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A SUPPLEMENTARY CONDITION FOR CALCULATING PERIODICAL VIBRATIONS

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Fourier series are often used to study structural vibrations induced by periodical loads. However, there is a hidden error when the response is expressed by introducing a phase lag, which makes the expression more concise. This article identifies what the error is and introduces a supplementary condition to avoid such an error.

Periodical vibrations can be expressed by a Fourier series as follows [1]:

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi t}{T} + b_n \sin\frac{2n\pi t}{T}\right). \tag{1}$$

Alternative, equation (1) can be represented in a more concise form as

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r_n \sin\left(\frac{2n\pi t}{T} + \phi_n\right), \tag{2a}$$

where

$$r_n = \sqrt{a_n^2 + b_n^2}, \quad \phi_n = \tan^{-1}\left(\frac{a_n}{b_n}\right). \tag{2b, 2c}$$

Equation (2) is often used instead of equation (1) in calculations. However, it should be noted that equations (1) and (2) are not identical. The difference between these equations is due to the definition of the angle, or the phase lag, ϕ_n, in equation (2c). When deriving equation (2) from equation (1), one has

$$\sin\phi_n = \frac{a_n}{\sqrt{a_n^2 + b_n^2}}, \quad \cos\phi_n = \frac{b_n}{\sqrt{a_n^2 + b_n^2}}. \tag{3a, 3b}$$

As there are no limitations on the signs of a_n and b_n, the phase lag ϕ_n in equation (3) may vary between 0 and 2π. However, the angle ϕ_n in equation (2a) defined by equation (2c) and

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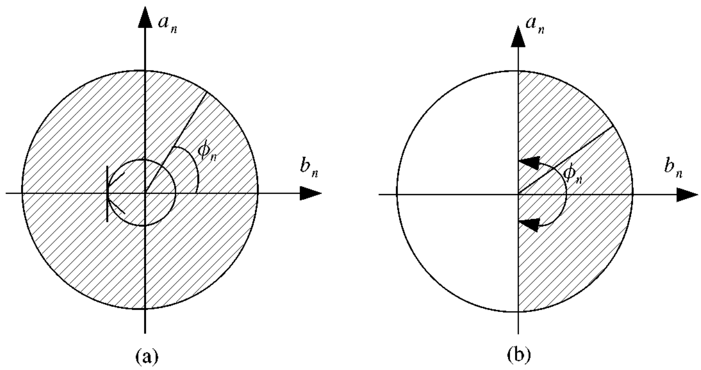

Figure 1. Definition of the range of the phase lag ϕ_n: (a) $-\pi < \phi_n < \pi$ in the actual summation; (b) $-\pi/2 < \phi_n < \pi/2$ in equation (2).

TABLE 1

Comparison of the phase lags defined in Figure 1(*a*) (*equation* (3)) *and Figure* 1(*b*) (*equation* (2*c*))

Situation	ϕ_n in Figure 1(a) or equation (3)	ϕ_n in Figure 1(b) or equation (2c)	Conclusion
When $a_n > 0$ and $b_n > 0$	$0 < \phi_n < \pi/2$	$0 < \phi_n < \pi/2$	Equation (3) = equation (2c)
When $a_n > 0$ and $b_n < 0$	$\pi/2 < \phi_n < \pi$	$-\pi/2 < \phi_n < 0$	Equation (3) $\neq$ equation (2c)
When $a_n < 0$ and $b_n > 0$	$-\pi/2 < \phi_n < 0$	$-\pi/2 < \phi_n < 0$	Equation (3) = equation (2c)
When $a_n < 0$ and $b_n < 0$	$-\pi < \phi_n < -\pi/2$	$0 < \phi_n < \pi/2$	Equation (3) $\neq$ equation (2c)

calculated by some computer software is assigned a value between $-\pi/2$ and $\pi/2$. This is because equation (2c) only considers the ratio rather than the individual signs of the numerator and denominator, i.e., there is no difference between $-a_n/b_n$ and $a_n/-b_n$ or between a_n/b_n and $-a_n/-b_n$. Therefore, the defined region of the phase lag ϕ_n in equation (2c) (Figure 1 (b)) does not match that in the original equation (equation (1) or equation (3)) as shown in Figure 1a. It can be examined in detail as listed in Table 1.

It can be noted that the range of the phase lag is incomplete in equation (2c) and has a difference of π with equation (3) when $b_n < 0$. To cure the problem, a supplementary condition to equation (2c) must be introduced as follows:

$$\text{If } b_n < 0 \quad \text{then } \phi_n = \tan^{-1}\left(\frac{a_n}{b_n}\right) + \pi. \tag{4}$$

The above finding was found to be important when the authors were studying floor responses to dance-type loads by Fourier analysis [2]. Dance-type loads can be described by a high contact force for a certain time t_p (contact duration) followed by zero force when the feet leave the floor. The contact duration t_p may vary from 0 to T_p (the period of dance type load) corresponding to different movements and activities. The contact ratio was then defined as follows:

$$\alpha = t_p/T_p\,. \tag{5}$$

Thus, different contact ratios characterize different rhythmic activities. The dance-type loads were defined in a period as follows [3]:

$$F(t) = \begin{cases} K_p G \sin(\pi t/t_p), & 0 \leqslant t \leqslant t_p, \\ 0, & t_p \leqslant t \leqslant T_p. \end{cases} \tag{6}$$

where K_p is the impact factor, F_{max}/G, and is equal to $\pi/2\alpha$, F_{max} is the peak dynamic load, G is the weight of the jumper, t_p is the contact duration and T_p is the period of the jumping load.

Alternatively, the above equation can be equivalently expressed as a function of the contact ratio by Fourier series [2]:

$$F(t) = G\left[1 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi t}{T_p} + b_n \sin\frac{2n\pi t}{T_p}\right)\right] = G\left[1 + \sum_{n=1}^{\infty} S_n(\alpha, T_p)\right], \tag{7a}$$

where

$$S_n(\alpha, T_P) = a_n(\alpha)\cos\frac{2n\pi t}{T_p} + b_n(\alpha)\sin\frac{2n\pi t}{T_p}, \tag{7b}$$

$$a_n(\alpha) = \begin{cases} 0, & n\alpha = \frac{1}{2}, \\ \frac{1 + \cos(2n\pi\alpha)}{(1 - 4n^2\alpha^2)}, & n\alpha \neq \frac{1}{2}, \end{cases} \tag{7c}$$

$$b_n(\alpha) = \begin{cases} \frac{\pi}{2}, & n\alpha = \frac{1}{2}, \\ \frac{\sin(2n\pi\alpha)}{(1 - 4n^2\alpha^2)}, & n\alpha \neq \frac{1}{2}. \end{cases} \tag{7d}$$

Equation (7a) indicates that human-induced loads equal the sum of the static-body weight and some dynamic components. Equation (7) can also be represented as follows:

$$F(t) = G\left[1 + \sum_{n=1}^{\infty} r_n \sin\left(\frac{2n\pi t}{T_p} + \phi_n\right)\right] = G\left[1 + \sum_{n=1}^{\infty} S_n^{\phi}(\alpha, T_p)\right], \tag{8a}$$

where

$$S_n^{\phi}(\alpha, T) = r_n \sin\left(\frac{2n\pi t}{T_p} + \phi_n\right), \tag{8b}$$

$$r_n = \sqrt{a_n(\alpha)^2 + b_n(\alpha)^2}, \quad \phi_n = \tan^{-1}\left(\frac{a_n(\alpha)}{b_n(\alpha)}\right). \tag{8c, 8d}$$

Table 2 provides the first six Fourier coefficients and the phase lags when $\alpha = \frac{2}{3}$.

It can be noted from Table 2 that when $n = 2$ or 5, $b_n < 0$ and since the ratio of a_n to b_n is positive, the phase lag would be of $\pi/6$ if only equation (2c) or equation (8d) is used. Following the supplementary condition equation (4), the phase lag should be $-5\pi/6$ (or $7\pi/6$).

Considering an example having the following data:

$$G = 1.0, \qquad T_p = 0.5, \qquad \alpha = \tfrac{2}{3}.$$

TABLE 2

The first six coefficients a_n, b_n, r_n and ϕ_n when $\alpha = \frac{2}{3}$

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
a_n	$-\frac{9}{14}$	$-\frac{9}{110}$	$-\frac{2}{15}$	$-\frac{9}{494}$	$-\frac{9}{782}$	$-\frac{2}{63}$
b_n	$\frac{9\sqrt{3}}{14}$	$-\frac{9\sqrt{3}}{110}$	0	$\frac{9\sqrt{3}}{494}$	$-\frac{9\sqrt{3}}{782}$	0
r_n	$\frac{9}{7}$	$\frac{9}{55}$	$\frac{2}{15}$	$\frac{9}{247}$	$\frac{9}{391}$	$\frac{2}{63}$
ϕ_n	$-\frac{\pi}{6}$	$-\frac{5\pi}{6}$	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	$-\frac{5\pi}{6}$	$-\frac{\pi}{2}$

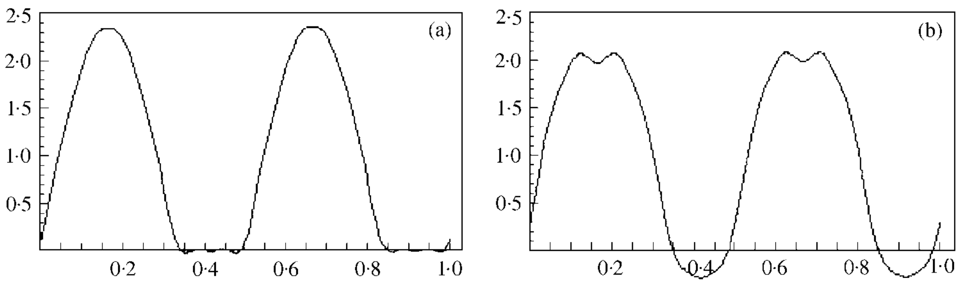

Figure 2. The comparison between the load models defined by equation (8): (a) using the supplementary condition (equation (4)); (b) without using equation (4).

The load curves (equation (8)), taking the first six Fourier terms, with and without using the supplementary condition (equation (4)) are shown in Figure 2.

It is obvious that Figure 2(b) gives an incorrect load model as it shows even negative loading and the peak does not occur at the centre of the applied load, which are different from the original definition of equation (6). Figure 2(a) provides a correct presentation of the load. As only the first six terms are used, there is a slight but negligible difference between equations (6) and equations (8) and (4) as expected.

To conclude this study and to avoid the mistake in the calculation of the phase lag, there are two ways to be considered:

1. Equation (1) is directly used to avoid the calculation of the phase lag, otherwise
2. the phase lag ϕ_n in equation (2a) should be determined, rather than equation (2c), using the following equations:

$$\phi_n = \begin{cases} \tan^{-1}\left(\dfrac{a_n}{b_n}\right) + \pi & \text{when } b_n < 0, \\ -\dfrac{\pi}{2} & \text{when } b_n = 0 \quad \text{and } a_n < 0, \\ \dfrac{\pi}{2} & \text{when } b_n = 0 \quad \text{and } a_n > 0, \\ \tan^{-1}\left(\dfrac{a_n}{b_n}\right) & \text{when } b_n > 0. \end{cases} \tag{9}$$

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