenko beam functions are enough to give satisfactory results.

The comparison study has been given in Table 3 for the first five dimensionless eigenfrequencies of Timoshenko beams with equal spans and a thickness ratio $h/l = 0{\cdot}15$ by

TABLE 2

The convergence study on the first six dimensionless eigenfrequencies of S–S and C–C Timoshenko beams with unequal spans for $h/l = 0{\cdot}1$

B C	N	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
				$J = 1$; $\xi_1 = 0{\cdot}4$			
S–S	6	3·1751	6·7840	10·5296	18·7767	20·6422	33·2617
	7	3·1751	6·7840	10·5294	18·7767	20·6057	29·7518
	8	3·1751	6·7840	10·5294	18·7767	20·5978	29·7017
	9	3·1751	6·7840	10·5293	18·7767	20·5910	29·6710
	10	3·1751	6·7840	10·5293	18·7767	20·5910	29·6710
C–C	6	4·5490	9·0570	12·1898	20·5255	22·4319	34·0617
	7	4·5490	9·0557	12·1897	20·4768	22·4035	30·8942
	8	4·5490	9·0557	12·1895	20·4754	22·3499	30·8803
	9	4·5490	9·0557	12·1891	20·4738	22·3337	30·7818
	10	4·5490	9·0557	12·1890	20·4737	22·3316	30·7797
				$J = 2$; $\xi_1 = 0{\cdot}3$; $\xi_2 = 0{\cdot}6$			
S–S	6	6·6148	10·2565	13·6864	20·0138	32·6463	33·4751
	7	6·6147	10·2565	13·6836	19·9896	29·6832	33·1374
	8	6·6147	10·2565	13·6801	19·9803	29·5714	32·1933
	9	6·6147	10·2564	13·6801	19·9781	29·5678	31·8339
	10	6·6147	10·2564	13·6801	19·9781	29·5678	31·8339
C–C	6	8·7265	12·1226	15·3668	21·9658	33·4839	34·5207
	7	8·7259	12·1221	15·3473	21·9650	30·8110	33·9586
	8	8·7258	12·1221	15·3368	21·9223	30·8080	33·1165
	9	8·7258	12·1219	15·3350	21·9116	30·6570	32·8440
	10	8·7258	12·1218	15·3348	21·9103	30·6561	32·8373
				$J = 3$; $\xi_1 = 0{\cdot}2$; $\xi_2 = 0{\cdot}5$; $\xi_3 = 0{\cdot}7$			
S–S	6	10·9227	12·3318	20·9822	23·3999	32·3176	33·2982
	7	10·9226	12·3261	20·8911	23·0924	31·3022	32·6694
	8	10·9224	12·3261	20·8778	23·0243	30·4147	32·2050
	9	10·9223	12·3261	20·8737	23·0013	30·3954	31·6505
	10	10·9223	12·3261	20·8737	23·0013	30·3954	31·6505
C–C	6	12·3312	14·1306	23·1402	25·4437	32·7319	34·8853
	7	12·3267	14·1162	23·0942	25·0967	32·7166	33·1707
	8	12·3261	14·1135	23·0329	25·0665	31·8201	32·8065
	9	12·3261	14·1127	23·0017	25·0524	31·6302	32·5107
	10	12·3261	14·1125	23·0011	25·0492	31·6271	32·5061

TABLE 3

The comparison study of the first five dimensionless eigenfrequencies of S–S; C–S and C–C Timoshenko beams with unequal spans for $h/l = 0{\cdot}15$

B C	Methods	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
				$J = 0$		
S–S	Present	0·9644	3·5194	7·0424	11·0702	15·3444
	DS†	0·9644	3·5194	7·0424	11·0702	15·3444
C–S	Present	1·4386	4·1633	7·6626	11·5814	15·7293
	DS	1·4386	4·1632	7·6625	11·5807	15·7273
C–C	Present	1·9814	4·7860	8·2462	12·0604	16·0889
	DS	1·9814	4·7859	8·2461	12·0580	16·0887
				$J = 1;\ \xi_1 = 1/2$		
S–S	Present	3·5194	4·7860	11·0702	12·0605	19·7316
	DS	3·5194	4·7859	11·0702	12·0580	19·7316
C–S	Present	3·9019	5·6874	11·3900	12·6552	19·8936
	DS	3·9019	5·6873	11·3898	12·6489	19·8661
C–C	Present	4·7860	6·0915	12·0604	12·8907	20·2670
	DS	4·7859	6·0915	12·0580	12·8821	20·2510
				$J = 2;\ \xi_1 = 1/3;\ \xi_2 = 2/3$		
S–S	Present	7·0424	7·9126	9·6619	19·7316	20·1726
	DS	7·0424	7·9123	9·6600	19·7316	20·1532
C–S	Present	7·2275	8·7777	10·3371	19·8577	20·5234
	DS	7·2774	8·7765	10·3358	19·8554	20·4854
C–C	Present	7·9127	9·6617	10·5932	20·1646	20·8210
	DS	7·9123	9·6600	10·5929	20·1532	20·7542
				$J = 3;\ \xi_1 = 1/4;\ \xi_2 = 1/2;\ \xi_3 = 3/4$		
S–S	Present	11·0702	11·5802	12·8884	14·3774	28·6069
	DS	11·0702	11·5784	12·8821	14·3664	28·6069
C–S	Present	11·2013	12·1646	13·6656	14·8869	28·6582
	DS	11·2010	12·1606	13·6550	14·8808	28·6511
C–C	Present	11·5799	12·8903	14·3778	15·0722	28·7874
	DS	11·5784	12·8821	14·3664	15·0708	28·7290
				$J = 4;\ \xi_1 = 1/5;\ \xi_2 = 2/5;\ \xi_3 = 3/5;\ \xi_4 = 4/5$		
S–S	Present	15·3444	15·6186	16·4172	17·6368	18·9395
	DS	15·3444	15·6157	16·4016	17·5890	18·8586
C–S	Present	15·4133	15·9562	16·9863	18·3124	19·3655
	DS	15·4126	15·9486	16·9578	18·2482	19·3089
C–C	Present	15·6187	16·4172	17·6338	18·9178	19·5235
	DS	15·6157	16·4016	17·5890	18·8586	19·4781

†Dynamic stiffness method.

using the present method and the dynamic stiffness (DS) method [9] respectively. Three kinds of boundary conditions: simply–simply supported; simply supported–clamped and clamped–clamped, are considered. The number of point supports is steadily increased from zero to four. The accuracy of eigenfrequencies given by dynamic stiffness method is 10^{-5} by using the method of successive bisection to the dynamic stiffness matrix. Excellent agreement has been observed for all cases, which shows that the present method has very high accuracy.

TABLE 4

The dimensionless fundamental eigenfrequencies of Timoshenko beams with equal spans for different thickness ratios

h/l	C–C	C–S	S–S	C–F	S–F	F–F
			$J=0$			
0·001	2·2669	1·5622	1·0000	0·3562	0·0	0·0
0·01	2·2653	1·5616	0·9998	0·3562	0·0	0·0
0·1	2·1249	1·5032	0·9836	0·3534	0·0	0·0
			$J=1;\ \xi_1=1/2$			
0·001	6·2487	4·6664	4·0000	1·0000	0·9191	0·0
0·01	6·2387	4·6618	3·9973	0·9996	0·9189	0·0
0·1	5·4457	4·2672	3·7586	0·9606	0·8969	0·0
			$J=2;\ \xi_1=1/3;\ \xi_2=2/3$			
0·001	11·5333	9·6942	8·9999	2·1668	2·1527	1·8176
0·01	11·5028	9·6761	8·9863	2·1651	2·1511	1·8171
0·1	9·3696	8·3170	7·9187	2·0165	2·0114	1·7642
			$J=3;\ \xi_1=1/4;\ \xi_2=1/2;\ \xi_3=3/4$			
0·001	18·6651	16·7048	15·9996	3·8409	3·8390	3·6769
0·01	18·5914	16·6540	15·9568	3·8357	3·8338	3·6732
0·1	14·1279	13·3232	13·0366	3·4190	3·4189	3·3567
			$J=4;\ \xi_1=1/5;\ \xi_4=2/5;\ \xi_3=3/5;\ \xi_4=4/5$			
0·001	27·7346	25·7093	24·9989	6·0023	6·0020	5·9322
0·01	27·5816	25·5927	24·8949	5·9897	5·9895	5·9211
0·1	19·5452	18·9738	18·7767	5·0740	5·0740	5·0677
			$J=5;\ \xi_1=1/6;\ \xi_2=1/3;\ \xi_3=1/2;\ \xi_4=2/3;\ \xi_5=5/6$			
0·001	38·7747	36·7112	35·9978	8·6459	8·6459	8·6173
0·01	38·4888	36·4781	35·7829	8·6201	8·6201	8·5925
0·1	25·4286	25·0398	24·9080	6·9190	6·9190	6·9168

5. NUMERICAL EXAMPLES

The effect of thickness ratio on dimensionless fundamental eigenfrequencies of beams with equal spans from one up to six is given in Table 4. Three different thickness ratios: $h/l = 0{\cdot}001$, 0·01, 0·1 and six kinds of boundary conditions are considered. It is shown that eigenfrequencies decrease with the increase of thickness ratio, however, increase with the increase of span number and boundary constraints. Moreover, one can find that the effect of thickness ratio on eigenfrequencies increases with the increase of the span number.

6. CONCLUDING REMARKS

The free vibration of multi-span Timoshenko beams is studied by the Rayleigh–Ritz method. The static Timoshenko beam functions are developed as the trial functions in the present analysis, which are the complete solutions of transverse deflections and rotational angles of the beam when a series of static sinusoidal loads acts on the beam. The high accuracy and low computational cost have been confirmed by convergence and comparison studies.

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