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VIBRATION FREQUENCIES OF A ROTATING FLEXIBLE ARM CARRYING A MOVING MASS

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A clamped–free flexible beam rotating in a horizontal plane and carrying a moving mass is modelled by the Euler–Bernoulli beam theory. The equation of motion is derived by Hamilton's principle including the effects of centrifugal stiffening arising from the rotation of the beam. The motion of the moving mass and the beam is coupled. The equation of motion is a coupled non-linear partial differential equation where the coupling terms have to be evaluated at the position of the moving mass. In order to obtain the mode shapes which account for the motion of the moving mass, the solution is discretized into space and time functions and the beam is divided into two separate regions with respect to the moving mass. This results in two non-homogeneous linear mode shape ordinary differential equations with four boundary, one discontinuity and three continuity conditions. The power series method is used to solve for the mode shape differential equations. A frequency equation is derived giving the relationship between the non-dimensional modal frequencies and the four non-dimensional parameters, i.e., the moving mass position, the moving mass, the beam angular velocity and the total moment of inertia about the hub. The numerical bisection method is used to solve for the vibration frequencies under different parameters. Results are presented for the first three modes of vibration.

1. INTRODUCTION

The dynamic response of beam-like structures subjected to moving mass has long been investigated by numerous researchers in the field of civil and mechanical engineering. In the field of civil engineering, typical examples include the dynamic response of a single or multi-span bridge under moving loads, vehicles and trains [1, 2], whereas in the field of mechanical engineering, examples include the dynamic response of cranes carrying moving loads or a robotic arm carrying a moving end effector (i.e., SCARA robot).

Most researchers on this subject used either a moving-force or a moving-mass model for the system and most of them used the assumed mode method in the formulation of the equation of motion [1–7]. The mode shape function is chosen so that it satisfies the prescribed geometric boundary conditions of the beam. The vibration behavior such as the beam deflection subjected to different values of moving force or mass, different moving mass velocities or position, etc., are usually analyzed. In recent years, the inertial effect of the mass and the interaction between the mass and the beam have attracted much attention. The mode shape function is required to satisfy not only the boundary conditions but also the transient conditions imposed by the moving mass. Stanisic [8] developed a method to obtain mode shape which accounts for the motion of the mass by dividing the beam into two separate regions with respect to the moving mass. Later, Khalily *et al.* [9] extended the work to obtain numerical solutions using two mode shapes for the system. Recently,

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Siddiqui *et al.* [10] investigated the dynamic behavior of a flexible cantilever beam carrying a moving mass-spring. In their work, the internal resonance behavior due to the coupling between the motion of the moving mass-spring and the beam is investigated. However, for all the above studies [1–12] the beam is not subjected to rotation and hence the effects of centrifugal stiffening [13] is not considered in their work.

This paper makes use of the method by Stanisic [8] but takes into account the effects of centrifugal stiffening [13] due to the rotation of the beam in the formulation of the equation of motion. The system is a clamped–free rotating flexible Euler–Bernoulli beam carrying a moving mass. The equation of motion is derived by Hamilton's principle. The motion of the moving mass and the beam is coupled. The beam is divided into two separate regions with respect to the moving mass (i.e., the left and right sides). The mode shapes have taken into account the motion of the moving mass. This results in two non-homogeneous linear ordinary differential equation with four boundary, one discontinuity and three continuity conditions. The power series method is used to solve the mode shape differential equations. A frequency equation is derived giving the relationship between the non-dimensional modal frequencies and the four non-dimensional parameters, i.e., the moving mass position, the moving mass, the beam angular velocity and the total moment of inertia about the hub. The numerical bisection method is used to solve the vibration frequencies under different parameters. Results are presented for the first three modes of vibration.

2. THEORY AND FORMULATION

A clamped–free flexible arm carrying a moving mass is shown in Figure 1. It is modelled by the Euler–Bernoulli beam theory in which rotary inertia and shear deformation effects are ignored. The arm is of length L, mass per unit length ρ and flexural rigidity EI. It rotates at an angular velocity of $\dot{\theta}$ in a horizontal plane about the clamped axis and has a mass m travelling along it. Let OXY and Oij represent the inertial and rotating Cartesian axes respectively. The moment of inertia of the hub is J. The transverse displacement of a spatial point on the beam at a distance r $(0 < r < L)$ from the origin is denoted by $w(r, t)$. The position vector $\mathbf{r}$ at a spatial position r is given by

$$\mathbf{r} = r\mathbf{i} - w(r, t)\mathbf{j}, \qquad \dot{\mathbf{r}} = w(r, t)\dot{\theta}\mathbf{i} + r\dot{\theta}\mathbf{j} - \dot{w}(r, t)\mathbf{j}. \tag{1}$$

Let $s(t)$ be the position of the mass with respect to the clamped end of the beam and $\dot{s}(t)$ be the velocity of the mass relative to the beam. The resultant velocity $\mathbf{V}_m$ of the mass is

$$\mathbf{V}_m = [\dot{\mathbf{r}} + \dot{\mathbf{s}}]_{r=s} = [(\dot{s} + w\dot{\theta})\mathbf{i} + (r\dot{\theta} - \dot{w} - \dot{s}w')\mathbf{j}]_{r=s}, \tag{2}$$

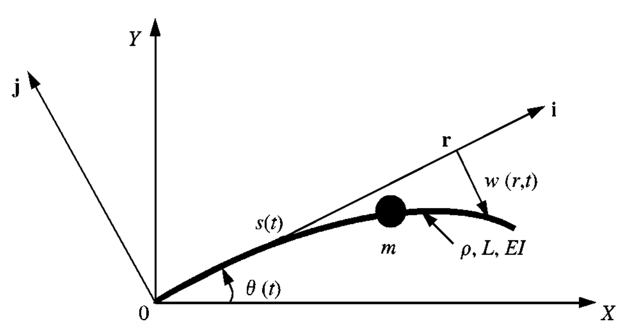

Figure 1. A rotating flexible beam carrying a moving mass.

where a dot and a prime denote the derivatives with respect to time t and the spatial variable r respectively. The kinetic energy of the beam is

$$T_b = \frac{1}{2}\int_0^L \rho \dot{\mathbf{r}}^{\mathrm{T}}\dot{\mathbf{r}}\,\mathrm{d}r + \frac{1}{2}J\dot{\theta}^2. \tag{3}$$

The kinetic energy of the moving mass is

$$T_m = \tfrac{1}{2}m\mathbf{V}_m^{\mathrm{T}}\mathbf{V}_m. \tag{4}$$

The total potential energy of the system is

$$V = \frac{EI}{2}\int_0^L w''^2\,\mathrm{d}r + \frac{1}{2}\int_0^L P(r,t)w'^2\,\mathrm{d}r, \tag{5}$$

where $P(r, t)$ is the centrifugal force arising from the centrifugal stiffening effect and is given by

$$P(r,t) = \begin{cases} ms\dot{\theta}^2 + \displaystyle\int_r^L \rho r\dot{\theta}^2\,\mathrm{d}r, & 0 \leqslant r \leqslant s, \\ \displaystyle\int_r^L \rho r\dot{\theta}^2\,\mathrm{d}r, & s < r \leqslant L. \end{cases} \tag{6}$$

The virtual work done by the applied motor torque τ is given by

$$\delta W = \tau\,\delta\theta. \tag{7}$$

By applying Hamilton's principle,

$$\int_{t_1}^{t_2} (\delta T_b + \delta T_m - \delta V + \delta W)\,\mathrm{d}t = 0. \tag{8}$$

Substituting equations (1)–(7) into equation (8), one obtains the governing equation of motion of the flexible beam as

$$\begin{aligned} &EIw'''' - P(r,t)w'' + \rho r\dot{\theta}^2 w' - \rho\dot{\theta}^2 w + \rho\ddot{w} - \rho r\ddot{\theta} \\ &\quad + m(\ddot{s}w' + 2\dot{s}\dot{w}' + \dot{s}^2 w'' - 2\dot{s}\dot{\theta} - w\dot{\theta}^2 + \ddot{w} - r\ddot{\theta})\delta(r-s) = 0, \end{aligned} \tag{9}$$

where $\delta(\cdot)$ is the Dirac delta function. The four boundary conditions are

$$w(0,t) = w'(0,t) = w''(L,t) = w'''(L,t) = 0. \tag{10a–d}$$

When the mass is located at the tip of the beam ($s = L$), the equation of motion of the flexible beam and the boundary conditions become

$$EIw'''' - mL\dot{\theta}^2 w'' - \tfrac{1}{2}\rho\dot{\theta}^2(L^2 - r^2)w'' + \rho r\dot{\theta}^2 w' - \rho\dot{\theta}^2 w + \rho\ddot{w} - \rho r\ddot{\theta} = 0 \tag{11}$$

and

$$w(0,t)=0, \quad w'(0,t)=0, \quad w''(L,t)=0, \tag{12a–d}$$

$$mw''''(L,t)+\rho w'''(L,t)=0.$$

Torque balance about the hub gives

$$J_t\ddot{\theta}=\tau-\mu_l, \tag{13}$$

where J_t is the total moment of inertia about the hub and

$$\mu_l=\int_0^L \rho r\ddot{w}(r,t)\,\mathrm{d}r+ms\ddot{w}(s,t). \tag{14}$$

Setting $\tau=0$ for free vibration of the beam, equation (13) becomes

$$\ddot{\theta}=-\frac{\mu_l}{J_t}. \tag{15}$$

3. DETERMINATION OF THE MODE SHAPE EQUATIONS

Let the solution of equation (9) for $w(r,t)$ be expressed as

$$w(r,t)=Y(r)\mathrm{e}^{\mathrm{i}\omega t}, \tag{16}$$

where ω is the natural frequency of the beam, $Y(r)$ is the eigenfunctions or mode shapes of the flexible beam. In order to obtain $Y(r)$, the method introduced by Stanisic [8] is used. Substituting the form of w given in equation (16) into equations (9), (14) and (15), and considering two separate regions on the beam with respect to the mass (i.e., the left and right sides), the following mode shape equations are obtained.

(1) $0\leqslant r<s$:

$$EIY_L'''' - ms\dot{\theta}^2Y_L''-\tfrac{1}{2}\rho\dot{\theta}^2(L^2-r^2)Y_L''+\rho r\dot{\theta}^2Y_L'-\rho\dot{\theta}^2Y_L-\rho\omega^2Y_L=-\frac{\rho r\mu}{J_t}. \tag{17}$$

(2) $s<r\leqslant L$:

$$EIY_R''''-\tfrac{1}{2}\rho\dot{\theta}^2(L^2-r^2)Y_R''+\rho r\dot{\theta}^2Y_R'-\rho\dot{\theta}^2Y_R-\rho\omega^2Y_R=-\frac{\rho r\mu}{J_t}, \tag{18}$$

where

$$\mu=-\int_0^L \rho r\omega^2Y(r)\,\mathrm{d}r-ms\omega^2Y(s) \tag{19}$$

and $Y_L(r,s)$ and $Y_R(r,s)$ correspond to the left and right parts with respect to the mass, i.e., $0\leqslant r<s$ and $s<r\leqslant L$ respectively. Note that the term containing $\delta(r-s)$ in equation (9)

has disappeared in equations (17) and (18) because it only come into effect at the position of the mass, i.e., $r = s$. Introducing the non-dimensional parameters

$$\xi = \frac{r}{L}, \quad s_0 = \frac{s}{L}, \quad N = \frac{m}{\rho L},$$

$$J_0 = \frac{J_t}{\rho L^3}, \quad \eta = \sqrt{\frac{\rho}{EI}}\,\dot{\theta} L^2, \quad \Omega = \sqrt{\frac{\rho}{EI}}\,\omega L^2. \tag{20}$$

Substituting equation (20) into equations (17)–(19) gives

(3) $0 \leqslant \xi < s_0$:

$$Y_L'''' - N s_0 \eta^2 Y_L'' - \tfrac{1}{2}\eta^2 (1 - \xi^2) Y_L'' + \eta^2 \xi Y_L' - \eta^2 Y_L - \Omega^2 Y_L = -\frac{\mu_0 \xi}{J_0}. \tag{21}$$

(4) $s_0 < \xi \leqslant 1$:

$$Y_R'''' - \tfrac{1}{2}\eta^2 (1 - \xi^2) Y_R'' + \eta^2 \xi Y_R' - \eta^2 Y_R - \Omega^2 Y_R = -\frac{\mu_0 \xi}{J_0}, \tag{22}$$

where

$$\mu_0 = -\int_0^1 \xi \Omega^2 Y(\xi)\,\mathrm{d}\xi - N s_0 \Omega^2 Y(s_0) \tag{23}$$

and a prime $(\ )'$ represents the derivative with respect to ξ.

The four boundary conditions in equations (10a–d) become

$$Y(0) = Y'(0) = Y''(1) = Y'''(1) = 0, \tag{24a–d}$$

where

$$Y(\xi, s_0) = \begin{cases} Y_L(\xi, s_0), & 0 \leqslant \xi < s_0, \\ Y_R(\xi, s_0), & s_0 < \xi \leqslant 1. \end{cases} \tag{25}$$

Since $Y(\xi, s_0)$ is a continuous function and the moving mass is being modelled as a particle with no point moment acting at $\xi = s_0$. Therefore, $Y(\xi, s_0)$ together with its first and second derivative should be continuous at $\xi = s_0$. The following three continuity conditions should hold:

$$Y_L(s_0, s_0) = Y_R(s_0, s_0), \tag{26a}$$

$$Y_L'(s_0, s_0) = Y_R'(s_0, s_0), \tag{26b}$$

$$Y_L''(s_0, s_0) = Y_R''(s_0, s_0). \tag{26c}$$

There is also a discontinuity condition imposed by the shearing force at the position of the mass. In order to facilitate the derivation of this condition, we first set both $\dot{s}$ and $\ddot{s}$ to zero in equation (9). Then inserting equations (14)–(16), (20) and (23) into it yields

$$Y''''(\xi) - P_0(\xi)Y''(\xi) + \eta^2\xi Y'(\xi) - \eta^2 Y(\xi) - \Omega^2 Y(\xi) + \frac{\mu_0\xi}{J_0}$$

$$= N\left[\eta^2 Y(\xi) + \Omega^2 Y(\xi) - \frac{\mu_0\xi}{J_0}\right]\delta(\xi - s_0), \tag{27}$$

where

$$P_0(\xi) = \begin{cases} Ns_0\eta^2 + \frac{1}{2}\eta^2(1-\xi^2), & 0 \leqslant \xi \leqslant s_0, \\ \frac{1}{2}\eta^2(1-\xi^2), & s_0 < \xi \leqslant 1. \end{cases} \tag{28}$$

Integrate equation (27) with respect to ξ over the interval [0, 1]; hence,

$$\lim_{\Delta\to 0}\int_0^{s_0-\Delta}\left[Y'''' - P_0(\xi)Y'' + \eta^2\xi Y' - \eta^2 Y - \Omega^2 Y + \frac{\mu_0\xi}{J_0}\right]\mathrm{d}\xi$$

$$+ \lim_{\Delta\to 0}\int_{s_0-\Delta}^{s_0+\Delta}\left[Y'''' - P_0(\xi)Y'' + \eta^2\xi Y' - \eta^2 Y - \Omega^2 Y + \frac{\mu_0\xi}{J_0}\right]\mathrm{d}\xi$$

$$+ \lim_{\Delta\to 0}\int_{s_0+\Delta}^{1}\left[Y'''' - P_0(\xi)Y'' + \eta^2\xi Y' - \eta^2 Y - \Omega^2 Y + \frac{\mu_0\xi}{J_0}\right]\mathrm{d}\xi$$

$$= \int_0^1 N\left(\eta^2 Y + \Omega^2 Y - \frac{\mu_0\xi}{J_0}\right)\delta(\xi - s_0)\,\mathrm{d}\xi. \tag{29}$$

Then equation (29) by means of equations (21) and (22) leads to

$$\langle Y'''(\xi, s_0)\rangle|_{\xi=s_0} = N\left[\eta^2 Y(s_0, s_0) + \Omega^2 Y(s_0, s_0) - \frac{\mu_0 s_0}{J_0}\right]. \tag{30}$$

Equation (30) represents the jump of the shearing force in the beam at $\xi = s_0$ and $\langle\cdot\rangle$ denotes jump of the function, i.e.,

$$\langle f(\xi)\rangle|_{\xi=s_0} = \lim_{\Delta\to 0}\left[f(s_0+\Delta) - f(s_0-\Delta)\right].$$

Defining

$$C_1 = -\tfrac{1}{2}\eta^2 - Ns_0\eta^2, \quad C_2 = \tfrac{1}{2}\eta^2,$$

$$C_3 = -\Omega^2 - \eta^2, \quad C_4 = -\frac{\mu_0}{J_0}. \tag{31}$$

Equations (21) and (22) of the flexible beam become

$$Y_L'''' + (C_1 + C_2\xi^2)Y_L'' + 2C_2\xi Y_L' + C_3 Y_L = C_4\xi, \tag{32}$$

$$Y_R'''' + (C_2\xi^2 - C_2)Y_R'' + 2C_2\xi Y_R' + C_3 Y_R = C_4\xi. \tag{33}$$

Equations (32) and (33) are non-homogeneous linear ordinary differential equations with variable coefficients. The total solution can be expressed in terms of a homogeneous solution and a particular solution in the form,

$$Y_L(\xi) = Y_{Lc}(\xi) + F\xi, \quad 0 \leqslant \xi < s_0, \tag{34}$$

$$Y_R(\xi) = Y_{Rc}(\xi) + F\xi, \quad s_0 < \xi \leqslant 1, \tag{35}$$

where $F\xi$ is the particular solution, $Y_{Lc}(\xi)$ and $Y_{Rc}(\xi)$ are the homogeneous solutions of $Y_L(\xi)$ and $Y_R(\xi)$ respectively. Substituting equations (34) and (35) into equations (32) and (33), respectively, gives

$$Y''''_{Lc} + (C_1 + C_2\xi^2)Y''_{Lc} + 2C_2\xi Y'_{Lc} + C_3 Y_{Lc} = 0, \tag{36}$$

$$Y''''_{Rc} + (C_2\xi^2 - C_2)Y''_{Rc} + 2C_2\xi Y'_{Rc} + C_3 Y_{Rc} = 0, \tag{37}$$

$$F = \frac{\mu_0}{\Omega^2 J_0} = \frac{1}{J_0}\left[-\int_0^l \xi Y(\xi)\,\mathrm{d}\xi - N s_0 Y(s_0) \right]. \tag{38}$$

4. POWER SERIES SOLUTION OF THE MODE SHAPE EQUATIONS

Equations (36) and (37) are homogeneous variable coefficient differential equation that cannot be solved analytically by using ordinary trigonometric or hyperbolic functions. Hence, the power series method is used in this case by expressing the homogeneous solution $Y_{Lc}(\xi)$ and $Y_{Rc}(\xi)$ as a power series in the independent variable ξ.
Let

$$u(\xi) = \sum_{k=0}^{\infty} a_k \xi^k, \quad 0 \leqslant \xi < s_0, \tag{39}$$

$$v(\xi) = \sum_{k=0}^{\infty} b_k \xi^k, \quad s_0 < \xi \leqslant 1. \tag{40}$$

Substituting equations (39) and (40) into the homogeneous equations (36) and (37) and equating coefficients of a like power of ξ yields the following recurrence formula:

$$a_{k+4} = -\frac{C_1 a_{k+2}}{(k+4)(k+3)} - \left[\frac{kC_2}{(k+4)(k+3)(k+2)} + \frac{C_3}{(k+4)(k+3)(k+2)(k+1)} \right] a_k, \quad k \geqslant 0, \tag{41}$$

$$b_{k+4} = \frac{C_2 b_{k+2}}{(k+4)(k+3)} - \left[\frac{kC_2}{(k+4)(k+3)(k+2)} + \frac{C_3}{(k+4)(k+3)(k+2)(k+1)} \right] b_k, \quad k \geqslant 0. \tag{42}$$

There are four arbitrary constants a_0, a_1, a_2, a_3 in equation (39). Four linearly independent solutions u_0, u_1, u_2, u_3 can be obtained by selecting these four arbitrary constants as follows:

$$\begin{aligned}
&\text{for } u_0, \quad a_0 = 1 \quad \text{and} \quad a_1 = a_2 = a_3 = 0,\\
&\text{for } u_1, \quad a_0 = 0 \quad \text{and} \quad a_1 = 1, \quad a_2 = a_3 = 0,\\
&\text{for } u_2, \quad a_0 = a_1 = 0 \quad \text{and} \quad a_2 = 1, \quad a_3 = 0,\\
&\text{for } u_3, \quad a_0 = a_1 = a_2 = 0 \quad \text{and} \quad a_3 = 1.
\end{aligned} \tag{43}$$

These four linearly independent functions can be written explicitly as

$$\begin{aligned}
u_0(\xi) &= 1 - \frac{C_3}{24}\xi^4 + \frac{C_1 C_3}{720}\xi^6 + \cdots,\\
u_1(\xi) &= \xi - \frac{2C_2 + C_3}{120}\xi^5 + \frac{(2C_2 + C_3)C_1}{5040}\xi^7 + \cdots,\\
u_2(\xi) &= \xi^2 - \frac{C_1}{12}\xi^4 + \frac{C_1^2 - 6C_2 - C_3}{360}\xi^6 + \cdots,\\
u_3(\xi) &= \xi^3 - \frac{C_1}{20}\xi^5 + \frac{C_1^2 - 12C_2 - C_3}{840}\xi^7 + \cdots.
\end{aligned} \tag{44}$$

Similarly, there are four arbitrary constants b_0, b_1, b_2, b_3 in equation (40). Four linearly independent solutions v_0, v_1, v_2, v_3 can be obtained by selecting these four arbitrary constants as follows:

$$\begin{aligned}
&\text{for } v_0, \quad b_0 = 1 \quad \text{and} \quad b_1 = b_2 = b_3 = 0,\\
&\text{for } v_1, \quad b_0 = 0 \quad \text{and} \quad b_1 = 1, \quad b_2 = b_3 = 0,\\
&\text{for } v_2, \quad b_0 = b_1 = 0 \quad \text{and} \quad b_2 = 1, \quad b_3 = 0,\\
&\text{for } v_3, \quad b_0 = b_1 = b_2 = 0 \quad \text{and} \quad b_3 = 1.
\end{aligned} \tag{45}$$

These four linearly independent functions can be written explicitly as

$$\begin{aligned}
v_0(\xi) &= 1 - \frac{C_3}{24}\xi^4 - \frac{C_2 C_3}{720}\xi^6 + \cdots,\\
v_1(\xi) &= \xi - \frac{2C_2 + C_3}{120}\xi^5 - \frac{(2C_2 + C_3)C_2}{5040}\xi^7 + \cdots,\\
v_2(\xi) &= \xi^2 + \frac{C_2}{12}\xi^4 + \frac{C_2^2 - 6C_2 - C_3}{360}\xi^6 + \cdots,\\
v_3(\xi) &= \xi^3 + \frac{C_2}{20}\xi^5 + \frac{C_2^2 - 12C_2 - C_3}{840}\xi^7 + \cdots.
\end{aligned} \tag{46}$$

The linear combination of these four linearly independent functions is the homogeneous solution of equations (36) and (37). Hence equations (34) and (35) can be written as

$$Y_L(\xi) = A_0 u_0(\xi) + A_1 u_1(\xi) + A_2 u_2(\xi) + A_3 u_3(\xi) + F\xi, \quad 0 \leqslant \xi < s_0, \tag{47}$$

$$Y_R(\xi) = B_0 v_0(\xi) + B_1 v_1(\xi) + B_2 v_2(\xi) + B_3 v_3(\xi) + F\xi, \quad s_0 < \xi \leqslant 1. \tag{48}$$

The eight solution constants A_0, A_1, A_2, A_3 and B_0, B_1, B_2, B_3 can be found by substituting equations (47) and (48) into the four boundary conditions (24a–d) and the four continuity conditions (26a–c) and (30). From equation (24a), we get

$$A_0 = 0. \tag{49}$$

The remaining seven constants can be expressed by the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{24} & D_{25} & D_{26} & D_{27} \\ 0 & 0 & 0 & D_{34} & D_{35} & D_{36} & D_{37} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} & D_{47} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} & D_{57} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} & D_{67} \\ D_{71} & D_{72} & D_{73} & D_{74} & D_{75} & D_{76} & D_{77} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ B_0 \\ B_1 \\ B_2 \\ B_3 \end{Bmatrix} = \begin{Bmatrix} -F \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ N s_0 \eta^2 F \end{Bmatrix}, \tag{50}$$

where

$$D_{24} = v_0''(1), \quad D_{25} = v_1''(1), \quad D_{26} = v_2''(1), \quad D_{27} = v_3''(1),$$

$$D_{34} = v_0'''(1), \quad D_{35} = v_1'''(1), \quad D_{36} = v_2'''(1), \quad D_{37} = v_3'''(1),$$

$$D_{41} = u_1(s_0), \quad D_{42} = u_2(s_0), \quad D_{43} = u_3(s_0), \quad D_{44} = -\,v_0(s_0),$$

$$D_{45} = -\,v_1(s_0), \quad D_{46} = -\,v_2(s_0), \quad D_{47} = -\,v_3(s_0),$$

$$D_{51} = u_1'(s_0), \quad D_{52} = u_2'(s_0), \quad D_{53} = u_3'(s_0), \quad D_{54} = -\,v_0'(s_0),$$

$$D_{55} = -\,v_1'(s_0), \quad D_{56} = -\,v_2'(s_0), \quad D_{57} = -\,v_3'(s_0),$$

$$D_{61} = u_1''(s_0), \quad D_{62} = u_2''(s_0), \quad D_{63} = u_3''(s_0), \quad D_{64} = -\,v_0''(s_0),$$

$$D_{65} = -\,v_1''(s_0), \quad D_{66} = -\,v_2''(s_0), \quad D_{67} = -\,v_3''(s_0),$$

$$D_{71} = -\,u_1'''(s_0) + C_3 N u_1(s_0), \quad D_{72} = -\,u_2'''(s_0) + C_3 N u_2(s_0),$$

$$D_{73} = -\,u_3'''(s_0) + C_3 N u_3(s_0), \quad D_{74} = v_0'''(s_0), \quad D_{75} = v_1'''(s_0),$$

$$D_{76} = v_2'''(s_0), \quad D_{77} = v_3'''(s_0). \tag{51}$$

Substituting equations (47) and (48) into equation (38), one obtains the following frequency equation relating the non-dimensional modal frequencies Ω_i (i is the vibration mode) to the moving mass N, the beam angular velocity η, the moving mass position s_0 and the total moment of inertia about the hub J_0:

$$\int_0^{s_0} [A_1^* \xi u_1(\xi) + A_2^* \xi u_2(\xi) + A_3^* \xi u_3(\xi)]\,\mathrm{d}\xi$$

$$+ \int_{s_0}^1 [B_0^* \xi v_0(\xi) + B_1^* \xi v_1(\xi) + B_2^* \xi v_2(\xi) + B_3^* \xi v_3(\xi)]\,\mathrm{d}\xi$$

$$+ N s_0 [A_1^* u_1(s_0) + A_2^* u_2(s_0) + A_3^* u_3(s_0) + s_0] + J_0 + \frac{1}{3} = 0, \tag{52}$$

where

$$A_1^* = \frac{A_1}{F}, \quad A_2^* = \frac{A_2}{F}, \quad A_3^* = \frac{A_3}{F}, \quad B_0^* = \frac{B_0}{F}, \quad B_1^* = \frac{B_1}{F}, \quad B_2^* = \frac{B_2}{F}, \quad B_3^* = \frac{B_3}{F}.$$

Using equations (39) and (43), the spatial derivatives and integrals of u_1, u_2 and u_3 can be obtained. The expressions of u_1, u_2 and u_3 and their integrals are given below:

$$u_1(\xi) = \xi + \sum_{k=0}^{\infty} a_{k+4} \xi^{k+4}, \tag{53}$$

$$u_2(\xi) = \xi^2 + \sum_{k=0}^{\infty} a_{k+4} \xi^{k+4}, \tag{54}$$

$$u_3(\xi) = \xi^3 + \sum_{k=0}^{\infty} a_{k+4} \xi^{k+4}, \tag{55}$$

$$\int_0^{s_0} \xi u_1(\xi)\,\mathrm{d}\xi = \frac{1}{3} s_0^3 + \sum_{k=0}^{\infty} \frac{a_{k+4}}{k+6} s_0^{k+6}, \tag{56}$$

$$\int_0^{s_0} \xi u_2(\xi)\,\mathrm{d}\xi = \frac{1}{4} s_0^4 + \sum_{k=0}^{\infty} \frac{a_{k+4}}{k+6} s_0^{k+6}, \tag{57}$$

$$\int_0^{s_0} \xi u_3(\xi)\,\mathrm{d}\xi = \frac{1}{5} s_0^5 + \sum_{k=0}^{\infty} \frac{a_{k+4}}{k+6} s_0^{k+6}. \tag{58}$$

Using equations (40) and (45), the spatial derivatives and integrals of v_0, v_1, v_2 and v_3 can be obtained. The expressions of v_0, v_1, v_2 and v_3 and their integrals are given below:

$$v_0(\xi) = 1 + \sum_{k=0}^{\infty} b_{k+4} \xi^{k+4}, \tag{59}$$

$$v_1(\xi) = \xi + \sum_{k=0}^{\infty} b_{k+4} \xi^{k+4}, \tag{60}$$

$$v_2(\xi) = \xi^2 + \sum_{k=0}^{\infty} b_{k+4}\xi^{k+4}, \tag{61}$$

$$v_3(\xi) = \xi^3 + \sum_{k=0}^{\infty} b_{k+4}\xi^{k+4}, \tag{62}$$

$$\int_{s_0}^{1} \xi v_0(\xi)\,\mathrm{d}\xi = \frac{1}{2} - \frac{s_0^2}{2} + \sum_{k=0}^{\infty} \frac{b_{k+4}}{k+6}(1 - s_0^{k+6}), \tag{63}$$

$$\int_{s_0}^{1} \xi v_1(\xi)\,\mathrm{d}\xi = \frac{1}{3} - \frac{s_0^3}{3} + \sum_{k=0}^{\infty} \frac{b_{k+4}}{k+6}(1 - s_0^{k+6}), \tag{64}$$

$$\int_{s_0}^{1} \xi v_2(\xi)\,\mathrm{d}\xi = \frac{1}{4} - \frac{s_0^4}{4} + \sum_{k=0}^{\infty} \frac{b_{k+4}}{k+6}(1 - s_0^{k+6}), \tag{65}$$

$$\int_{s_0}^{1} \xi v_3(\xi)\,\mathrm{d}\xi = \frac{1}{5} - \frac{s_0^5}{5} + \sum_{k=0}^{\infty} \frac{b_{k+4}}{k+6}(1 - s_0^{k+6}), \tag{66}$$

Table 1

Non-dimensional first modal frequencies Ω_1 under different moving masses N, mass positions s_0 and beam angular velocities η for $J_0 = 3$

		$J_0 = 3$		First mode Ω_1				
N	s_0	$\eta = 0$	$\eta = 0·5$	$\eta = 1·0$	$\eta = 1·5$	$\eta = 2·0$	$\eta = 2·5$	$\eta = 3·0$
1	0	3·340	3·347	3·366	3·398	3·441	3·494	3·556
	0·2	3·339	3·344	3·361	3·388	3·425	3·469	3·521
	0·4	3·042	3·038	3·028	3·009	2·980	2·940	2·885
	0·6	2·433	2·410	2·341	2·222	2·041	1·780	1·388
	0·8	1·806	1·761	1·618	1·347	0·832	0	0
	1·0	1·298	1·311	1·348	1·404	1·473	1·550	1·631
2	0	3·340	3·347	3·366	3·398	3·441	3·494	3·556
	0·2	3·306	3·311	3·324	3·346	3·376	3·411	3·450
	0·4	2·767	2·755	2·716	2·650	2·550	2·411	2·218
	0·6	1·943	1·902	1·773	1·537	1·125	0	0
	0·8	1·302	1·232	0·995	0·351	0	0	0
	1·0	0·870	0·886	0·929	0·990	1·060	1·133	1·206
3	0	3·340	3·347	3·366	3·398	3·441	3·494	3·556
	0·2	3·273	3·276	3·287	3·304	3·325	3·350	3·376
	0·4	2·536	2·516	2·453	2·343	2·176	1·931	1·566
	0·6	1·631	1·577	1·403	1·056	0	0	0
	0·8	1·032	0·946	0·626	0	0	0	0
	1·0	0·664	0·681	0·726	0·787	0·854	0·921	0·985
4	0	3·340	3·347	3·366	3·398	3·441	3·494	3·556
	0·2	3·239	3·242	3·249	3·261	3·275	3·289	3·300
	0·4	2·341	2·314	2·230	2·080	1·846	1·483	0·822
	0·6	1·412	1·348	1·136	0·655	0	0	0
	0·8	0·859	0·761	0·339	0	0	0	0
	1·0	0·539	0·557	0·603	0·662	0·725	0·786	0·844

where a_{k+4} and b_{k+4} can be determined by the recurrence formula given by equations (41) and (42) respectively. Numerical bisection method [15] for the root finding is then used to solve the non-dimensional modal frequencies Ω_i of the frequency equation (52) under different values of J_0, s_0, N and η. The whole calculation is performed using double-precision FORTRAN programs.

5. RESULTS

Equation (15) shows that as $J_t \to \infty$, $\ddot{\theta}$ approaches zero. When $J_0 = 10\,000$ and $\eta = 0$, the frequencies Ω_i obtained from equation (52) for various values of N and s_0 agree with results in reference [12] for the clamped–free and free–clamped stationary beams. When $J_0 = 3$ and $s_0 = 0$, the results agree with those in reference [13] for $N = 0$, U (axial force) $= 0$ and various η.

In this paper, numerical results and Figures 2–9 are presented for $J_0 = 3$. Tables 1–3 show the calculated values of the non-dimensional first modal frequencies Ω_1, second modal frequencies Ω_2 and third modal frequencies Ω_3, respectively, for different values of moving masses N, mass position s_0 and beam angular velocities η. Figures 2 and 3 show the 2-D

TABLE 2

Non-dimensional second modal frequencies Ω_2 under different moving masses N, mass positions s_0 and beam angular velocities η for $J_0 = 3$

		$J_0 = 3$ Second mode Ω_2						
N	s_0	$\eta = 0$	$\eta = 0.5$	$\eta = 1.0$	$\eta = 1.5$	$\eta = 2.0$	$\eta = 2.5$	$\eta = 3.0$
1	0	22·007	22·038	22·132	22·286	22·500	22·773	23·102
	0·2	19·567	19·594	19·676	19·811	19·999	20·240	20·531
	0·4	15·283	15·306	15·376	15·491	15·650	15·851	16·092
	0·6	18·010	18·050	18·170	18·365	18·632	18·965	19·355
	0·8	21·943	22·024	22·262	22·652	23·178	23·828	24·585
	1·0	16·222	16·343	16·701	17·280	18·058	19·010	20·110
2	0	22·007	22·038	22·132	22·286	22·500	22·773	23·102
	0·2	17·423	17·445	17·510	17·618	17·768	17·959	18·192
	0·4	12·872	12·891	12·948	13·041	13·169	13·328	13·516
	0·6	16·819	16·865	17·000	17·220	17·517	17·878	18·294
	0·8	21·930	22·065	22·461	23·095	23·933	24·939	26·074
	1·0	15·838	16·051	16·674	17·661	18·952	20·485	22·208
3	0	22·007	22·038	22·132	22·286	22·500	22·773	23·102
	0·2	15·686	15·703	15·751	15·832	15·945	16·088	16·261
	0·4	11·629	11·646	11·699	11·786	11·904	12·050	12·219
	0·6	16·207	16·259	16·413	16·659	16·986	17·379	17·821
	0·8	21·926	22·117	22·674	23·551	24·688	26·016	27·478
	1·0	15·700	16·005	16·886	18·254	20·003	22·036	24·275
4	0	22·007	22·038	22·132	22·286	22·500	22·773	23·102
	0·2	14·313	14·325	14·361	14·420	14·502	14·606	14·731
	0·4	10·868	10·886	10·939	11·025	11·143	11·286	11·452
	0·6	15·820	15·879	16·052	16·325	16·684	17·107	17·576
	0·8	21·923	22·173	22·894	24·011	25·429	27·055	28·808
	1·0	15·630	16·026	17·157	18·883	21·048	23·521	26·208

TABLE 3

Non-dimensional third modal frequencies Ω_3 under different moving masses N, mass positions s_0 and beam angular velocities η for $J_0 = 3$

		$J_0 = 3$	Third mode Ω_3					
N	s_0	$\eta = 0$	$\eta = 0{\cdot}5$	$\eta = 1{\cdot}0$	$\eta = 1{\cdot}5$	$\eta = 2{\cdot}0$	$\eta = 2{\cdot}5$	$\eta = 3{\cdot}0$
1	0	61·687	61·721	61·823	61·994	62·233	62·536	62·903
	0·2	57·215	57·261	57·393	57·615	57·921	58·314	58·789
	0·4	59·519	59·560	59·667	59·857	60·124	60·442	60·861
	0·6	61·044	61·094	61·220	61·427	61·768	62·177	62·649
	0·8	61·437	61·532	61·800	62·005	62·209	62·881	63·295
	1·0	50·887	51·026	51·443	52·130	53·076	54·268	55·690
2	0	61·687	61·721	61·823	61·994	62·233	62·536	62·903
	0·2	54·791	54·842	54·994	55·249	55·600	56·049	56·589
	0·4	58·950	58·988	59·099	59·291	59·548	59·893	60·292
	0·6	60·743	60·806	60·959	61·255	61·614	62·167	62·717
	0·8	61·396	61·526	61·827	61·986	62·266	62·713	63·017
	1·0	50·440	50·687	51·419	52·617	54·249	56·277	58·658
3	0	61·687	61·721	61·823	61·994	62·233	62·536	62·903
	0·2	53·079	53·136	53·304	53·579	53·965	54·454	55·046
	0·4	58·642	58·677	58·797	58·968	59·236	59·569	59·965
	0·6	60·505	60·544	60·772	61·086	61·579	62·177	62·883
	0·8	61·056	61·110	61·346	61·553	62·089	62·266	63·205
	1·0	50·284	50·638	51·683	53·379	55·664	58·470	61·721
4	0	61·687	61·721	61·823	61·994	62·233	62·536	62·903
	0·2	51·723	51·781	51·961	52·256	52·665	53·186	53·814
	0·4	58·413	58·457	58·566	58·753	59·007	59·336	59·731
	0·6	60·266	60·359	60·613	60·976	61·569	62·252	63·055
	0·8	61·112	61·226	61·338	61·916	62·062	62·750	63·251
	1·0	50·205	50·665	52·020	54·201	57·112	60·646	64·698

plots of the non-dimensional modal frequencies Ω_i as functions of moving mass N and mass position s_0 for beam angular velocity $\eta = 0$ and 3 respectively. Figures 4 and 5 show the 2-D plots of the non-dimensional modal frequencies Ω_i as function of beam angular velocity η for different mass position s_0 and moving mass N. Figures 6 and 7 show the 3-D plots of the non-dimensional modal frequencies Ω_i as functions of beam angular velocity η and moving mass N for different mass position s_0. Figures 8 and 9 show the 3-D plots of the non-dimensional modal frequencies Ω_i as functions of beam angular velocity η and mass position s_0 for different moving mass N.

Table 1 shows that as the mass approaches the tip of the beam for $1{\cdot}5 \leqslant \eta \leqslant 3{\cdot}0$, the non-dimensional first modal frequencies become zero. However, at $s_0 = 1$, they reappear again. The modal frequencies at $s_0 = 1$ are obtained using the equation of motion given by equation (11) and the boundary conditions given by equations (12a–d).

Figures 2, 6 and 7 show that when $\eta = 0$ the non-dimensional frequencies decrease or remain approximately constant with increase in moving mass N for different values of s_0. Figures 3 and 6 show that for $\eta = 3$ and $s_0 = 0{\cdot}2$ the modal frequencies decrease with increase in N. Figure 3 also reveals that for $s_0 = 0{\cdot}5$ the third modal frequency increases with increase in N whereas the first and second modal frequencies decrease with increase in N. Figure 3 also shows that at $s_0 = 0{\cdot}8$, the first and third modal frequencies remain zero

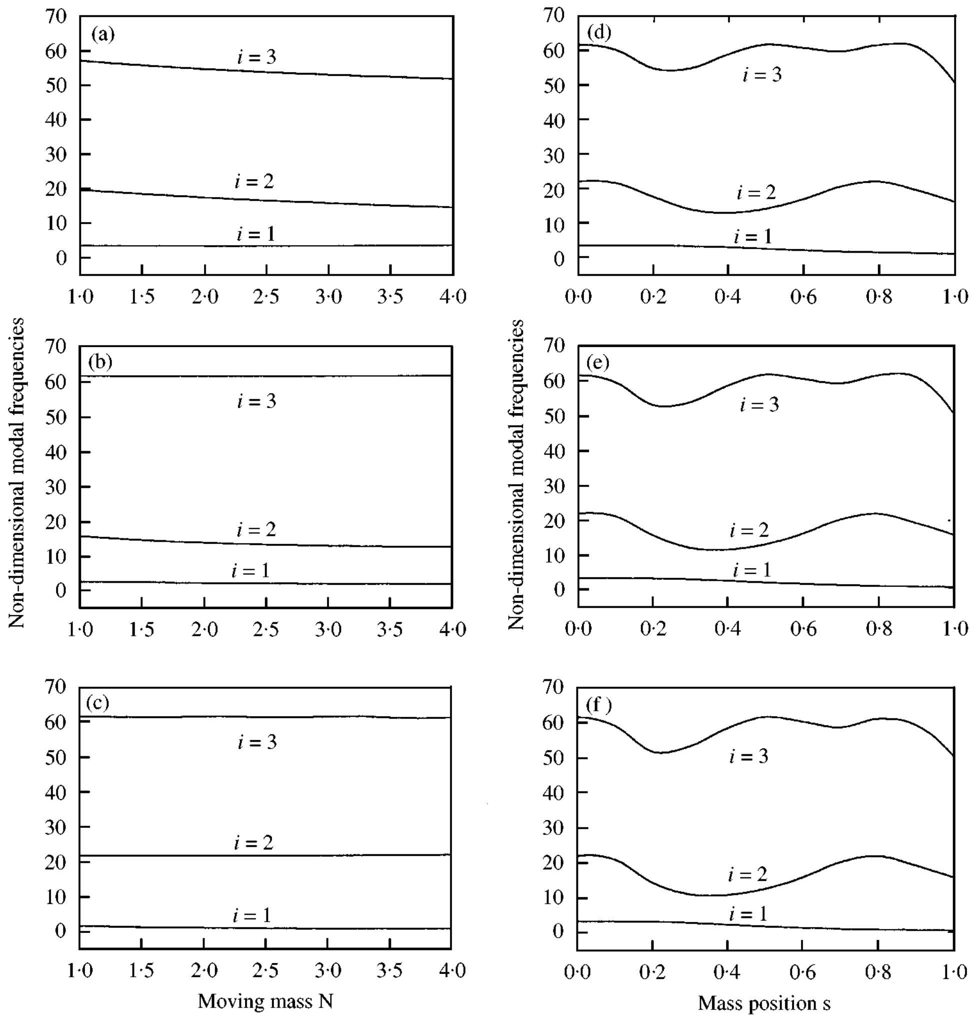

Figure 2. Non-dimensional modal frequencies Ω_i as functions of moving mass N and mass position s_0 for beam angular velocity $\eta = 0$. Values of s_0: (a) 0·2; (b) 0·5; (c) 0·8. Values of N: (d) 2; (e) 3; (f) 4.

and approximately constant, respectively, with increase in N, whereas the second modal frequencies increase with increase in N.

Figures 4 and 5 show that in general the non-dimensional second and third modal frequencies increase steadily with increase in η for different values of N and s_0. However, for the first vibration mode with various N, Figures 4 and 6 show that at $s_0 = 0{\cdot}2$ the frequencies increase with increase in η. Figures 4, 5 and 7 show that at $s_0 = 0{\cdot}3, 0{\cdot}5, 0{\cdot}6$ and 0·7, the first modal frequencies decrease with increase in η for $N = 2$ and 4.

Figures 2 and 3 and Figures 8 and 9 show that for different values of N and η, the non-dimensional second and third modal frequencies increase and decrease repeatedly as s_0 varies from zero to one. For the first vibration mode under different values of N and η, the frequencies decrease steadily as s_0 increases from zero.

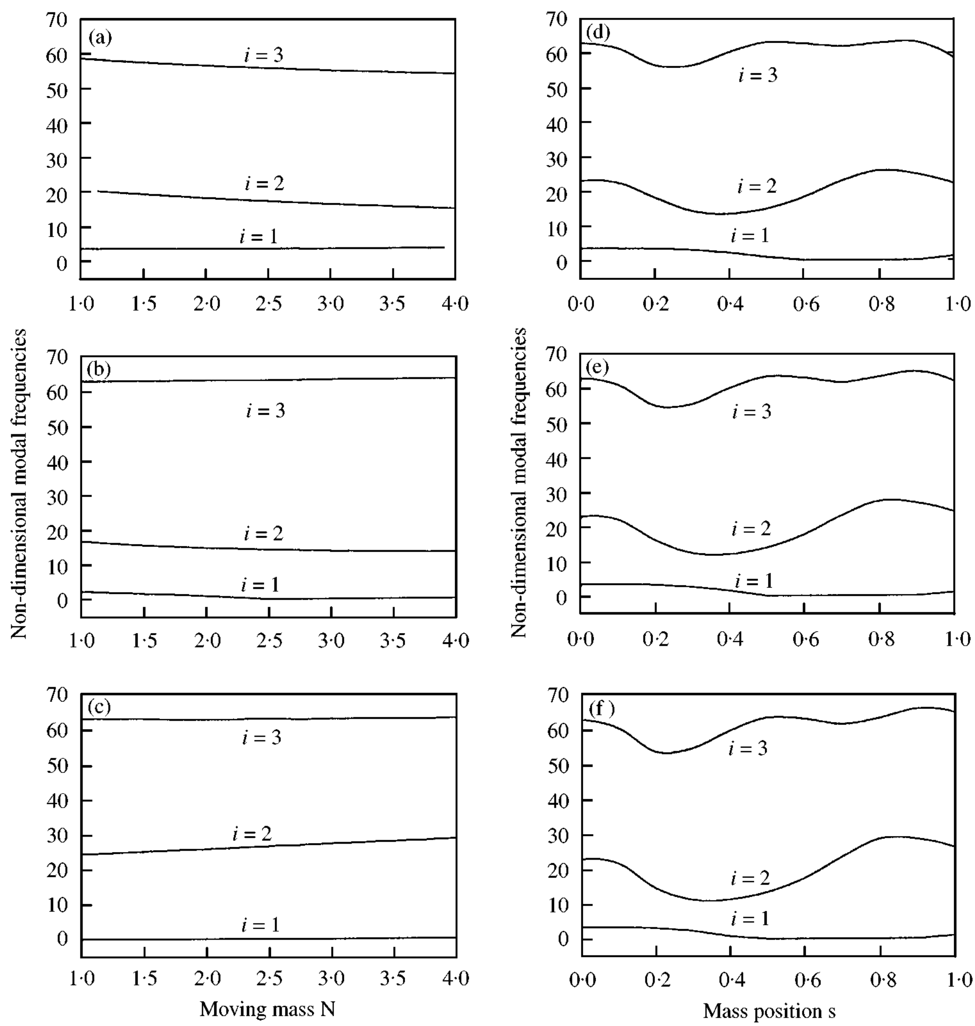

Figure 3. Non-dimensional modal frequencies Ω_i as functions of moving mass N and mass position s_0 for beam angular velocity $\eta = 3{\cdot}0$. Values of s_0: (a) 0·2; (b) 0·5; (c) 0·8. Values of N: (d) 2; (e) 3; (f) 4.

6. CONCLUSIONS

In this paper, the equation of motion of a clamped–free flexible Euler–Bernoulli beam rotating in a horizontal plane and carrying a moving mass is derived by Hamilton's principle including the effects of centrifugal stiffening. The beam is divided into two separate regions with respect to the moving mass. Two mode shape differential equations are derived with four boundary, one discontinuity and three continuity conditions. The power series method is used to solve the mode shape differential equations. A frequency equation is derived giving the relationship between the non-dimensional modal frequencies and the four non-dimensional parameters namely the moving mass position, the moving mass, the beam angular velocity and the total moment of inertia about the hub. Numerical bisection method with double-precision FORTRAN programs are used to solve the frequency equation. Results are presented for the first three modes of vibration. These results are

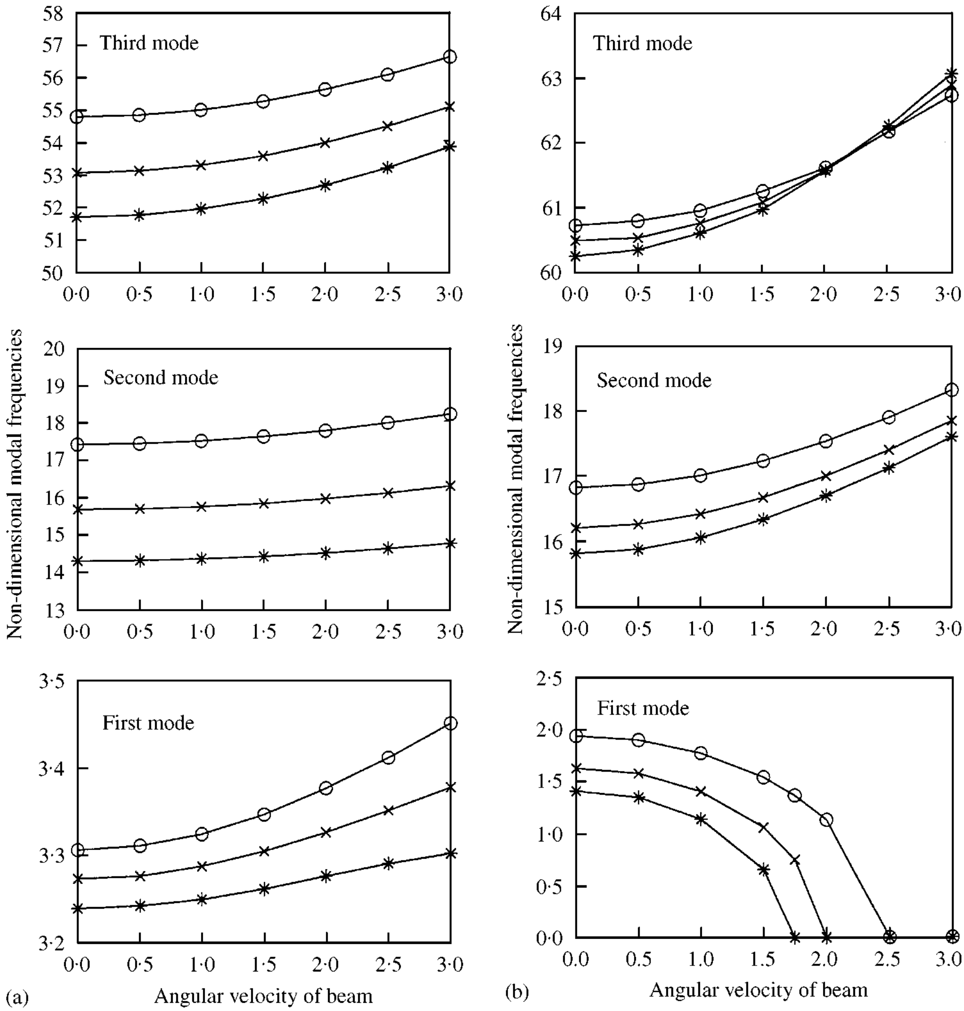

Figure 4. Non-dimensional modal frequencies Ω_i as function of beam angular velocity η. Values of mass position s_0: (a) $s_0 = 0{\cdot}2$; (b) $s_0 = 0{\cdot}6$. Values of moving mass N: —○— $N = 2$; —×— $N = 3$; —∗— $N = 4$.

useful in understanding the dynamic behavior of many practical engineering problems that involve rotation of flexible arm carrying a moving mass.

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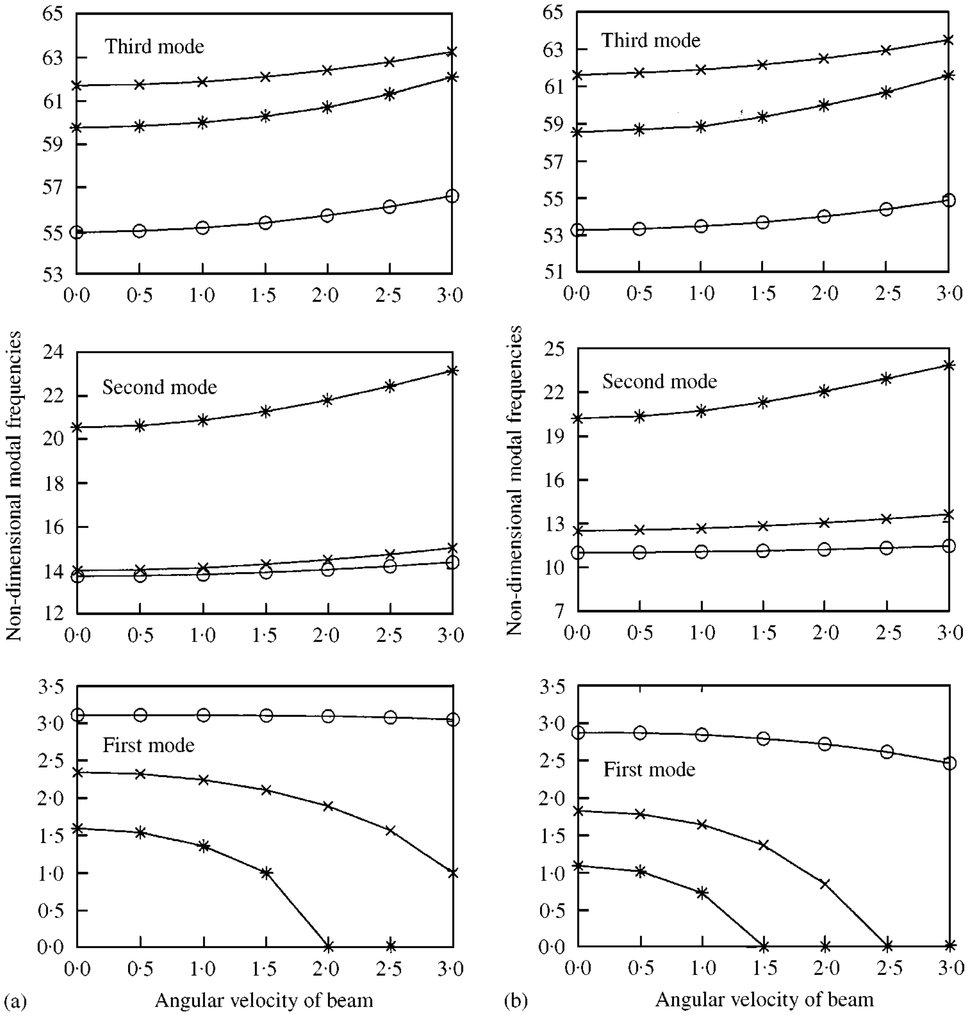

Figure 5. Non-dimensional modal frequencies Ω_i as function of beam angular velocity η. Values of moving mass N: (a) $N = 2$; (b) $N = 4$. Values of mass position s_0: —○— $s_0 = 0{\cdot}3$; —×— $s_0 = 0{\cdot}5$; —∗— $s_0 = 0{\cdot}7$.

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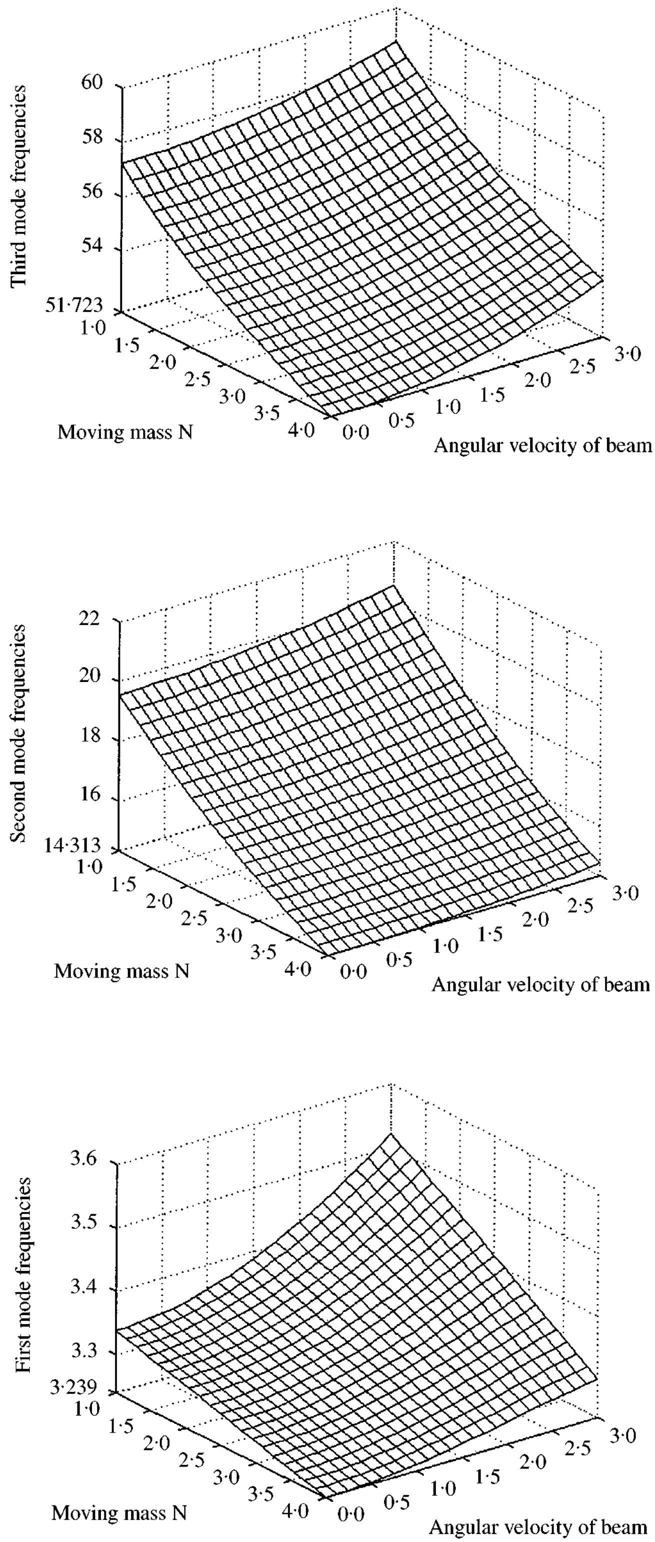

Figure 6. Non-dimensional modal frequencies Ω_i as functions of beam angular velocity η and moving mass N for mass position $s_0 = 0{\cdot}2$.

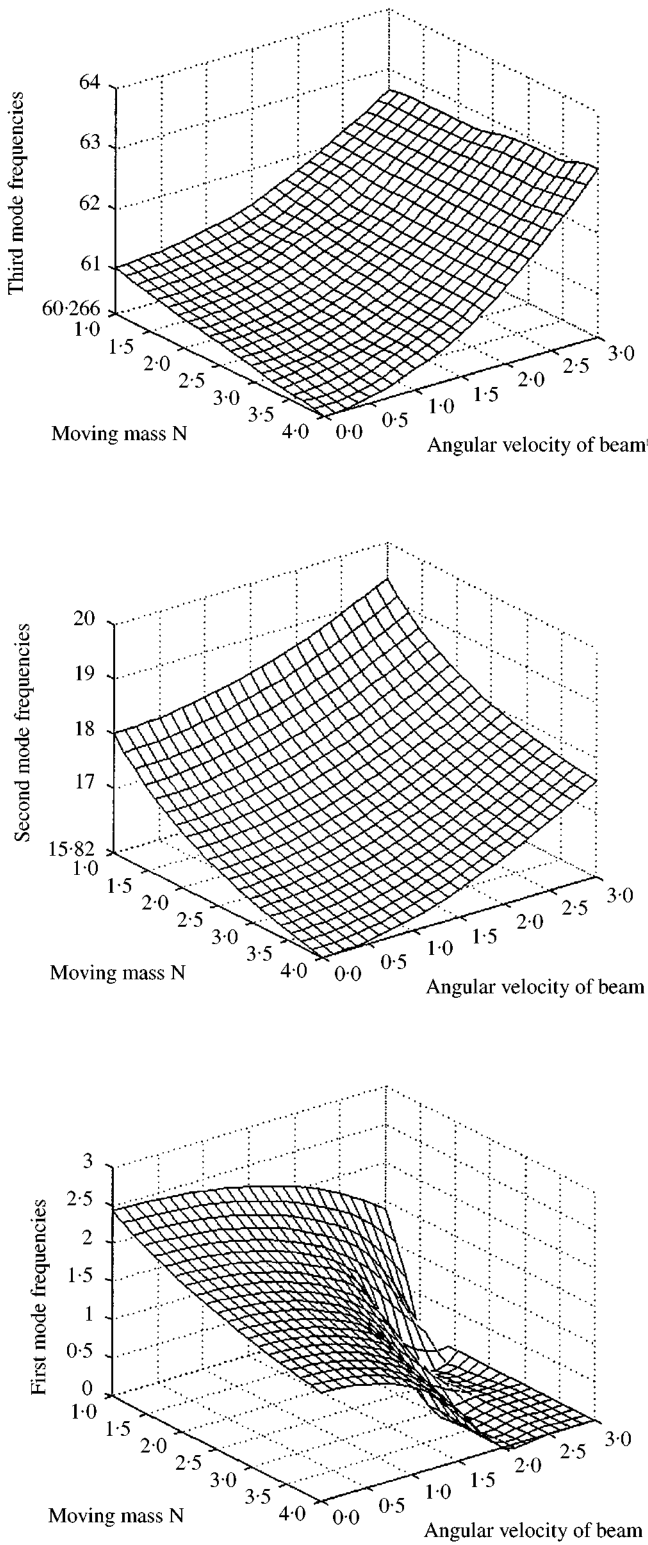

Figure 7. Non-dimensional modal frequencies Ω_i as functions of beam angular velocity η and moving mass N for mass position $s_0 = 0{\cdot}6$.

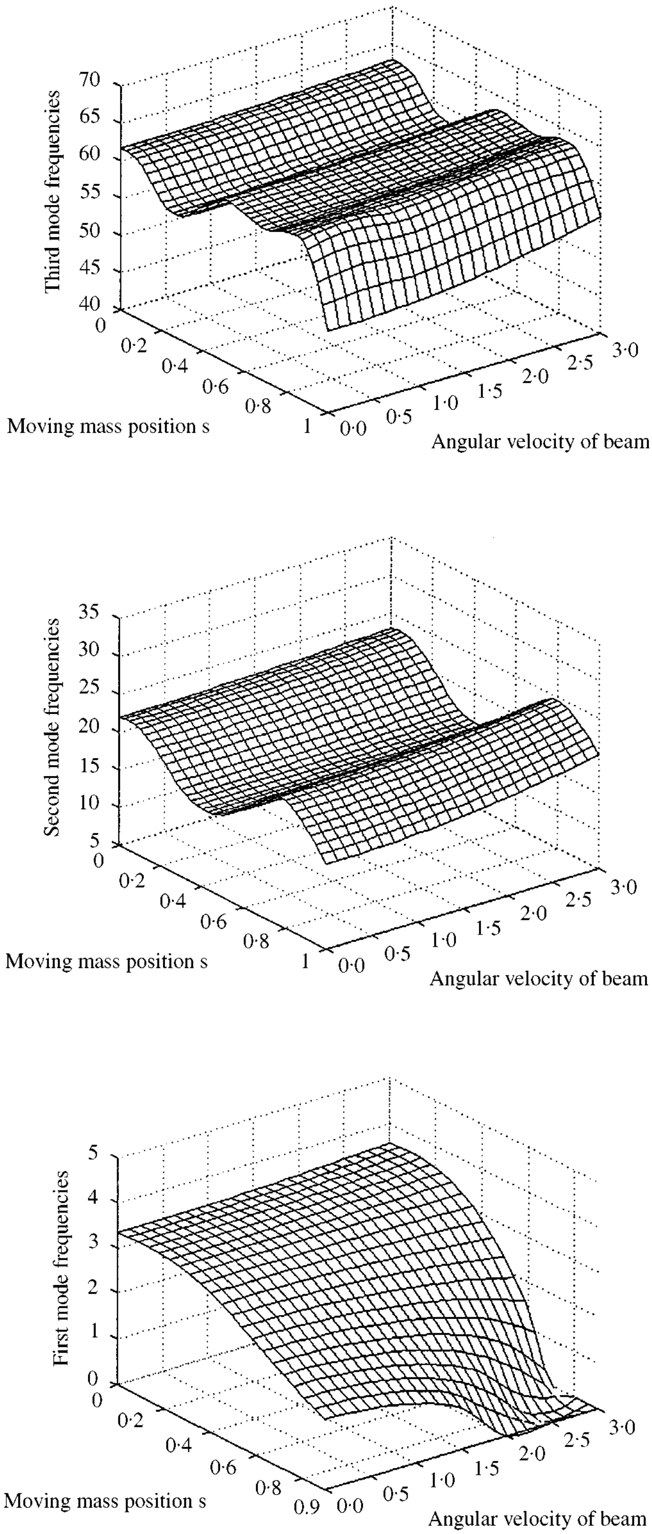

Figure 8. Non-dimensional modal frequencies Ω_i as functions of beam angular velocity η and mass position s_0 for moving mass $N = 1$.

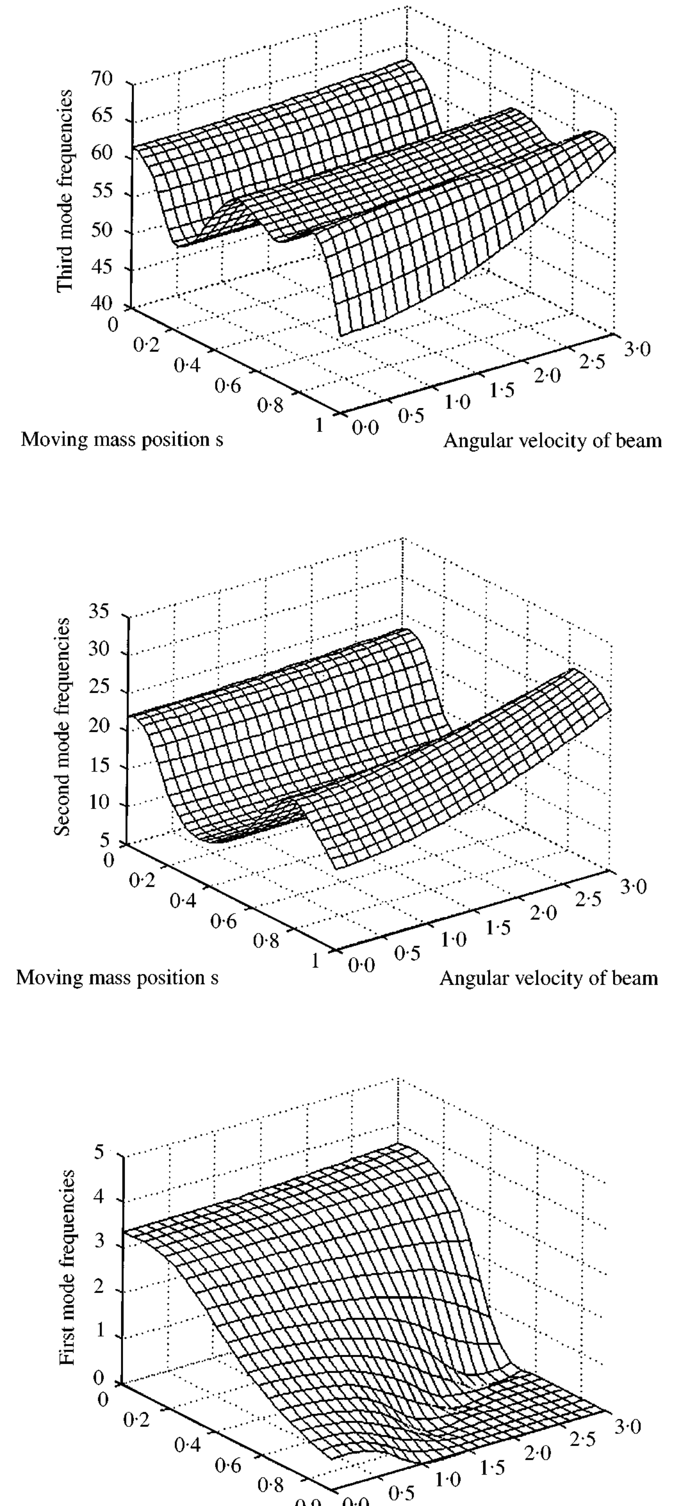

Figure 9. Non-dimensional modal frequencies Ω_i as functions of beam angular velocity η and mass position s_0 for moving mass $N = 4$.

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APPENDIX A: NOMENCLATURE

EI	flexural rigidity of flexible beam
J	moment of inertia of the hub
J_t	total moment of inertia about the hub
J_0	non-dimensional form of J_t
L	length of flexible beam
m	moving mass
N	non-dimensional moving mass
η	non-dimensional angular velocity of flexible beam
$P(r, t)$	centrifugal force arising from centrifugal effect
$P_0(\xi)$	non-dimensional centrifugal force defined in equation (28)
r	position of a point on flexible beam
s	position of moving mass with respect to the clamped axis of beam
$\dot{s}$	velocity of moving mass relative to flexible beam
s_0	non-dimensional form of s
t	time
T_b	kinetic energy of flexible arm
T_m	kinetic energy of moving mass
τ	applied torque developed by motor
μ_l	load torque developed by flexible beam and moving mass
μ	torque defined in equation (19)
μ_0	non-dimensional form of μ
V	total potential energy of flexible arm
w	transverse displacement of flexible beam
Y	mode shape function defined in equation (16)
θ	hub angle of flexible beam
$\dot{\theta}$	angular velocity of flexible beam
ρ	mass per unit length of flexible beam
$\delta(\cdot)$	Dirac delta function
δW	virtual work
ω_i	modal frequency of flexible beam
Ω_i	non-dimensional modal frequency of flexible beam
ξ	non-dimensional spatial co-ordinate
$\mathbf{r}$	position vector of a point on flexible beam
$(\mathbf{i}, \mathbf{j})$	a pair of orthogonal unit vectors for flexible beam
$\mathbf{V}_m$	resultant velocity of moving mass