THE GEOMETRICAL MODE AND FREQUENCY ANALYSES OF A VIBRATING SYSTEM WITH PLANES OF SYMMETRY

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The geometrical mode and frequency analyses of a vibrating system can be performed via the theory of screw. From a screw theoretical standpoint, a vibration mode can be geometrically interpreted as a pure rotation about the center of vibration in a plane and as a twisting motion on a screw in a three-dimensional space. This paper presents a method to diagonalize a spatial stiffness matrix by use of a parallel axis congruence transformation when the stiffness matrix satisfies some conditions. It also describes that the diagonalized stiffness matrix can have the planes of symmetry depending on the location of the center of elasticity. For a system with the planes of symmetry, the vibration modes can be expressed by the axes of vibration. Analytical solutions for the axes of vibrations have been derived. The set of axes of vibrations yields the modal matrix and the response at the mass center is expressed by the reciprocal product between the axes of vibration and the applied wrench. A numerical example of an application to the vibrational analysis of an optical disc drive has been presented.

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1. INTRODUCTION

The vibration modes and frequency responses of a vibrating system can be explained more clearly via the screw theory, which deals with the dynamics of a rigid body based on the geometrical theories. From the geometrical point of view, the eigenvectors in the planar vibration analysis can be interpreted as pure rotations about the center of vibration or pure translations. In a three-dimensional space, they represent repetitive twisting motions on the axes of vibration.

Based on the screw theory, Griffs and Duffy [1] derived the mapping of stiffness represented in a correlation form of screws with a symmetric positive-definite 6×6 matrix and presented the general model of a spatial stiffness by line springs. It has been known that a general stiffness matrix cannot be diagonalized by a rigid-body transformation [2]. Lipkin and Patterson [3] developed a geometrical decomposition method diagonalizing the stiffness matrix by a congruence transformation. The decomposition method was formulated through two singular eigenvalue problems yielding eigenwrenches and eigentwists. Ciblak [4] defined the centers of elasticity, stiffness, and compliance representing the geometrical properties of an elastically suspended system. Blanchet [5]

derived a cubic equation for three vibration centers in a plane where those three centers coalesce.

In this paper, the premises of diagonalize a spatial stiffness matrix by use of a parallel axis congruence transformation and their geometrical meanings are presented. It is also shown that the planes of symmetry of decouple the vibration modes of a system depend on the location of the center of elasticity. For a system with the planes of symmetry, the analytical solutions for the axes of vibrations have been derived. In addition, the response of the system to an applied force has been expressed in terms of the reciprocal product of the axes of vibrations and the applied wrench, which, in turn, provides one with more clear geometrical interpretation of the frequency response of the forced vibration.

As a numerical example, an optical disk drive with spring suspensions has been modelled and its vibration modes and frequency responses have been analyzed so as to illustrate the applicability of the theoretical development presented in this paper.

2. DIAGONALIZATION OF A STIFFNESS MATRIX

2.1. REPRESENTATION OF A STIFFNESS MATRIX

For an elastically suspended rigid body at the unloaded position in a three-dimensional space, the mapping of stiffness can be expressed by a general correlation of screws [1] as follows:

$$\hat{\mathbf{w}} = \mathbf{K}_G \hat{\mathbf{X}}, \tag{1}$$

where the wrench and infinitesimal twist are expressed, respectively, in Plücker's ray co-ordinates $\hat{\mathbf{w}} = [\mathbf{f}^T \, \mathbf{m}^T]^T$ and $\hat{\mathbf{X}} = [\boldsymbol{\delta}^T \, \boldsymbol{\phi}^T]^T$ in the axis co-ordinates [6]. Here, $\mathbf{f}$ and $\mathbf{m}$ are used to denote the force and moment, and $\boldsymbol{\delta}$ and $\boldsymbol{\phi}$ are used to denote the small translational and angular displacements respectively. $\mathbf{K}_G$ can be expressed in the matrix form with four 3×3 submatrices by

$$\mathbf{K}_G = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{D} \end{bmatrix}. \tag{2}$$

Griffis and Duffy [1] derived the stiffness matrix for a rigid body supported by n line springs only acting along their axial directions. The stiffness matrix can be expressed by

$$\mathbf{K}_G = \mathbf{j} \, \mathbf{K}_c \, \mathbf{j}^T, \tag{3}$$

where the $6 \times n$ matrix $\mathbf{j}$ is called the Jacobian matrix and expressed by $\mathbf{j} = [\hat{\mathbf{s}}_1, \dots, \hat{\mathbf{s}}_n]$, and $\hat{\mathbf{s}}_i$ denotes the line-bound vector of the ith line spring. $\mathbf{K}_c$ is the diagonal matrix whose diagonal elements are the spring constants k_i. The line-bound vector can be defined by the unit direction vector of the ith line, $\mathbf{s}_i$, and its moment vector about the mass center G with the position vector $\mathbf{r}_i$ to the line:

$$\hat{\mathbf{s}}_i = \begin{bmatrix} \mathbf{s}_i \\ \mathbf{s}_{oi} \end{bmatrix} = \begin{bmatrix} \mathbf{s}_i \\ \mathbf{r}_i \times \mathbf{s}_i \end{bmatrix}. \tag{4}$$

Now, substituting equation (4) into equation (3) yields

$$\mathbf{K}_G = \sum_{i=1}^{n} k_i \begin{bmatrix} \mathbf{s}_i \, \mathbf{s}_i^T & -\mathbf{s}_i \, \mathbf{s}_i^T \, \mathbf{r}_i \times \\ \mathbf{r}_i \times \mathbf{s}_i \, \mathbf{s}_i^T & -\mathbf{r}_i \times \mathbf{s}_i \, \mathbf{s}_i^T \, \mathbf{r}_i \times \end{bmatrix}, \tag{5}$$

where $\times$ represents the cross product operator.

2.2. FREE VECTOR DECOMPOSITION

Lipkin and Patterson [3] developed a congruence transformation in formed from eigenwrenches and eigentwists that diagonalizes a stiffness matrix, which was given by

$$\mathbf{K}_G = \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{M} & \mathbf{\Phi} \end{bmatrix} \begin{bmatrix} \mathbf{K}_f & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_\phi \end{bmatrix} \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{M} & \mathbf{\Phi} \end{bmatrix}^T = \begin{bmatrix} \mathbf{F}\mathbf{K}_f\,\mathbf{F}^T & \mathbf{F}\mathbf{K}_f\,\mathbf{M}^T \\ \mathbf{M}\mathbf{K}_f\,\mathbf{F}^T & \mathbf{M}\mathbf{K}_f\,\mathbf{M}^T + \mathbf{\Phi}\mathbf{K}_\phi\,\mathbf{\Phi}^T \end{bmatrix}, \tag{6}$$

where $\mathbf{K}_f = diag\,(k_{f1},\ k_{f2},\ k_{f3})$ and $\mathbf{K}_\phi = diag\,(k_{\phi1}, k_{\phi2}, k_{\phi3})$. $\mathbf{F} = [\mathbf{f}_1\ \mathbf{f}_2\ \mathbf{f}_3]$ and $\mathbf{\Phi} = [\mathbf{\phi}_1\ \mathbf{\phi}_2\ \mathbf{\phi}_3]$ are the 3×3 submatrices. The vectors $\mathbf{f}_i$ and $\mathbf{\phi}_i$ are the eigenvectors obtained from the definitions of the eigenwrench and eigentwist as follows:

$$k_f\,\mathbf{f} = \mathbf{A}\,\mathbf{f}, \qquad k_\phi\mathbf{\phi} = (\mathbf{D} - \mathbf{B}^T\,\mathbf{A}^{-1}\mathbf{B})\mathbf{\phi}. \tag{7, 8}$$

In equations (7) and (8), the eigenvalues k_f and k_ϕ are the linear and angular stiffnesses respectively. In the 3×3 submatrix $\mathbf{M} = [\mathbf{m}_1\ \mathbf{m}_2\ \mathbf{m}_3]$, $\mathbf{m}_i$ is obtained from the relation $\mathbf{m}_i = \mathbf{B}^T\,\mathbf{f}_i\,k_{fi}^{-1}$. $\mathbf{F}$ and $\mathbf{\Phi}$ are the orthogonal matrices. The full details of the derivation can be found in Lipkin and Patterson [3].

2.3. DIAGONALIZATION OF A STIFFNESS MATRIX BY A PARALLEL AXIS CONGRUENCE TRANSFORMATION

This section establishes the geometric conditions that can be used to diagonalize a stiffness matrix by the use of a parallel axis congruence transformation. The first condition is that the directions of the eigenwrenches $\mathbf{F}$ should be parallel to the axes of the co-ordinate system, i.e., $\mathbf{F}$ is the identity matrix. It can be easily shown that the submatrix $\mathbf{A}$ in equation (7) becomes a diagonal matrix when $\mathbf{F}$ is the identity matrix.

The second condition is that $\mathbf{A}^{-1}\mathbf{B}$ should be skew symmetric. In this case, the following can be obtained from equation (6):

$$\mathbf{A}^{-1}\mathbf{B} + \mathbf{B}^T\mathbf{A}^{-1} = \mathbf{F}\mathbf{K}_f^{-1}\mathbf{F}^T\mathbf{F}\mathbf{K}_f\mathbf{M}^T - \mathbf{M}\mathbf{K}_f\mathbf{F}^T\mathbf{F}\mathbf{K}_f^{-1}\mathbf{F}^T = \mathbf{F}\mathbf{M}^T + \mathbf{M}\mathbf{F}^T = 0. \tag{9}$$

From this, it can be seen that when $\mathbf{A}^{-1}\mathbf{B}$ is skew symmetric, $\mathbf{M}$ is also skew symmetric because $\mathbf{F}$ is the identity matrix. The geometrical relation of $\mathbf{M}$ to the eigenwrenches is given by

$$\mathbf{m}_i = p_i\,\mathbf{f}_i + \mathbf{r}_i \times \mathbf{f}_i, \tag{10}$$

where p_i is the pitch of the ith eigenwrench and $\mathbf{r}_i$ is the perpendicular vector from G to the ith eigenwrench. Since the eigenwrenches are parallel to the axes of the co-ordinate system, $\mathbf{r}_i$ can be expressed by

$$[\mathbf{r}_1\ \mathbf{r}_2\ \mathbf{r}_3] = \begin{bmatrix} 0 & r_{x2} & r_{x3} \\ r_{y1} & 0 & r_{y3} \\ r_{z1} & r_{z2} & 0 \end{bmatrix}. \tag{11}$$

Substituting equation (11) into equation (10) yields

$$\mathbf{M} = \begin{bmatrix} p_1 & -r_{z2} & r_{y3} \\ r_{z1} & p_2 & -r_{x3} \\ -r_{y1} & r_{x2} & p_3 \end{bmatrix}. \tag{12}$$

Since $\mathbf{M}$ is skew symmetric, all the eigenwrenches must intersect at a common point by equation (12). Geometrically, this implies that the three centers, i.e., the centers of elasticity, stiffness, and compliance, coalesce.

Ciblak [4] defined the center of elasticity at which $\mathbf{A}^{-1}\mathbf{B}_E$ is symmetric and derived the expression for the vector $\mathbf{h}$ from the mass center G to the center of elasticity E as follows:

$$\mathbf{h}\times = -\frac{1}{2}[\mathbf{A}^{-1}\mathbf{B} - \mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1}]. \tag{13}$$

On the assumption that $\mathbf{A}^{-1}\mathbf{B}$ is skew symmetric, equation (13) can be reduced to

$$\mathbf{h}\times = -\mathbf{A}^{-1}\mathbf{B}. \tag{14}$$

Using equation (14), the parallel axis congruence transformation can be expressed by

$$\mathbf{K}_E = \mathbf{E}_h^{\mathrm{T}}\mathbf{K}_G\mathbf{E}_h = \begin{bmatrix} \mathbf{A} & \mathbf{B}_E \\ \mathbf{B}_E^{\mathrm{T}} & \mathbf{D}_E \end{bmatrix}, \tag{15}$$

where $\mathbf{E}_h = \begin{bmatrix} \mathbf{I}_3 & \mathbf{h}\times \\ \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix}$ in which $\mathbf{I}_3$ and $\mathbf{0}_3$ are, respectively, 3×3 identity and zero matrices and

$$\mathbf{B}_E = \mathbf{A}\mathbf{h}\times + \mathbf{B}, \tag{16}$$

$$\mathbf{D}_E = (-\mathbf{h}\times\mathbf{A} + \mathbf{B}^{\mathrm{T}})\mathbf{h}\times - \mathbf{h}\times\mathbf{B} + \mathbf{D}. \tag{17}$$

Substituting equation (14) into equations (16) and (17) yields

$$\mathbf{B}_E = \mathbf{0}_3, \qquad \mathbf{D}_E = -\mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{B} + \mathbf{D}. \tag{18, 19}$$

If $\mathbf{\Phi} = \mathbf{I}_3$, then comparing equation (19) with equation (8) gives

$$\mathbf{D}_E = \mathbf{K}_\phi. \tag{20}$$

In summary, when the conditions that $\mathbf{F} = \mathbf{\Phi} = \mathbf{I}_3$ and $\mathbf{A}^{-1}\mathbf{B}$ is skew symmetric are satisfied, it has been shown that the spatial stiffness matrix $\mathbf{K}_G$ can be diagonalized by a parallel axis congruence transformation to the center of elasticity E. Here, the elements of the diagonalized stiffness matrix may be expressed as follows:

$$k_{f1} = k_x, \quad k_{f2} = k_y, \quad k_{f3} = k_{z2}, \quad k_{\phi1} = k_\alpha, \quad k_{\phi2} = k_\beta, \quad k_{\phi3} = k_\gamma, \tag{21}$$

where the subscripts x, y, z denote the translational directions and α, β, γ denote the rotational directions respectively.

3. CONDITIONS OF THE PLANES OF SYMMETRY

For an elastically supported single rigid body with the 6×6 inertia matrix, $\mathbf{M}_G$, at the mass center G, the equation of motion for free vibration can be expressed at the point G by

$$\mathbf{M}_G\ddot{\mathbf{X}}_G + \mathbf{K}_G\mathbf{X}_G = \mathbf{0}. \tag{22}$$

The infinitesimal twist representing a general displacement in a three-dimensional space can be expressed by

$$\mathbf{X}_G = \hat{\mathbf{X}}\, e^{j\Omega t}, \tag{23}$$

where Ω denotes the natural frequency of the system and $\hat{\mathbf{X}}$ may be given by

$$\hat{\mathbf{X}} = [\delta_x\ \delta_y\ \delta_z\ \phi_x\ \phi_y\ \phi_z]^{\mathrm{T}}. \tag{24}$$

When the axes of the co-ordinate frames are chosen to be coincident with the principal axes of inertia, the inertia matrix can be expressed by the following diagonal matrix:

$$\mathbf{M}_G = diag\,(m\ m\ m\ I_x\ I_y\ I_z). \tag{25}$$

Now, substituting equation (23) into equation (22) yields

$$(\mathbf{K}_G - \Omega^2 \mathbf{M}_G)\,\hat{\mathbf{X}} = \mathbf{0}. \tag{26}$$

If the stiffness matrix $\mathbf{K}_G$ can be diagonalized by the use of equation (15), using the vector from E to G, $\mathbf{h}'$, i.e., $\mathbf{h}' = -\mathbf{h}$, equation (15) can be rewritten in a form as

$$\mathbf{K}_G = \mathbf{E}_{h'}^{\mathrm{T}}\, \mathbf{K}_E\, \mathbf{E}_{h'}. \tag{27}$$

Substituting equation (27) into equation (26) and using equation (25) yields

$$\begin{bmatrix} \bar{\mathbf{A}} & \bar{\mathbf{B}} \\ \bar{\mathbf{B}}^{\mathrm{T}} & \bar{\mathbf{D}} \end{bmatrix} \hat{\mathbf{X}} = \mathbf{0}, \tag{28}$$

where

$$\bar{\mathbf{A}} = \begin{bmatrix} k_x - \Omega^2 m & 0 & 0 \\ 0 & k_y - \Omega^2 m & 0 \\ 0 & 0 & k_z - \Omega^2 m \end{bmatrix}, \quad \bar{\mathbf{B}} = \begin{bmatrix} 0 & h_z k_x & -h_y k_x \\ -h_z k_y & 0 & h_x k_y \\ h_y k_z & -h_x k_z & 0 \end{bmatrix},$$

$$\bar{\mathbf{D}} = \begin{bmatrix} k_\alpha + h_z^2 k_y + h_y^2 k_z - I_x \Omega^2 & -h_x h_y k_z & -h_x h_z k_y \\ -h_x h_y k_z & k_\beta + h_z^2 k_x + h_x^2 k_z - I_y \Omega^2 & -h_y h_z k_x \\ -h_x h_z k_y & -h_y h_z k_x & k_\gamma + h_y^2 k_x + h_x^2 k_y - I_z \Omega^2 \end{bmatrix}. \tag{29}$$

Depending on the arrangement of the stiffness of a system, the system can have the planes(s) of symmetry. Harris [7] defined the plane of symmetry, but no detailed conditions for a system to have the planes of symmetry have been given.

Here, the close observation of equation (29) reveals the fact that the vector $\mathbf{h}$ provides the conditions of the planes of symmetry that make the vibration modes of a system decoupled. That is, if $\mathbf{h}$ has a zero element, then vibration modes can be decoupled into two separate groups. For example, if the z element of $\mathbf{h}$ is zero, i.e., $h_z = 0$, then x–y plane becomes the plane of symmetry and the vibrations in the co-ordinates δ_x, δ_y, ϕ_z are coupled but independent of any vibrations in the other co-ordinates and also the vibrations in the co-ordinates δ_z, ϕ_x, ϕ_y are coupled but independent of the other co-ordinates. For the case

that $h_z = 0$, the axes of vibration can be obtained by solving equation (28), which are expressed in the following forms of line-bound vectors:

$$\mathbf{X}^{in} = [\delta_x \ \delta_y \ 0 \ 0 \ 0 \ 1]^{\mathrm{T}}, \tag{30}$$

$$\mathbf{X}^{out} = [0 \ 0 \ \delta_z \ \phi_x \ \phi_y \ 0]^{\mathrm{T}}. \tag{31}$$

Here, $\mathbf{X}^{in}$ are the perpendicular axes to the x–y plane and $\mathbf{X}^{out}$ are lying in the x–y plane. The terms $\mathbf{X}^{in}$ and $\mathbf{X}^{out}$ are called, respectively, in-plane and out-of-plane modes.

In summary, it can be said that, when the stiffness matrix is diagonalized by a parallel axis transformation using the vector $\mathbf{h}$ (from G to E) with a zero element, there exists one plane of symmetry to split the solution into two groups: in-plane and out-of-plane vibration modes.

4. AXES OF VIBRATIONS FOR THE PLANES OF SYMMETRY

The solutions of the axes of vibrations can be obtained from the parallel axis congruence transformation of the stiffness and inertia matrix to a point on the axis of vibration. Rewriting equation (26) gives

$$(\mathbf{E}_v^{\mathrm{T}} \mathbf{K}_E \mathbf{E}_{v'} - \Omega^2 \mathbf{E}_v^{\mathrm{T}} \mathbf{M}_G \mathbf{E}_v) \mathbf{X}_v = \mathbf{0}. \tag{32}$$

where $\mathbf{v}'$ denotes the vector from the point E to any point on the axis of vibration and $\mathbf{v}$ is the vector from the point G to any point on the axis of vibration, and the transformations are given by

$$\mathbf{E}_v = \begin{bmatrix} \mathbf{I}_3 & \mathbf{v} \times \\ \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix}, \qquad \mathbf{E}_{v'} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{v}' \times \\ \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix}. \tag{33}$$

Thus, $\mathbf{v} = \mathbf{v}' + \mathbf{h}$.

In the following two subsections, in order to derive the analytical solution for the axis of vibration, only the case that the x–y plane is the plane of symmetry is considered since, for the other cases, the solutions can be obtained in the same manner. In this case, the axes of vibrations for the in-plane modes intersect the x–y plane perpendicularly and those for the out-of-plane modes lie in the plane. Thus, the vectors $\mathbf{v}'$ and $\mathbf{v}$ can be lain in the x–y plane by choosing the endpoints of these vectors as the intersecting points for the in-plane modes and as any point for the out-of-plane modes, respectively. Expanding equation (32) in terms of the vector $\mathbf{v}$ gives two cubic equations, one for the in-plane and the other for the out-of-plane modes (see Figures 1 and 2).

4.1. FOR IN-PLANE MODES WITH ONE PLANE OF SYMMETRY

The locations of the axes of vibrations of a system with a plane of symmetry can be expressed by the vector $\mathbf{v}$. When the axis of vibration represented by equation (30) is rewritten at the intersecting point between the axis of vibration and the x–y plane, it can be expressed in the form by

$$\mathbf{X}_V^{in} = [0 \ 0 \ 0 \ 0 \ 0 \ 1]^{\mathrm{T}}. \tag{34}$$
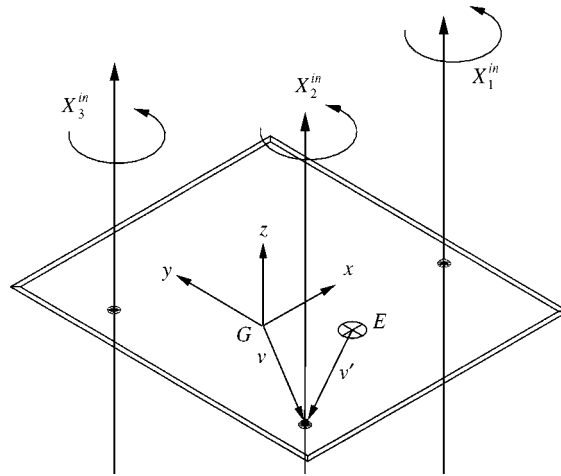
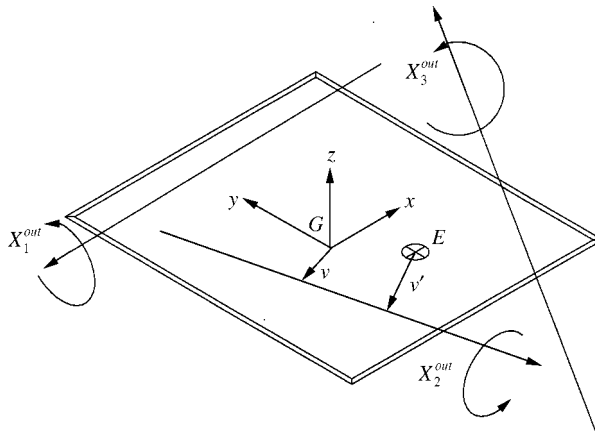Figure 1. The axes of vibration for in-plane modes.

Figure 2. The axes of vibration for out-of-plane modes.

To obtain the solution, equation (34) is substituted into equation (32) to yield three equations in three unknowns Ω, v_x, and v_y:

$$\Omega^2 = \frac{(v_y - h_y)k_x}{v_y m}, \quad \Omega^2 = \frac{(v_x - h_x)k_y}{v_x m}, \quad \Omega^2 = \frac{k_y + k_x(v_y - h_y)^2 + k_y(v_x - h_x)^2}{I_z + m(v_x^2 + v_y^2)}. \quad (35)$$

Eliminating Ω between the first and second equations in equation (35) gives the relation v_x and v_y,

$$v_y = \frac{v_x h_y k_x}{h_x k_y + v_x(k_x - k_y)}. \quad (36)$$

Again, eliminating Ω from the first and the third equations in equations (35) and using equation (36) gives the following cubic equation in terms of v_x:

$$[k_y h_x m(k_y - k_x)]v_x^3 + [(k_x - k_y)(k_y h_x^2 m + k_y \, m - k_y I_z) - m k_y(k_y h_x^2 + k_x h_y^2)]v_x^2$$

$$+ h_x k_y [I_z(k_x - 2k_y) + (k_y + h_y^2 k_x + h_x^2 k_y)m] v_x + h_x^2 I_z k_y^2 = 0. \quad (37)$$

Similarly, eliminating Ω and v_x from these three equations gives another cubic equation in terms of v_y.

From equation (37), when $h_x = 0$ and/or $k_x = k_y$, the coefficient of v_x^3 vanishes and this corresponds to one or two pure translational modes. Recalling that we are now consider the case that $h_z = 0$ (the plane of symmetry is the x–y plane), the condition $h_x = 0$ implies that there exists another plane of symmetry, which becomes the case of two planes of symmetry. When $k_x = k_y$, the axis of vibration corresponding to this translational mode lies on the line joining the points G and E.

4.2. FOR OUT-OF-PLANE MODES WITH ONE PLANE OF SYMMETRY

In this case, the vector $\mathbf{v}$ can be chosen, without loss of generality, as

$$\mathbf{v} = [v_x \ 0 \ 0]^{\mathrm{T}} \tag{38}$$

When the axis of vibration represented by equation (31) is rewritten at any point on the axis of vibration, it can be expressed in the form by

$$\mathbf{X}_V^{out} = [0 \ 0 \ 0 \ \phi_x \ \phi_y \ 0]^{\mathrm{T}}. \tag{39}$$

Substituting equations (38) and (39) into equation (32) yields the following three equations in three unknowns Ω, ϕ_x, and ϕ_y:

$$\frac{\phi_y}{\phi_x} = \frac{h_y k_z}{(h_x - v_x)\,k_z + v_x m \Omega^2}, \tag{40}$$

$$\frac{\phi_y}{\phi_x} = \frac{k_\alpha + h_y^2 k_z - I_x \Omega^2}{(h_x - v_x)\,h_y k_z}, \tag{41}$$

$$\frac{\phi_y}{\phi_x} = \frac{(h_x - v_x)\,h_y\,k_z}{k_\beta + (v_x - h_x)^2\,k_z - (I_y + v_x^2 m)\Omega^2}. \tag{42}$$

Eliminating ϕ_y/ϕ_x from equations (40) and (42) yields

$$\Omega^2 = \frac{k_\beta}{I_y + h_x m\,v_x}. \tag{43}$$

Also, eliminating ϕ_y/ϕ_x from equations (40) and (41) and using equation (43) gives the following cubic equation in terms of v_x:

$$h_x^2 k_\alpha k_z m^2 v_x^3 - h_x m \left[I_x k_\beta k_z - 2I_y k_\alpha k_z + (k_\alpha k_\beta + h_x^2 k_\alpha k_z + h_y^2 k_\beta k_z) m \right] v_x^2$$

$$+ \left[I_y^2 k_\alpha k_z + I_x k_\beta (k_\beta + h_x^2 k_z) m - I_y (I_x k_\beta k_z + (k_\alpha k_\beta + 2h_x^2 k_\alpha k_z + h_y^2 k_\beta k_z) m) \right] v_x$$

$$+ h_x I_y k_z (I_x k_\beta - I_y k_\alpha) = 0. \tag{44}$$

Now, the locations of the axes of vibrations, i.e., the values of v_x, can be obtained from equation (44) and thereby the natural frequencies Ω from equation (43). The directions of the axes of vibrations can be determined by any one of equations (40)–(42).

In order to completely express the axis of vibration, the value δ_z in equation (31) can be computed from $\boldsymbol{\delta} = \mathbf{v} \times \boldsymbol{\phi}$ as

$$\boldsymbol{\delta} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_x & 0 & 0 \\ \phi_x & \phi_y & 0 \end{bmatrix} = v_x \phi_y \mathbf{k} = \delta_z \mathbf{k}. \tag{45}$$

In the same manner, by choosing $\mathbf{v}$ on the y-axis, another cubic equation can be obtained in the form that the subscripts x and α in equation (44) are replaced by y and β respectively.

4.3. AXES OF VIBRATION FOR TWO AND THREE PLANES OF SYMMETRY

As discussed in the previous section, the numbers of the planes of symmetry are related to the numbers of zero components in $\mathbf{h}$. When two components of $\mathbf{h}$ become zeros, there exist two planes of symmetry. In this case, the completely decoupled modes, i.e., a pure translation and pure rotation mode, are generated along the intersecting line of the planes of symmetry. In each plane of symmetry, two modes coupled with one translation and one rotation are generated.

If the x–y and z–x planes are the planes of symmetry, one translational and one rotational modes about the x-axis are completely independent of the other modes with the natural frequencies of $\sqrt{k_x/m}$ and $\sqrt{k_\alpha/I_x}$. All the remaining axes of vibrations are intersecting the z-axis.

For the x–y and z–x planes of symmetry, it is clear that $h_y = h_z = 0$. The vector $\mathbf{v}$ can be so chosen that $v_y = v_z = 0$. For the in-plane modes, substituting equation (34) into equation (32) yields two equations in two unknowns Ω, v_x:

$$\Omega^2 = \frac{(v_x - h_x) k_y}{v_x m}, \quad \Omega^2 = \frac{k_\gamma + (v_x - h_x)^2 k_y}{I_z + v_x^2 m}. \tag{46}$$

Eliminating Ω^2 from two equations in equation (46) gives

$$(mk_y h_x) v_x^2 + (I_z k_y - k_\gamma m - h_x^2 k_y m) v_x - h_x I_z k_y = 0. \tag{47}$$

Solving equation (47) for v_x gives the locations of the axes of vibrations for the in-plane modes and the corresponding natural frequencies can be determined from either of the two equations in equation (46).

For the out-of-plane modes, equation (39) can be expressed in the form

$$\mathbf{X}^{out} = [0 \ 0 \ 0 \ 0 \ 1 \ 0]^T. \tag{48}$$

Substituting equation (48) into equation (32) yields two equations in two unknowns Ω, v_x:

$$\Omega^2 = \frac{(v_x - h_x) k_z}{v_x m}, \quad \Omega^2 = \frac{k_\beta + (v_x - h_x)^2 k_z}{I_y + v_x^2 m}. \tag{49}$$

Eliminating Ω^2 from two equations in equation (49) gives

$$(mk_z h_x) v_x^2 + (I_y k_z - k_\beta m - h_x^2 k_z m) v_x - h_x I_y k_z = 0. \tag{50}$$

Now, solving equation (50) for v_x yields the locations of the axes of vibrations for the out-of-plane modes and the corresponding natural frequencies are determined from either

of the two equations in equation (49). For the three planes of symmetry, the vector **h** becomes the zero vector, which is the case that the point G and E coalesce, there exist three planes of symmetry and all the vibration modes are decoupled.

5. FREQUENCY RESPONSE TO FORCED VIBRATION

For an elastically supported single rigid body, equation (22) can be expressed for forced vibration by

$$\mathbf{M}_G \ddot{\mathbf{X}}_G + \mathbf{K}_G \mathbf{X}_G = \mathbf{Q}(t). \tag{51}$$

When harmonic excitation is assumed, the applied force can be expressed by [8]

$$\mathbf{Q}(t) = \hat{\mathbf{w}}\, \mathrm{e}^{\mathrm{j}\omega t}. \tag{52}$$

where $\hat{\mathbf{w}}$ is the time-independent applied wrench and ω is the angular frequency. The infinitesimal twist response can be expressed by

$$\mathbf{X}_G = \hat{\mathbf{X}}\, \mathrm{e}^{\mathrm{j}\omega t}. \tag{53}$$

Substituting equations (52) and (53) into equation (51) yields

$$(\mathbf{K}_G - \omega^2 \mathbf{M}_G)\, \hat{\mathbf{X}} = \hat{\mathbf{w}}. \tag{54}$$

Here, it is noted that the modal matrix is defined by the set of the axes of vibrations as

$$\mathbf{\Psi} = [\mathbf{X}_1 \ \cdots \ \mathbf{X}_6]. \tag{55}$$

Using these axes of vibrations in equation (55), the response to the applied wrench can be expressed by

$$\hat{\mathbf{X}} = \sum_{r=1}^{6} \frac{\mathbf{X}_r^{\mathrm{T}} \hat{\mathbf{w}} \mathbf{X}_r}{k_r - \omega^2 m_r} = \sum_{r=1}^{6} \frac{R_r \mathbf{X}_r}{k_r - \omega^2 m_r}, \tag{56}$$

where k_r and m_r are the rth diagonal elements in the diagonal matrices $\mathbf{\Psi}^{\mathrm{T}}\mathbf{K}_G\mathbf{\Psi}$ and $\mathbf{\Psi}^{\mathrm{T}}\mathbf{M}_G\mathbf{\Psi}$ respectively. From equation (56), it can be seen that the rth residue R_r representing the influence of rth mode on the response is in fact the reciprocal product of the rth-axis of vibration and the applied wrench.

It is noted that the response function in equation (56) is expressed at the mass center. When the response at any point G' on the body is interested, the response at G' can be obtained by transforming $\hat{\mathbf{X}}$ to the point G' using the vector $\mathbf{g}'$ from G to G'. This can be expressed by

$$\hat{\mathbf{X}}_{G'} = \begin{bmatrix} \mathbf{I}_3 & -\mathbf{g}' \times \\ \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix}, \quad \hat{\mathbf{X}} = \sum_{r=1}^{6} \frac{R_r \begin{bmatrix} \mathbf{I}_3 & -\mathbf{g}' \times \\ \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \mathbf{X}_r}{k_r - \omega^2 m_r}. \tag{57}$$

From equation (57), it can be seen that when the interested point G' is lying on the rth-axis of vibration, there is no response in translation direction due to that axis mode.

6. EXAMPLE: AN OPTICAL DISC DRIVE

6.1. DIAGONALIZATION BY A PARALLEL AXIS CONGRUENCE TRANSFORMATION

Figure 3 shows the vibrating system of a rigid body supported by three rubbers which is one of the typical models of optical disc drives such as Digital Video Disk (DVD), CD-ROM, and Mini Disk (MD). The equation of motion for this model can be expressed by

$$\mathbf{M}_G\ddot{\mathbf{X}}_G + \mathbf{C}_G\dot{\mathbf{X}}_G + \mathbf{K}_G\mathbf{X}_G = \mathbf{Q}(t), \tag{58}$$

where the difference between equations (51) and (58) is the damping matrix. If a proportional damping is assumed, the undamped mode analysis can be directly applied. Choosing the co-ordinate axes along the principal axes of inertia, the mass matrix can be expressed by equation (25).

Each of the three pieces of rubbers placed between the base and the optical disc drive can be modelled as three linear springs when the elasticity of rubber is assumed to be constant as shown in Figure 3. The stiffness matrix can be obtained from equation (3) and the submatrices of the stiffness matrix in equation (2) are computed as follows:

$$\mathbf{A} = \begin{bmatrix} 3k_1 & 0 & 0 \\ 0 & 3k_1 & 0 \\ 0 & 0 & 3k_2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & k_1(2l_3 - l_4) \\ 0 & 0 & k_1(l_1 - l_2 + l_5) \\ -k_2(2l_3 - l_4) & -k_2(l_1 - l_2 + l_5) & 0 \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} k_2(2l_3^2 + l_4^2) & k_2(l_1l_2 - l_2l_3 - l_4l_5) & 0 \\ k_2(l_1l_3 - l_2l_3 - l_4l_5) & k_2(l_1^2 + l_2^2 + l_5^2) & 0 \\ 0 & 0 & k_1(l_1^2 + l_2^2 + 2l_3^2 + l_4^2 + l_5^2) \end{bmatrix}.$$

The conditions that the stiffness matrix can be diagonalized by a parallel axis congruence transformation are examined. It can be easily shown that $\mathbf{F} = \mathbf{I}_3$ and $\mathbf{A}^{-1}\mathbf{B}$ is skew symmetric from the above stiffness matrix. However, the condition that $\mathbf{\Phi} = \mathbf{I}_3$ is not satisfied. Here, after some algebra, the relation $l_5 = (l_1 - l_2)/2$ can be obtained from equation (19), which then satisfies this condition. Some physical properties and dimensions of the optical disc drive used in this example are given in Tables 1 and 2. Using the relation $l_5 = (l_1 - l_2)/2$, the dimension l_5 can be modified from its original value 0·0237 to 0·0052 m so that the stiffness matrix can be diagonalized by a parallel axis transformation from the point G to E.

Now, the co-ordinates of the point E can be computed from equation (14) as

$$E = (0·0052, \ -0·0328, 0),$$

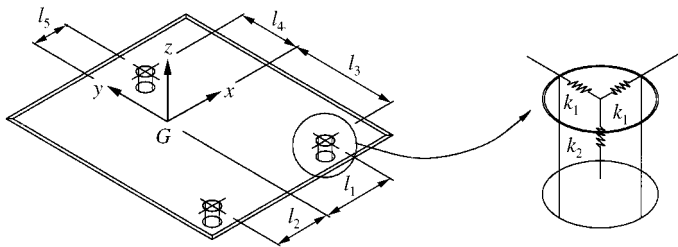which shows that the $x-y$ plane becomes the plane of symmetry.

Figure 3. The optical disc drive model.

Inertial and stiffness properties of the optical disc drive

m	0·168 kg
I_x	$0{\cdot}9662 \times 10^{-4}$ kg m^2
I_y	$2{\cdot}4447 \times 10^{-4}$ kg m^2
I_z	$3{\cdot}3026 \times 10^{-4}$ kg m^2
k_1	$2{\cdot}7 \times 10^3$ N/m
k_2	$4{\cdot}5 \times 10^3$ N/m

TABLE 2

Dimensions of the optical disc drive

l_1	0·0467 m
l_2	0·0363 m
l_3	0·0693 m
l_4	0·0401 m
l_5	0·237 m (modified dimension: 0·0052 m)

The stiffness matrix at the point E in equation (15) is computed as

$$\mathbf{K}_E = 1{\cdot}0\,e^4 \times diag(0{\cdot}8100 \;\; 0{\cdot}8100 \;\; 1{\cdot}3500 \;\; 0{\cdot}0036 \;\; 0{\cdot}0016 \;\; 0{\cdot}0031).$$

Since $k_x = k_y$ in $\mathbf{K}_E$, one of the axes of vibrations for in-plane modes is corresponding to the translational mode lying on the line joining the points G and E, and the natural frequency is $\sqrt{k_x/m}$.

The damping matrix can be obtained from the scalar multiplication to the stiffness matrix, and the damping ratios are computed as

$$\varsigma = (0{\cdot}0091 \;\; 0{\cdot}01840 \;\; 0{\cdot}0110 \;\; 0{\cdot}0113 \;\; 0{\cdot}0130 \;\; 0{\cdot}0370).$$

6.2. THE AXES OF VIBRATIONS

For the in-plane modes, solving equation (37) for v_x gives

$$[v_{xi}] = [0{\cdot}0617 - 0{\cdot}0029].$$

The co-ordinates of the intersecting points between the axes of vibrations and the x–y plane are $(0{\cdot}0167, \; -0{\cdot}1057, 0)$ and $(-0{\cdot}0029, 0{\cdot}0181, 0)$. Therefore, the line co-ordinates of the axes of vibrations can be expressed by

$$[\mathbf{X}_i^{in}] = \begin{bmatrix} -0{\cdot}1057 & 0{\cdot}0181 & -0{\cdot}1561 \\ -0{\cdot}0167 & 0{\cdot}0029 & 0{\cdot}9877 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Clearly, $\mathbf{X}_1^{in}$ and $\mathbf{X}_2^{in}$ represent the rotational modes about the axes of vibration in view of equation (30), and $\mathbf{X}_3^{in}$ represents a pure translation mode. The corresponding natural frequencies are obtained from one of the three equations in equation (35) as

$$[\Omega_i^{in}] = [29\cdot02 \;\; 58\cdot58 \;\; 34\cdot95] \text{ Hz.}$$

For out-of-plane modes, the solutions of equation (44) are computed as

$$[v_{xi}] = [-0\cdot2480 \;\; 0\cdot0705 \;\; -0\cdot0194],$$

and from equation (43), the corresponding natural frequencies are

$$[\Omega_i^{out}] = [117\cdot90\cdot \;\; 35\cdot82 \;\; 41\cdot54] \text{ Hz.}$$

From one of equations (40)–(42), the directions of the axes can be determined as

$$\left[\tan^{-1}\left(\frac{\phi_y}{\phi_x}\right)_i\right] = [1\cdot3060° \;\; 57\cdot5698° \;\; -70\cdot0741°],$$

and the line co-ordinates of the axes of vibrations are

$$[\mathbf{X}_3^{out}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0\cdot0057 & 0\cdot0595 & 0\cdot0188 \\ 0\cdot9997 & 0\cdot5363 & 0\cdot2407 \\ 0\cdot0228 & 0\cdot8440 & -0\cdot9706 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is also clear that $\mathbf{X}_1^{out}$, $\mathbf{X}_2^{out}$ and $\mathbf{X}_3^{out}$ represent the rotational modes about the axes of vibration in view of equation (31).

Combining the above two matrices representing the in-plane and out-of-plane modes gives the well-known modal matrix as

$$\mathbf{\Psi} = [\mathbf{X}_1^{in} \;\; \mathbf{X}_2^{in} \;\; \mathbf{X}_3^{in} \;\; \mathbf{X}_1^{out} \;\; \mathbf{X}_2^{out} \;\; \mathbf{X}_3^{out}].$$

The axes of vibrations are shown in Figures 4 and 5.

6.3. FREQUENCY RESPONSE TO FORCED VIBRATION

When a proportional damping is added, from the modification of equation (56) the response can be expressed by

$$\hat{\mathbf{X}} = \sum_{r=1}^{6} \frac{\mathbf{X}_r^T \hat{\mathbf{w}} \mathbf{X}_r}{k_r + \omega j c_r - \omega^2 m_r} = \sum_{r=1}^{6} \frac{R_r \mathbf{X}_r}{k_r + \omega j c_r - \omega^2 m_r}, \tag{59}$$

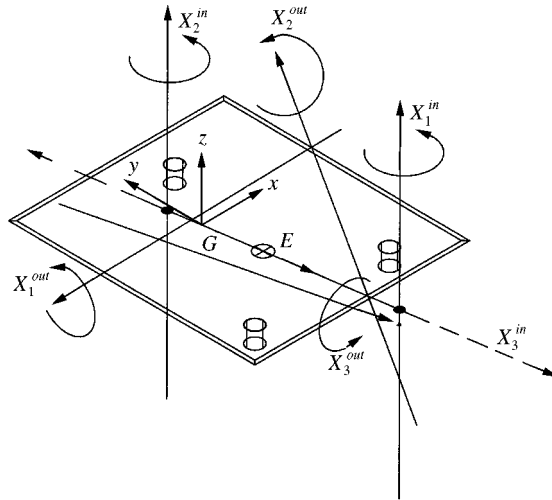where c_r is the rth diagonal element in $\mathbf{\Psi}^T \mathbf{C}_G \mathbf{\Psi}$.

Figure 4. The axes of vibration in the optical disc drive.

If the harmonic force is applied to the body of the disc drive due to a harmonic base excitation along the z-axis, it can be expressed by

$$\mathbf{Q}(t) = \mathbf{K}_G\, X_b \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathrm{e}^{\mathrm{j}\omega t}, \tag{60}$$

where X_b denotes the amplitude of the base excitation. The time-inepdendent applied wrench can be expressed by

$$\hat{\mathbf{w}} = X_b\,(k_2 + \mathrm{i}c_2) \begin{bmatrix} 0 \\ 0 \\ 1 \\ h_y \\ -h_x \\ 0 \end{bmatrix}. \tag{61}$$

In the case of the proportional damping, the direction of the wrench is stationary. Equation (61) show the direction of the wrench is parallel to the z-axis and it passes through the center of elasticity as shown in Figure 6. Here, it can be seen that the axes of vibrations for in-plane modes are parallel to the wrench. Therefore, the mutual moments between the axes of vibrations for in-plane modes and the wrench become zero and there is no influence of the axes of vibrations for in plane on the responses.

The displacement in the direction of the z-axis corresponds to the values of the third row in equation (59) and has the three resonances associated with the axes of the vibration for out of plane as shown in Figure 7. When the intersecting point between $\mathbf{X}_2^{out}$ and $\mathbf{X}_3^{out}$ is chosen to observe the response in the direction of the z-axis at that point, it can be seen that there exists only one resonance (see Figure 8). This is because no translational
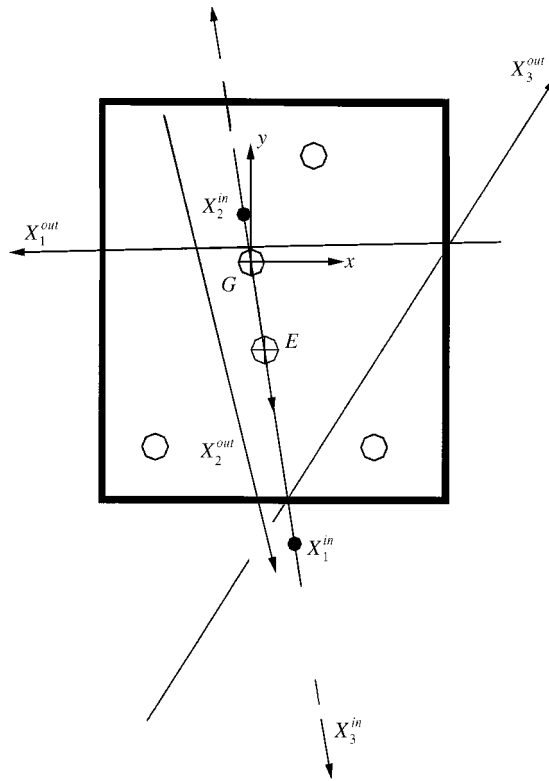
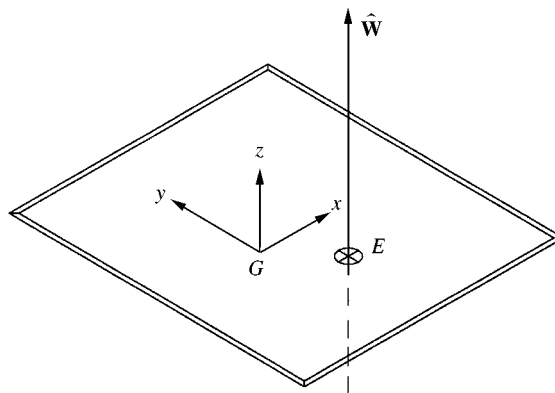Figure 5. Top view of the axes of vibration.

Figure 6. The applied wrench.

displacements in the direction of the z-axis can be generated by $\mathbf{X}_2^{out}$ and $\mathbf{X}_3^{out}$ as discussed in section 5.

7. CONCLUSION

In this study, the geometrical conditions to diagonalize a spatial stiffness matrix by the use of a parallel axis congruence transformation and their geometrical meanings are
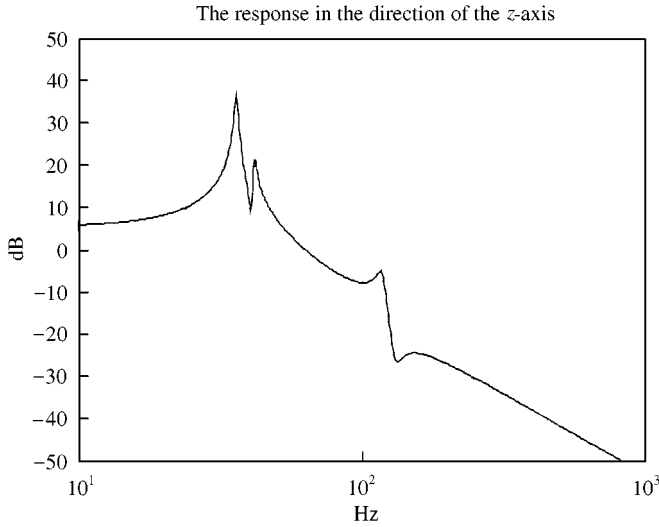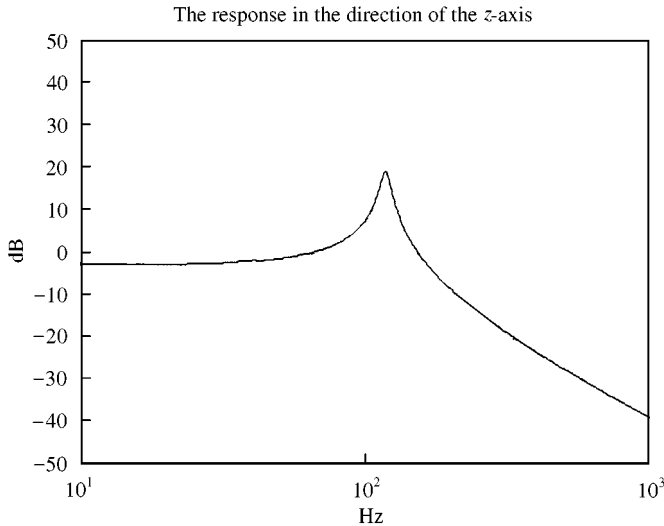
The response in the direction of the z-axis

Figure 7. The response at the mass center.

The response in the direction of the z-axis

Figure 8. The response at the intersecting point of $\mathbf{X}_2^{out}$ and $\mathbf{X}_3^{out}$.

presented. It is shown that the planes of symmetry to decouple the vibration modes of a system depend on the location of the center of elasticity. For a system with the planes of symmetry, the analytical solution for the axes of vibrations have been derived. In addition, the relation between the applied force and the response of the system is expressed by the reciprocal product.

As a numerical example, an optical disk drive with spring suspensions has been modelled and its vibration modes have been analyzed so as to illustrate the applicability of the theoretical development.

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